Statistica Sinica **6**(1996), 289-297

ON THE CONSTRUCTION OF TREND RESISTANT ASYMMETRICAL ORTHOGONAL ARRAYS

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Abstract. In this paper we show how to use difference matrices to construct trend resistant asymmetrical orthogonal arrays. In particular, by modifying a method of constructing asymmetrical orthogonal arrays given by Wang and Wu (1991), it is shown that the treatments in many previously constructed asymmetrical orthogonal arrays can be ordered to give various degrees of trend resistance.

Key words and phrases: Orthogonal main effects plan, orthogonal array, trend resistant, polynomial trend.

1. Introduction

In this paper we consider experimental situations where the treatments that make up a mixed level orthogonal main effects plan (OMEP) are to be applied sequentially to experimental units over space or time and where there may be an unknown or uncontrollable trend effect which is highly correlated with the order in which the observations are obtained. Any ordered application of treatments to experimental units over space or time is called a run order. In situations such as described above, the experimenter may prefer to assign treatments to experimental units in such a way that the usual estimates for the factorial effects of interest are not affected by the unknown trend. Such run orders are called trend resistant. A good deal of work has been done on the construction of trend resistant run orders of factorial designs when all factors have the same number of levels, (see Bailey, Cheng and Kipnis (1992) for a summary of the work done on these problems). The only work known to the author on the construction of trend resistant mixed level factorial run orders is that done in Coster (1993), Bailey, Cheng and Kipnis (1992) and Jacroux (1992). Unfortunately, the construction methods given in these papers require fairly large numbers of experimental units when the number of factors involved is large. In this paper, we show that by modifying a method of constructing asymmetrical OMEP's given in Wang and Wu (1991), a number of trend resistant mixed level OMEP's can be constructed which can handle fairly large numbers of factors with smaller numbers of observations.

2. Notation and Definitions

In this section, the notation and main definitions which are used throughout the sequel are given. It is assumed that an OMEP is to be constructed for n factors, denoted by $A_1, \ldots, A_n$ where factor A_i has p_i levels, $i = 1, \ldots, n$. An allocation of factor level combinations to available experimental units will be called a design and denoted by $\boldsymbol{d}$. We denote the levels of a given factor having p_i levels by 0, 1, ..., $p_i - 1$ and assume throughout that p_i is a prime or power of a prime. The model assumed here for analyzing the data from a given design $\boldsymbol{d}$ is one which utilizes orthogonal polynomials.

Definition 2.1. The system of orthogonal polynomials on m equally spaced points $i = 0, 1, \ldots, m-1$ is the set $\{P_{km}, k = 0, \ldots, m-1\}$ of polynomials satisfying

$$\sum_{i=0}^{m-1} P_{k'm}(i)\, P_{km}(i) = 0 \quad \text{for all } k \neq k',$$

where $P_{om}(i) = 1$ for $i = 0, 1, \ldots, m-1$ and $P_{km}(i)$ is a polynomial of degree k. We assume that each polynomial in the system is scaled so that its values are always integers.

With each factor A_i having p_i levels, we associate $p_i - 1$ main effect component parameters A_i^j, $j = 1, \ldots, p_i - 1$. A_i^j is called the jth order main effect of A_i. For a given ordered allocation of the treatments in $\boldsymbol{d}$ to experimental units, suppose we let $\boldsymbol{y} = (y_1, \ldots, y_N)'$ denote the ordered vector of observations obtained. The model for $\boldsymbol{d}$ can be written as

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{\epsilon} = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_2\boldsymbol{\beta}_2 + \boldsymbol{\epsilon}, \tag{2.1}$$

where ϵ is an $N \times 1$ vector of independent error terms having expectation zero and constant variance σ^2. The parameters in $\boldsymbol{\beta}_1$ correspond to the factorial effects defined above and the parameters in $\boldsymbol{\beta}_2$ to possible trend effects. If we let $\boldsymbol{X} = (\boldsymbol{x}_0, \ldots, \boldsymbol{x}_p) = (x_{ij})$, we shall assume that the first column of $\boldsymbol{X}$, $\boldsymbol{x}_0$, corresponds to an overall mean effect and that it is a column of 1's. If $\boldsymbol{x}_t$ of $\boldsymbol{X}_1$ corresponds to main effect component A_i^j and factor A_i occurs at level w in run y_l, then $x_{lt} = P_{jp_i}(w)$. It is assumed that any trend effect can be represented as a polynomial of the form trend effect $= \alpha_0 + \alpha_1 x + \cdots + \alpha_\ominus\, x^\ominus$, $x = 1, \ldots, N$, and the values that x assumes correspond to the positions in which observations are obtained in the run order.

In this paper we restrict our attention to OMEP's. A design $\boldsymbol{d}$ is said to be an OMEP if the columns of $\boldsymbol{X}_1$ in model (2.1) corresponding to the main effect parameters form a mutually orthogonal set of vectors.

Definition 2.2. Let $\boldsymbol{d}$ be an OMEP such as described above. Let $\boldsymbol{x}_j$ be the column of $\boldsymbol{X}_1$ in model (2.1) corresponding to the main effect component A_i^k. We say $\boldsymbol{x}_j$ or A_i^k is t-trend free or t-trend resistant if $\sum_{i=1}^{N} x_{ij}\, i^Z = 0$ for $z = 0, 1, \ldots, t$. We say factor A_i is t-trend free if all $p_i - 1$ main effect components of factor A_i are at least t-trend free.

In this paper we primarily consider the construction of trend resistant OMEP's which can be constructed from orthogonal arrays of strength two. In describing these orthogonal arrays, we adopt much of the notation used in Wang and Wu (1991). Formally, an orthogonal array of strength two, denoted by $L_N\,(p_1^{n_1} \cdots p_r^{n_r})$, is an $N \times n$ matrix, $n = n_1 + \cdots + n_r$, having n_i columns with p_i levels, p_i being unequal, such that for any two columns all of their level combinations appear equally often. If $r > 1$, the array is said to be asymmetrical or have mixed levels. We also use $L_N(p_1)$ to denote an array in which the p_1 levels of some factor occur N/p_1 times in a column. An orthogonal array of strength two is saturated if $\sum_{i=1}^{r} n_i(p_i - 1) = N - 1$.

In Section 3 we give a systematic method for constructing asymmetrical OMEP's which is given in Wang and Wu (1991). By modifying this method of constructing OMEP's and combining it with a result given in Jacroux (1992), we show how to construct a large number of trend resistant OMEP's.

3. Construction Method

Wang and Wu (1991) have given a systematic method of constructing asymmetrical orthogonal arrays of strength two using difference matrices. Thus, we first give a brief review of difference matrices. Let G be an additive group of p elements denoted $\{0, 1, \ldots, p-1\}$. A $\lambda p \times k$ matrix with elements from G, denoted by $D_{\lambda p,k;p}$, is called a difference matrix if, among the differences modulus p, of the corresponding elements of any two columns, each element of G occurs exactly λ times. We note that any difference matrix $D_{\lambda p,k;p}$ can be written in the form $D_{\lambda p,k;p} = (0J^{(1)}_{\lambda p,1}, L^{(2)}_{\lambda p}(p), \ldots, L^{(k)}_{\lambda p}(p))$ where J_{mn} is an $m \times n$ matrix of ones and $L^{(i)}_{\lambda p}(p)$ denotes the ith column of $D_{\lambda p,k;p}$. We note that any subset of the columns which make up $D_{\lambda p,k;p}$ is also a difference matrix.

Let $A = [A_1, \ldots, A_n]$ and $B = [B_1, \ldots, B_n]$ be two partitioned matrices such that for each i, both A_i and B_i have entries from an additive group $G_i = \{0, 1, \ldots, p_i - 1\}$. The generalized Kronecker sum of A and B, denoted by $A \otimes B$ and defined in Wang and Wu (1991), is

$$A \circledast B = [A_1 * B_1, \ldots, A_n * B_n],$$

where for $l \times r$ and $m \times s$ matrices $A_q = (a_{ij}^{(q)})$ and $B_q = (b_{ij}^{(q)})$ whose elements are from the same additive group G_q, $A_q * B_q = [B_q^{(a_{ij}^{(q)})}]$ and $B_q^y = (B_q + yJ_{ms}) (\operatorname{mod} p_q)$

where B_q^y is obtained by adding y to each element of B_q modulus p_q. With the above definitions in mind, Wang and Wu (1991) prove the following result.

Lemma 3.1. *If A is an orthogonal array $L_N(p_1^{n_1}\cdots p_r^{n_r})$ with the partition $A=[L_N(p_1^{n_1}),\ldots,L_N(p_r^{n_r})]$ and $B=[D_{M,k_1;p_1},\ldots,D_{M,k_r;p_r}]$ where $D_{M,k_i;p_i}$ is a difference matrix and N and M are both multiples of the p_i's, then their generalized Kronecker sum*

$$A\circledast B=[L_n(p_1^{n_1})*D_{M,k_1;p_1},\ldots,L_N(p_r^{n_r})*D_{M,k_r,p_r}]$$

is an orthogonal array $L_{NM}(p_1^{k_1n_1}\cdots p_r^{k_rn_r})$.

Jacroux (1992) has proven the following result.

Lemma 3.2. *Suppose $L_n(p_1)$ and $L_M(p_1)$ are arrays which are t_1 and t_2-trend free, respectively, and let $\mathbf{0}_l$ denote an $l\times 1$ vector of zeros. Then*
(a) *$L_{NM}(p_1)=L_N(p_1)*L_M(p_1)$ is (t_1+t_2+1)-trend free.*
(b) *$L_{NM}(p_1)=\mathbf{0}_N*L_M(p_1)$ is t_2-trend free.*
(c) *$L_{NM}(p_1)=L_N(p_1)*\mathbf{0}_M$ is t_1-trend free.*

Using Lemma 3.2, we obtain the following easy generalization.

Theorem 3.3. *Let $D_{N,k_1;p_1}=(\mathbf{0}_N^{(1)},L_N^{(2)}(p_1),\ldots,L_N^{(k_1)}(p_1))$ and $D_{N,k_2;p_1}=(\mathbf{0}_M^{(1)},L_M^{(2)}(p_1),\ldots,L_M^{(k_2)}(p_1))$. Now let $D_{NM,k_1k_2;p_1}=D_{N,k_1;p_1}*D_{M,k_2;p_1}$. Then*
(a) *If the ith column of $D_{N,k_1;p_1}$, $i=2,\ldots,k_1$, is a_i-trend free and the jth column of $D_{M,k_2;p_1}$, $j=2,\ldots,k_2$, is b_j-trend free, then column $(i-1)k_2+j$ of $D_{NM,k_1k_2;p_1}$ is (a_i+b_j+1)-trend free.*
(b) *The first column of $D_{NM,k_1k_2;p_1}$ has all zeros and all remaining columns of $D_{NM,k_1k_2;p_1}$ corresponding to $\mathbf{0}_N*L_M^{(i)}(p_1)$ or $L_N^{(j)}(p_1)*\mathbf{0}_M$ have the same level of trend resistance in $D_{NM,k_1k_2;p_1}$ as $L_M^{(i)}(p_1)$ or $L_N^{(j)}(p_1)$ do in $D_{M,k_2;p_1}$ and $D_{N,k_1;p_1}$, respectively.*

Proof. Simply observe that with the partitions given above for $D_{N,k_1;p_1}$ and $D_{M,k_2;p_1}$,

$$\begin{aligned}D_{NM;k_1k_2;p_1}&=D_{N,k_1;p_1}*D_{M,k_2;p_1}\\&=\left(\left(0_N^{(1)}*D_{M,k_2;p_1}\right),\left(L_N^{(2)}(p_1)*D_{M,k_2;p_1}\right),\ldots,\left(L_N^{(k_1)}(p_1)*D_M\right)\right)\\&=\Big(\Big(0_N^{(1)}*0_M^{(1)},0_N^{(1)}*L_M^{(2)}(p_1),\ldots,0_M^{(1)}*L_M^{(k_2)}(p_1),L_N^{(2)}(p_1)*0_M^{(1)},\\&\qquad L_N^{(2)}(p_1)*L_M^{(2)}(p_1),\ldots,L_N^{(2)}(p_1)*L_M^{(k_2)}(p_1),\ldots,L_N^{(k_1)}(p_1)*0_M^{(1)},\\&\qquad L_N^{(k_1)}(p_1)*L_M^{(2)}(p_1),\ldots,L_N^{(k_1)}(p_1)*L_M^{(k_2)}(p_1)\Big)\Big).\end{aligned}$$

Parts (a) and (b) now follow from lemma 3.2 (a), (b) and (c).

Using the preceding Lemmas, we have the following method of construction which is a modification of the method given in Wang and Wu (1991).

1. Construct matrices A and B as described in Lemma 3.1 so that the columns of A and B have maximal levels of trend resistance.
2. Construct the orthogonal array $A \circledast B$ as described in Lemma 3.1.
3. Let $\boldsymbol{L} = \mathbf{0}_N * L_M(M)$ be a matrix consisting of N copies of the array $L_M(M)$ as its rows. By adding the columns of $\boldsymbol{L}$ to $\boldsymbol{A} \circledast \boldsymbol{B}$, the resulting matrix $[A \circledast B, L]$ is an orthogonal array $L_{NM}(p_1^{k_1 n_1} \cdots p_r^{k_r n_r} M)$. (3.1)
4. (optional) Replace the M levels of the last factor in $L_{NM}(p_1^{k_1 n_1} \cdots p_r^{k_r n_r} M)$ by the corresponding rows in $L_M(q_1^{r_1} \cdots q_m^{r_m})$ to obtain $L_{NM}(p_1^{k_1 n_1} \cdots p_r^{k_r n_r} q_1^{r_1} \cdots q_m^{r_m})$. The orthogonal array $L_M(q_1^{r_1} \cdots q_m^{r_m})$ should be constructed so that its columns have maximal trend resistance.

With regard to the construction process just described, we now give some additional information which is useful in performing the various steps.

In Step 1 of Process (3.1), there are as yet no results available to indicate when a given column of a difference matrix has a maximal degree of trend resistance. However, Theorem 3.3 is useful for the construction of difference matrices having different levels of trend resistance and the following observations are also useful in performing Step 1:

1. Every orthogonal array $L_N(p^n)$ is a difference matrix $D_{N,n;p}$.
2. If $D_{p,k;p} = (0_p^{(1)}, L_p^{(2)}(p), \ldots, L_p^{(k)}(p))$, then each column $L_p^{(i)}(p)$, $i = 2, \ldots, k$, is at least 0-trend free.
3. Let $D_{N,k;p} = (0_N^{(1)}, L_N^{(2)}(p), \ldots, L_N^{(k)}(p))$. We note that changing the order of rows in $D_{N,k;p}$ does not change the fact that $D_{N,k;p}$ is a difference matrix. Thus, if p is odd and $N = \lambda p$, $\lambda = 2, \ldots, p$, one can use the results of Phillips (1968) to order the elements in $L_N^{(2)}(p)$ (and hence the rows of $D_{N,k;p}$) so that $L_N^{(2)}(p)$ is 1-trend free.
4. Let $D_{N,k;p} = (0_N^{(1)}, L_N^{(2)}(p), \ldots, L_N^{(k)}(p))$ and let $\bar{D}_{N,k;p}$ be the difference matrix obtained by reversing the order of the rows in $D_{N,k;p}$. Then all columns in $D_{2N,k;p} = \begin{pmatrix} D_{N,k;p} \\ \bar{D}_{N,k;p} \end{pmatrix}$ except the first column are 1-trend free. (3.2)
5. If p is a prime and $D_{p,p;p} = (0_p^{(1)}, L_p^{(2)}(p), \cdots, L_p^{(p)}(p))$, then $L_N\left(p^{p^{(n-1)}}\right) = L_p(p) * D_{p,p;p}^{(1)} * \cdots * D_{p,p;p}^{(n-1)}$ is an orthogonal array with $N = p^n$ where $D_{p,p;p}^{(i)}$ denotes the ith copy of $D_{p,p;p}$ used in the given generalized Kronecker sum. Using Lemma 3.2 and the facts given above, it is easily seen that $L_N\left(p^{p^{(n-1)}}\right)$ has $\binom{n-1}{x}(p-1)^x$ columns that are x-trend free for $x = 0, 1, \ldots, n-1$. We note that Wang (1991) gives a result similar to that indicated here.

Steps 2 and 3 of Process (3.1) are self explanatory.

After Step 3, the last factor of $L_{NM}(p_1^{k_1 n_1} \cdots p_r^{k_r n_r} M)$ has M levels and has 0-trend resistance. In Step 4 of (3.1), it is often possible to replace the M levels of the last factor by the corresponding rows of an orthogonal array $L_M(q_1^{r_1} \cdots q_m^{r_m})$ and achieve some level of trend resistance for the factors in $L_M(q_1^{r_1} \cdots q_1^{r_m})$. For example, if factor j has q_i levels and is t_j-trend resistant in $L_M(q_1^{r_1} \cdots q_m^{r_m})$ and the M levels of the last factor in $L_{NM}(p_1^{k_1 n_1} \cdots p_r^{k_r n_r} M)$ are replaced by the corresponding rows of $L_M(q_1^{r_1} \cdots q_m^{r_m})$, then factor j will still be t_j-trend free in $L_{NM}(p_1^{k_1 n_1} \cdots p_r^{k_r n_r} q_1^{r_1} \cdots q_m^{r_m})$ by Lemma 3.2. Thus $L_M(q_1^{r_1} \cdots q_m^{r_m})$ should be constructed so that it also has a maximal level of trend resistance. In this regard, the observations made in (3.2) are useful.

We now give several examples to illustrate these ideas.

Example 3.4. Suppose we wish to construct a trend free orthogonal array having 18 runs in which all factors have two or three levels. To begin, we use the difference matrix

$$D_{6,6;3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 2 & 1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 & 0 \\ 0 & 0 & 2 & 1 & 2 & 1 \end{pmatrix}.$$

We note that the rows of $D_{6,6;3}$ have been arranged so that the second column is 1-trend free. Using (3.1) down to Step 3, we obtain the orthogonal array $L_{18}(3^6 \cdot 6) = (L_3(3) * D_{6,6;3}, \mathbf{0}_3 * L_6(6))$. In Step 4, by replacing the six levels of the last factor in $L_{18}(3^6 \cdot 6)$ by the corresponding rows in $L_6(3 \cdot 2) = (00, 10, 20, 21, 11, 01)'$, we obtain $L_{18}(3^7 \cdot 2)$. We note that factor one in $L_6(3 \cdot 2)$ is 1-trend free; hence $L_{18}(3^7 \cdot 2)$ has five 3-level factors that are 1-trend free, one 3-level factor that is 2-trend free, and has the remaining factors 0-trend free. We also note that the design given here is slightly better than the design $L_{18}(3^7)$ given in John (1990) which has six 3-level factors that are 1-trend free and one 3-level factor that is 0-trend free.

Example 3.5. Suppose we wish to construct a trend free orthogonal array having 27 rows and all factors with three levels. Using

$$D_{3,3;3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix},$$

(3.1) and (3.2(3)), we obtain $L_{27}(3^9 \cdot 9) = (L_3(3) * D_{3,3;3} * D_{3,3,;3}, 0_3 * L_9(9))$ which, by (3.2(5)), has four 3-level factors that are 2-trend free, four 3-level factors that are 1-trend free, and has the remaining factors that are 0-trend free. Upon replacing the nine levels of the last factor in $L_{27}(3^9 \cdot 9)$ by the corresponding levels of $L_9(3^4) = (L_3(3) * D_{3,3;3}, 0_3 * L_3(3))$ (which has two 1-trend free 3 level factors and two 0-trend free 3-level factors), we obtain $L_{27}(3^{13})$ which has four 2-trend free 3-level factors, six 1-trend free 3-level factors and three 0-trend free 3-level factors. This design has the same level of trend resistance as the design $L_{27}(3^{13})$ given in John (1990).

In many cases, if the number of factors having a specified number of levels is relatively small, it is often possible to increase the level of trend resistance for the factors used. This is illustrated in the following example.

Example 3.6. Suppose we wish to construct a trend free orthogonal array having 18 runs in which three factors have three levels. To begin, use the difference matrix

$$D_{6,2;3} = \begin{pmatrix} 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 2 & 1 & 1 & 2 & 0 \end{pmatrix}'$$

which is obtained from $D_{3,3;3}$ of Example 3.5 using (3.2(4)) after omitting the column of zeros in $D_{3,3;3}$. We note that the two columns in $D_{6,2;3}$ are 1-trend free. Using (3.1) down to Step 3, we obtain the orthogonal array $L_{18}(3^2 \cdot 6) = (L_3 * D_{6,2;3}, \mathbf{0}_3 * L_6(6))$. In Step 4, by replacing the six levels of the last factor in $L_{18}(3^2 \cdot 6)$ by the corresponding levels of $L_6(3) = (0,1,2,2,1,0)$, we obtain $L_{18}(3^3)$. We note that the 3-level factor in $L_6(3)$ is 1-trend free, hence $L_{18}(3^3)$ has two 3-level factors that are 2-trend free and one 3-level factor that is 1-trend free. When comparing $L_{18}(3^3)$ constructed in this example with $L_{18}(3^7 \cdot 2)$ obtained in Example 3.4, we see that $L_{18}(3^3)$ has fewer factors but that these factors have slightly higher trend resistance than those in $L_{18}(3^7 \cdot 2)$.

Example 3.7. In this example we use the generalized Kronecker sum to construct an array having a larger number of runs. In particular, we now set about constructing an orthogonal array having 72 runs and having all factors with two or three levels. To begin, let

$$D_{12,12;2} = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&1&1&0&1&0&1&1&0&0 \\ 0&1&1&1&0&1&0&1&0&1&0&0 \\ 0&1&0&1&1&1&0&0&1&0&1&0 \\ 0&1&1&0&0&0&0&0&1&1&1&1 \\ 0&1&0&0&1&0&1&1&0&1&1&0 \\ 0&0&1&0&0&1&1&1&1&0&1&0 \\ 0&0&0&0&1&1&0&1&1&1&0&1 \\ 0&1&0&1&0&0&1&1&1&0&0&1 \\ 0&0&1&1&1&0&0&1&0&0&1&1 \\ 0&0&0&1&0&1&1&0&0&1&1&1 \\ 0&1&1&0&1&1&1&0&0&0&0&1 \end{bmatrix}, \quad D_{12;12;3} = \begin{bmatrix} 0&0&0&1&1&0&0&1&0&2&2 \\ 0&0&0&0&2&0&2&0&2&0&0 \\ 0&1&2&2&0&0&1&1&2&0&2 \\ 0&1&2&1&2&1&2&2&2&2&1 \\ 0&2&1&2&1&0&0&2&2&1&1 \\ 0&2&1&0&0&1&2&1&1&2&2 \\ 0&2&2&1&2&2&1&1&0&1&0 \\ 0&2&0&1&1&1&1&0&1&0&1 \\ 0&1&0&0&2&2&0&2&1&1&2 \\ 0&1&1&2&1&2&2&0&0&2&0 \\ 0&0&1&0&0&2&1&2&0&0&1 \\ 0&0&2&2&0&1&0&0&1&1&0 \end{bmatrix}$$

and $B = (D_{12,12;2}, D_{12,12;3})$. We note that columns two, three and four of $D_{12,12;2}$ are 1-trend free and column two of $D_{12,12;3}$ is 1-trend free. Now let $L_6(2 \cdot 3)$ be given by

$$L_6(2 \cdot 3) = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 2 & 1 & 0 \end{pmatrix}' = (L_6(2), L_6(3)) = A.$$

In $L_6(2 \cdot 3)$, $L_6(3)$ is 1-trend free. Upon applying Steps 1, 2 and 3 of (3.1), we get $L_{72}(2^{12} \cdot 3^{12} \cdot 12) = (A \circledast B,\ 0_6 * L_{12}(12)) = (L_6(2) * D_{12,12;2},\ L_6(3) * D_{12,12;3},\ 0_6 * L_{12}(12))$. Using the comments made previously, we see that $L_{72}(2^{12} \cdot 3^{12} \cdot 12)$ has three 2-trend free 2-level factors, eight 1-trend free 2-level factors, one 3-trend free 3-level factor, ten 2-trend free 3-level factors, and remaining factors 0-trend free. We can now replace the 12 levels of the last factor in $L_{72}(2^{12} \cdot 3^{12} \cdot 12)$ by the corresponding rows of

$$L_{12}(3 \cdot 2^4) = \begin{pmatrix} 0 & 0 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}'$$

given in Wang and Wu (1991) which has one 1-trend free 3-level factor and one 1-trend free 2-level factor or by the corresponding rows of

$$L_{12}(6 \cdot 2) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 5 & 4 & 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}'$$

which has a 1-trend free 6-level factor or by $L_{12}(2^{11})$ which is obtained from $D_{12,12;2}$ by eliminating the first column. Upon making these replacements, we obtain the orthogonal arrays $L_{72}(2^{16} \cdot 3^{13})$, $L_{72}(2^{13} \cdot 3^{12} \cdot 6)$ and $L_{72}(2^{23} \cdot 3^{12})$ which have the levels of trend resistance previously described.

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(Received November 1993; accepted November 1994)