Supplemental Materials: Inference of Bivariate Long-memory Aggregate Time Series

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This supplement consists of three sections. In the first section, we prove Equation (4). The second section contains the detailed proof of Theorem 1, and the third section contains the proof of Claim 2.

S1 Proof of Equation (4)

By Equations (2) and (3), we have

$$Y_n^{\Delta} = S_n - S_{n-1},$$

where

$$\begin{aligned} S_n &= \int_0^{n\Delta} Y_u du \\ &= \int_0^{n\Delta} e^{\Phi u} Y_0 du + \int_0^{n\Delta}\int_0^{u} e^{\Phi(u-v)}\Sigma d\bar{B}_v^H du \\ &= \left(e^{\Phi n\Delta} - I_2\right)\Phi^{-1}Y_0 + \int_0^{n\Delta}\int_v^{n\Delta} e^{\Phi(u-v)}\Sigma du d\bar{B}_v^H \\ &= \left(e^{\Phi n\Delta} - I_2\right)\Phi^{-1}Y_0 + \int_0^{n\Delta}\left(e^{\Phi(n\Delta - v)} - I_2\right)\Phi^{-1}\Sigma d\bar{B}_v^H, \end{aligned}$$

and I_2 is the 2×2 identity matrix. Thus,

$$\begin{aligned} Y_n^{\Delta} &= \left(e^{\Phi n\Delta} - e^{\Phi(n-1)\Delta}\right)\Phi^{-1}Y_0 + \int_0^{n\Delta} e^{\Phi(n\Delta - v)}\Phi^{-1}\Sigma d\bar{B}_v^H - \int_0^{(n-1)\Delta} e^{\Phi((n-1)\Delta - v)}\Phi^{-1}\Sigma d\bar{B}_v^H \\ &\quad -\Phi^{-1}\Sigma\left(\bar{B}_{n\Delta}^H - \bar{B}_{(n-1)\Delta}^H\right) \\ &= \Phi^{-1}(Y_{n\Delta} - Y_{(n-1)\Delta}) - \Phi^{-1}\Sigma\left(\bar{B}_{n\Delta}^H - \bar{B}_{(n-1)\Delta}^H\right). \end{aligned}$$

By the self-similarity property of the fractional Brownian motion, Y_n^{Δ} has the same distribution as

$$\Phi^{-1}(Y_{n\Delta} - Y_{(n-1)\Delta}) - \Phi^{-1}\Sigma \begin{bmatrix} \Delta^{H_1} & 0 \\ 0 & \Delta^{H_2} \end{bmatrix} \begin{bmatrix} B_n^{H_1} - B_{n-1}^{H_1} \\ B_n^{H_2} - B_{n-1}^{H_2} \end{bmatrix}.$$

Therefore, as $\Delta \to \infty$,

$$-\begin{bmatrix} \Delta^{-H_1} & 0 \\ 0 & \Delta^{-H_2} \end{bmatrix}\Sigma^{-1}\Phi Y_n^{\Delta} \xrightarrow{d} \begin{bmatrix} B_n^{H_1} - B_{n-1}^{H_1} \\ B_n^{H_2} - B_{n-1}^{H_2} \end{bmatrix}.$$

This completes the proof of Equation (4).

S2 Proof of Theorem 1

Theorem 1 follows from Theorem 2 of Hosoya (1996) if we can verify Conditions A, C, and D listed there. These conditions can be verified by a line of arguments quite parallel to those in Chan and Tsai (2008). In particular, let $\delta(x, y) = 1$ if $x = y$, and $\delta(x, y) = 0$ otherwise.

First, write

$$f(\omega, H_1, H_2, A) = 2(1 - \cos\omega) AG(\omega, H_1, H_2) A' = k(\omega) I_2 k^*(\omega)$$

where

$$\begin{aligned} k(\omega) &= \sqrt{2(1-\cos\omega)} A \; diag\left(R_0^{1/2}(\omega, H_1), R_0^{1/2}(\omega, H_2) \right) \quad \text{(S2.1)} \\ &= \sqrt{2(1-\cos\omega)} \begin{pmatrix} a_{11} R_0^{1/2}(\omega, H_1) & a_{21} R_0^{1/2}(\omega, H_2) \\ a_{12} R_0^{1/2}(\omega, H_1) & a_{22} R_0^{1/2}(\omega, H_2) \end{pmatrix}. \end{aligned}$$

As in Chan and Tsai (2008), we define, for $t = 1, ..., N$, a bivariate process $\tilde{y}_t$ by

$$\tilde{y}_t = \sum_{j=0}^{\infty} G_j e_{t-j},$$

where e_j is an iid $\text{N}(0_2, \ I_2)$, 0_2 is the 2×1 vector having each component equal to 0, and G_j is defined by $k(\omega) = \sum_{j=0}^{\infty} G_j \exp(ij\omega)$ with $k(\omega)$ given in (S2.1). We can write $\tilde{y}_t$ as a one sided rather than a two sided moving average representation, as would follow from the fact that the spectral transfer function $k(\omega)$ is defined through he square root of $\text{R}_0(\omega)$, because of Gaussianity and the resulting indistinguishability between the causal and noncausal representation. The new process $\tilde{y}_t$ has the same autocovariance structure as y_t. Thus, because of Gaussianity, the estimators obtained from $\tilde{y}_t$ correspond to the ones for y_t.

Condition A is clearly satisfied for the new error term e_t.

We now verify condition C.

(i) (a) We will show that $\int_{-\pi}^{\pi} |k_{\alpha\beta}(\omega)|^{2u} d\omega < \infty$ for some u such that $1 < u \leq 2$. We consider $k_{ij}(\omega) = \sqrt{2(1-\cos\omega)} a_{ji} R_0^{1/2}(\omega, H_j)$ for $i, j = 1, 2$. Let $\kappa_1 = 2H_1 - 1$ and $\kappa_2 = 2H_2 - 1$. Since $|k_{ij}|^2 = O(|\omega|^{-\kappa_j})$, there exist non-negative constants b_0 and b_1 such that

$$\int_{-\pi}^{\pi} |k_{ij}|^{2u} d\omega \leq b_0 \int_0^{\varepsilon} \omega^{-2u\kappa_j} d\omega + b_1 < \infty.$$

(b) We need to show that there exists $\gamma > 0$ such that

$$\sup_{|\lambda| < \varepsilon} \| \left[f^{-1}(\cdot) \{ f(\cdot) - f(\cdot - \lambda) \} \right]_{\alpha\beta} \|_u = O(\varepsilon^{\gamma}), \quad \text{(S2.2)}$$

where $\|g\|_p = \{\int_{-\pi}^{\pi} |g(\omega)|^p d\omega\}^{1/p}$. First, for $\omega \to 0$,

$$
\begin{aligned}
f(\omega) &\sim \begin{pmatrix} a_{11}^2|\omega|^{-\kappa_1} + a_{12}^2|\omega|^{-\kappa_2} & a_{11}a_{21}|\omega|^{-\kappa_1} + a_{12}a_{22}|\omega|^{-\kappa_2} \\ a_{11}a_{21}|\omega|^{-\kappa_1} + a_{12}a_{22}|\omega|^{-\kappa_2} & a_{21}^2|\omega|^{-\kappa_1} + a_{22}^2|\omega|^{-\kappa_2} \end{pmatrix} \\
&\sim \begin{pmatrix} a_{11}^2 & a_{11}a_{21} \\ a_{11}a_{21} & a_{21}^2 \end{pmatrix} |\omega|^{-\kappa_1}. \qquad \text{(S2.3)}
\end{aligned}
$$

From the same two inequalities as on page 27 in Chan and Tsai (2008) it follows that (S2.2) holds if it holds for a smaller interval containing only one pole of the spectral density function. Thus, it suffices to show that (S2.2) holds for $f_{ij}(\omega) \sim |\omega|^{-\kappa_1}$. Let $2 \geq u > 1$ be a constant such that $0 < u\kappa_1 < 1$. We have for $\alpha, \beta = 1, 2$,

$$
\begin{aligned}
& \| \left[f^{-1}(\cdot)\{f(\cdot) - f(\cdot - \lambda)\}\right]_{\alpha\beta} \|_u^u \\
\leq\ & 2\int_0^{|\lambda|} \left| \frac{\omega^{-\kappa_1} - (|\lambda| - \omega)^{-\kappa_1}}{\omega^{-\kappa_1}} \right|^u d\omega \\
& +2\int_{|\lambda|}^{\pi} \left| \frac{\omega^{-\kappa_1} - (\omega - |\lambda|)^{-\kappa_1}}{\omega^{-\kappa_1}} \right|^u d\omega. \qquad \text{(S2.4)}
\end{aligned}
$$

Using a change of variable $\omega = x|\lambda|$ and therefore $d\omega = dx|\lambda|$ we can write (S2.4) as

$$
2|\lambda| \int_0^1 \left| \frac{x^\kappa - (1-x)^\kappa}{(1-x)^\kappa} \right|^u dx + 2|\lambda| \int_1^{\pi/|\lambda|} \left| \frac{x^\kappa - (x-1)^\kappa}{(x-1)^\kappa} \right|^u dx. \qquad \text{(S2.5)}
$$

The first integral is bounded by $\int_0^1 (1-x)^{-\kappa u} < \infty$ since the numerator is bounded by 1 and $0 < \kappa u < 1$. We write the second integral as

$$
\int_1^{1+\varepsilon} \left| \frac{x^\kappa - (x-1)^\kappa}{(x-1)^\kappa} \right|^u dx + \int_{1+\varepsilon}^{\pi/|\lambda|} \left| \frac{x^\kappa - (x-1)^\kappa}{(x-1)^\kappa} \right|^u dx
$$

for some $\varepsilon > 0$. The first term is again bounded and in the second term, we approximate the numerator by $\kappa\,(x-\eta)^{\kappa-1}$, with $0 < \eta < 1$, which is bounded by $\kappa\,(x-1)^{\kappa-1}$. Since $u > 1$, the second term is therefore bounded by $\int_1^{\pi/|\lambda|} (x-1)^{-u} < \infty$. Thus (S2.5) is bounded and (S2.2) is of order $O\left(|\varepsilon|^\gamma\right)$ with $\gamma = 1/u > 1/2$.

(ii) Let $h_j(\omega, \theta) = \partial f^{-1}(\omega;\theta)/\partial\theta_j$, where θ_j is the jth component of $\theta = (H_1, H_2, a_{11}, a_{12}, a_{21}, a_{22})$. Here, we need to show that for any $\varepsilon > 0$, there exists $a > 0$ and Hermitian-valued bounded functions $\tilde{h}_j$ and $\bar{h}_j$ such that, if $|\theta_1 - \theta| < a$, $\tilde{h}_j(\omega) \leq h_j(\omega, \theta_1) \leq \bar{h}_j(\omega)$ and $\left\| \left[\{\bar{h}_j(\cdot) - \tilde{h}_j(\cdot)\}f(\cdot)\right]_{\alpha\beta} \right\|_v < \varepsilon$, where $v = u/(u-1)$ for u given in (S2.2) above. Note that $\partial f^{-1}/\partial\theta_j = -f^{-1}\partial \log f/\partial\theta_j$ and that both factors are uniformly continuous in ω and θ in a sufficiently small neighborhood of θ_0. Thus, the requirement in (ii) can be readily verified element by element.

(iii) Recall from equation (15) and below this equation, $V_j(\theta) = H_j(\theta) + \int_{-\pi}^{\pi} tr\{h_j(\omega,\theta)f(\omega)\}d\omega$, where $f(\omega)$ is the true spectral density function, and $H_j(\theta) = \partial \int_{-\pi}^{\pi} \log\det f(\omega;\theta)d\omega/\partial\theta_j$. Here, we need to show that $V_j(\theta)$ has a unique zero for all j at $\theta = \theta_0$, where θ_0 is an interior point of θ.

The true spectral density function is, by assumption, equal to $f(\omega,\theta_0)$. Consider the function $Q(\theta) = \int_{-\pi}^{\pi} \log\det f(\omega,\theta)d\omega - \int_{-\pi}^{\pi} \log\det f(\omega,\theta_0)d\omega + \int_{-\pi}^{\pi} tr\{f^{-1}(\omega,\theta)f(\omega)\}d\omega$. Note that the partial derivative of Q with respect to the jth component of θ equals $V_j(\theta)$, for all j. Condition (iii) holds if Q attains its unique minimum at $\theta = \theta_0$, which is shown below. Define $T(x) = \exp(x) - x$ which is a convex function that is always ≥ 1. Further note that $\log\det f^{-1}(\omega,\theta)f(\omega) = \sum_{i=1}^{2} \log\lambda_i(\omega)$ and $tr(f^{-1}(\omega,\theta)f(\omega)) = \sum_{i=1}^{2} \lambda_i(\omega)$ where $\lambda_i(\omega), i = 1, 2$ are the Eigenvalues of $f^{-1}(\omega,\theta)f(\omega)$. Jensen's inequality implies that

$$\begin{aligned} Q(\theta)/(2\pi) &= \int_{-\pi}^{\pi} \left(-\sum_{i=1}^{2} \log\lambda_i(\omega) + \sum_{i=1}^{2} \lambda_i(\omega) \right) d\omega/(2\pi) \\ &= \int_{-\pi}^{\pi} \sum_{i=1}^{2} T(\log\lambda_i(\omega))\, d\omega/(2\pi) \\ &\geq \sum_{i=1}^{2} T\left(\int_{-\pi}^{\pi} \log\lambda_i(\omega) \right) d\omega/(2\pi) \geq 2, \end{aligned}$$

with both equalities obtained if and only if $\theta = \theta_0$. This is true since a symmetric matrix with all eigenvalues being *one* must be the identity matrix. The unique zero of $V_j(\theta)$ follows then from convexity of $T(\cdot)$ and the fact that sums and integrals of convex functions remain convex.

(iv) $H_j(\theta)$ is continuous on θ.
This condition holds trivially.

Parts (i)–(ii) of condition D can be proved by arguments similar to those used in proving conditions (i) and (ii) of condition C. In particular, (i) follows from $\gamma = 1/u > 1/2$ in (ii) of Condition C (i) b).

(ii) We need to show that

$$\lim_{r\to 0} \sup_{|\theta-\theta_0|\leq r} \| [\{h_j(\cdot,\theta) - h_j(\cdot,\theta_0)\}f(\cdot)]_{\alpha\beta} \|_v < C$$

for some $C > 0$, $j = 1, ..., s$ and for $v = u/(u-1) \geq 2$. We use again the two inequalities on page 27 in Chan and Tsai (2008) to replace h_j by its pole asymptotics. Thus, for $H_1 \geq H_2$, at the pole

$$h_j(\cdot,\theta) = \begin{cases} O(|\omega|^{\kappa_1} \log|\omega|), \text{ for } j = 1 \\ O\left(|\omega|^{2\kappa_1-\kappa_2} \log|\omega|\right), \text{ for } j = 2. \end{cases}$$

Combining with $f(\cdot)$ from (S2.3), we obtain for $j=1$,

$$\begin{aligned}
& \| \left[\{ h_1(\cdot,\theta) - h_1(\cdot,\theta_0) \} f(\cdot) \right]_{\alpha\beta} \|_v^v \\
\leq\ & 2C \int_0^{\pi} \left| \log(|\omega|) \frac{(|\omega|^{\kappa_1} - |\omega|^{\kappa_{1,0}})}{|\omega|^{\kappa_{1,0}}} \right|^v d\omega \\
=\ & 2C \int_0^{\pi} \left| \log(|\omega|) \left(|\omega|^{|\kappa_1 - \kappa_{1,0}|} - 1 \right) \right|^v d\omega,
\end{aligned}$$

for $\alpha, \beta = 1, 2$. Applying a Taylor approximation

$$|\omega|^{|\kappa_1 - \kappa_{1,0}|} - 1 = |\kappa_1 - \kappa_{1,0}| |\omega|^{|\kappa_1^* - \kappa_{1,0}|} \log|\omega|,$$

with κ_1^* lying between κ_1 and $\kappa_{1,0}$, we obtain

$$\begin{aligned}
& \| \left[\{ h_1(\cdot,\theta) - h_1(\cdot,\theta_0) \} f(\cdot) \right]_{\alpha\beta} \|_v^v \\
\leq\ & 2|\kappa_1 - \kappa_{1,0}|^v \left(\int_0^{\pi} \left| \log^2(|\omega|) |\omega|^{|\kappa_1^* - \kappa_{1,0}|} \right|^v d\omega \right) < Cr^v
\end{aligned}$$

for $\alpha, \beta = 1, 2$, since the integral is finite. Next, for $j=2$,

$$\begin{aligned}
& \| \left[\{ h_2(\cdot,\theta) - h_2(\cdot,\theta_0) \} f(\cdot) \right]_{\alpha\beta} \|_v^v \\
\leq\ & 2C_1 \int_0^{\pi} | \ \log(|\omega|) \left(|\omega|^{|2\kappa_1 - \kappa_2 - \kappa_{1,0}|} - |\omega|^{|\kappa_{2,0} - \kappa_{1,0}|} \right) |^v d\omega \\
\leq\ & 2C_1 |2\kappa_1 - \kappa_2 - \kappa_{2,0}|^v \left(\int_0^{\pi} \left| \log^2(|\omega|) |\omega|^{|2\kappa_1^* - \kappa_2^* - \kappa_{1,0}|} \right|^v d\omega \right) \\
<\ & C_2 r^v.
\end{aligned}$$

Finally, the derivative with respect to any element of the weighting matrix A behaves as $|\omega|^{\kappa_1}$. For this case, we find a bound using parallel arguments to the ones for the memory parameters.

(iii) Given ε, we divide the radius a of a ball around $\theta_{j,0}$ into $m(\varepsilon)$ partitions of length $r(\varepsilon) = a/m(\varepsilon)$ with $\underline{\theta}_j^i$ and $\bar{\theta}_j^i$ denoting the lowest and highest value in partition i. In addition to the behavior of $h_j(\cdot)$ from Part (ii) and the uniform continuity of f^{-1} and consequently of h, we use monotonicity of $h_j(.)$ in the sense that for any j,

$$\kappa_j' \geq \kappa_j'' \Longleftrightarrow h_j(\kappa_j') \leq h_j(\kappa_j''),$$

implying $h_j\left(\underline{\kappa}_j^i\right) \geq h_j^i \geq h_j\left(\bar{\kappa}_j^i\right)$ for $\underline{\kappa}_j^i \leq \kappa_j^i \leq \bar{\kappa}_j^i$. Let $\bar{h}_j^i$ and $\tilde{h}_j^i$ be a pair of Hermitian bracketing functions. For similar reasons as in part (ii), we obtain an inequality similar to the one used there. In particular, for $j=1$,

$$\begin{aligned}
& \| k^* \{ \bar{h}_1^i - h_1^0 \} k \|_v^v \\
\leq\ & 2C_1 |\underline{\kappa}_1 - \kappa_{1,0}|^v \left(\int_0^{\pi} | \ \log^2(|\omega|) |\omega|^{|\kappa_1^* - \kappa_{1,0}|} |^v d\omega \right) \leq C_2 r^v(\varepsilon),
\end{aligned}$$

with κ_1^* lying between $\underline{\kappa}_1$ and $\kappa_{1,0}$. Taking $m(\varepsilon) = C_3/\varepsilon$ provides the result. Next, for $j = 2$,

$$\begin{aligned} & \|k^*\{\bar{h}_2^i - h_2^0\}k\|_v^v \\ \leq \; & 2C_4|2(\underline{\kappa}_1 - \kappa_{1,0}) - (\underline{\kappa}_2 - \kappa_{2,0})|^v \left(\int_0^\pi | \log^2(|\omega|)\, |\omega|^{|2\kappa_1^* - \kappa_2^* - \kappa_{1,0}|}|^v d\omega \right) \\ \leq \; & C_5 r^v(\varepsilon). \end{aligned}$$

Both results hold for $\alpha, \beta = 1, 2$. Derivatives with respect to elements of the weighting matrix can be bounded in a similar manner. Finally, $\|k^*\{\tilde{h}_1^i - h_1^0\}k\|_v^v \leq Cr^v(\varepsilon)$ holds similarly.

(iv) $\left|V(\theta)\right| \geq \alpha_1 \left|\theta_1 - \theta_0\right|$ for some $\alpha_1 > 0$ and a parameter vector θ_1 in the neighborhood of θ_0, where V is the vector consisting of all the first partial derivatives V_j.
This condition holds because

$$\begin{aligned} \frac{\partial^2 Q}{\partial\theta_i \partial\theta_j} &= \int_{-\pi}^{\pi} tr \left[f(\omega,\theta)^{-1} \frac{\partial f(\omega,\theta)}{\partial\theta_i} f(\omega,\theta)^{-1} \frac{\partial f(\omega,\theta)}{\partial\theta_j} \right. \\ & \left. + f^{-1}(\omega,\theta) \frac{\partial^2 f(\omega,\theta)}{\partial\theta_i \partial\theta_j} \left(I - f^{-1}(\omega,\theta) f(\omega) \right) \right] d\omega, \end{aligned}$$

where the second term vanishes for $\theta \to \theta_0$. Thus, $\frac{\partial^2 Q}{\partial\theta_i \partial\theta_j}$ converges to

$$\Gamma_{ij}(\theta) = \int_{-\pi}^{\pi} tr \left[f(\omega,\theta_0)^{-1} \left. \frac{\partial f(\omega,\theta)}{\partial\theta_i} \right|_{\theta=\theta_0} f(\omega,\theta_0)^{-1} \left. \frac{\partial f(\omega,\theta)}{\partial\theta_j} \right|_{\theta=\theta_0} \right] d\omega,$$

which is positive definite since the partial derivatives are linearly independent. In particular,

$$\begin{aligned} L_k(H) &= \int_{-\pi}^{\pi} \left\{ \frac{\partial}{\partial H} \log R_0(\omega; H) \right\}^k d\omega, & k = 1, 2, \\ G_{ij} &= \int_{-\pi}^{\pi} \frac{R_0(\omega; H_j)}{R_0(\omega; H_i)} d\omega, & i, j = 1, 2, \end{aligned}$$

then it can be verified that

$$\begin{aligned} \Gamma(\theta) &= \frac{1}{(\det A)^2} \widetilde{\Gamma}(\theta) \\ &= \frac{1}{(\det A)^2} [\Gamma_{(1)}(\theta), \Gamma_{(2)}(\theta)], \end{aligned}$$

where

$$\Gamma_{(1)}(\theta) = \begin{bmatrix} (\det A)^2 L_2(H_1) & 0 & 2a_{22}(\det A) L_1(H_1) \\ 0 & (\det A)^2 L_2(H_2) & 0 \\ 2a_{22}(\det A) L_1(H_1) & 0 & 8\pi a_{22}^2 + 2a_{21}^2 G_{21} \\ 0 & -2a_{21}(\det A) L_1(H_2) & -4\pi a_{21} a_{22} \\ -2a_{12}(\det A) L_1(H_1) & 0 & -8\pi a_{12} a_{22} - 2a_{11} a_{21} G_{21} \\ 0 & 2a_{11}(\det A) L_1(H_2) & 4\pi a_{12} a_{21} \end{bmatrix},$$

and

$$\Gamma_{(2)}(\theta) = \begin{bmatrix} 0 & -2a_{12}(\det A)L_1(H_1) & 0 \\ -2a_{21}(\det A)L_1(H_2) & 0 & 2a_{11}(\det A)L_1(H_2) \\ -4\pi a_{21}a_{22} & -8\pi a_{12}a_{22} - 2a_{11}a_{21}G_{21} & 4\pi a_{12}a_{21} \\ 8\pi a_{21}^2 + 2a_{22}^2 G_{12} & 4\pi a_{11}a_{22} & -8\pi a_{11}a_{21} - 2a_{12}a_{22}G_{12} \\ 4\pi a_{11}a_{22} & 8\pi a_{12}^2 + 2a_{11}^2 G_{21} & -4\pi a_{11}a_{12} \\ -8\pi a_{11}a_{21} - 2a_{12}a_{22}G_{12} & -4\pi a_{11}a_{12} & 8\pi a_{11}^2 + 2a_{12}^2 G_{12} \end{bmatrix}.$$

Let $\widetilde{\Gamma}[i:j,i:j]$ be the submatrix of $\widetilde{\Gamma}(\theta)$ formed by rows $i,..,j$, and columns $i,...,j$. By the Cauchy-Schwarz inequality, we have $2\pi L_2(H_i) > L_1^2(H_i)$, for $i=1,2$ and $G_{12}G_{21} > 4\pi^2$. Furthermore, $G_{ij} > 0$, for $i,j=1,2$. To show that $\Gamma(\theta)$ is positive definite, it suffices to show that $\det\widetilde{\Gamma}[i:6,i:6] > 0$, for $i=1,...,6$, and this is seen as follows,

$$\begin{aligned}
\det\widetilde{\Gamma}[6:6,6:6] &= 8\pi a_{11}^2 + 2a_{12}^2 G_{12} > 0, \\
\det\widetilde{\Gamma}[5:6,5:6] &= 4a_{11}^2 a_{12}^2 G_{12}G_{21} + 16\pi a_{12}^4 G_{12} + 16\pi a_{11}^4 G_{21} + 48\pi^2 a_{11}^2 a_{12}^2 > 0, \\
\det\widetilde{\Gamma}[4:6,4:6] &= 32\pi(\det A)^2\{4\pi a_{12}^2 G_{12} + a_{11}^2(G_{12}G_{21} - 4\pi^2)\} > 0, \\
\det\widetilde{\Gamma}[3:6,3:6] &= 256\pi^2(\det A)^4(G_{12}G_{21} - 4\pi^2) > 0, \\
\det\widetilde{\Gamma}[2:6,2:6] &= 128\pi(\det A)^6(2\pi L_2(H_2) - L_1^2(H_2))(G_{12}G_{21} - 4\pi^2) > 0, \\
\det\widetilde{\Gamma}[1:6,1:6] &= 64(\det A)^8(2\pi L_2(H_1) - L_1^2(H_1))(2\pi L_2(H_2) - L_1^2(H_2))(G_{12}G_{21} - 4\pi^2) > 0.
\end{aligned}$$

This completes the proof that $\Gamma(\theta)$ is positive definite under $H_1 > H_2$.

(v) This condition can be easily verified if the spectral density function admits no poles but otherwise it can be proved by adapting the arguments presented in Example 3.1 of Hosoya (1996).

This completes the proof of Theorem 1.

S3 Proof of Claim 2

First, write

$$\begin{aligned}
& f(\omega;\vartheta) \\
= \;& 2(1-\cos\omega)\,A\begin{pmatrix} R_0(H_1) & 0 \\ 0 & R_0(H_2) \end{pmatrix} A' \\
= \;& 2(1-\cos\omega)\,A\left\{\begin{pmatrix} R_0(H) & 0 \\ 0 & R_0(H) \end{pmatrix}\right. \qquad \text{(S3.6)} \\
& \left. + \begin{pmatrix} R_0(H_1) - R_0(H) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & R_0(H_2) - R_0(H) \end{pmatrix}\right\} A', \qquad \text{(S3.7)}
\end{aligned}$$

which converges to $2(1-\cos\omega) R_0(H) B$, as $H_2 \to H_1$. Next,

$$\begin{aligned}
V_3(\vartheta) &= H_3(\vartheta) + \int_{-\pi}^{\pi} tr \left\{ \frac{\partial f^{-1}(\omega;\vartheta)}{\partial H} f(\omega) \right\} d\omega, \\
V_4(\vartheta) &= H_4(\vartheta) + \int_{-\pi}^{\pi} tr \left\{ \frac{\partial f^{-1}(\omega;\vartheta)}{\partial b_{11}} f(\omega) \right\} d\omega, \\
V_5(\vartheta) &= H_5(\vartheta) + \int_{-\pi}^{\pi} tr \left\{ \frac{\partial f^{-1}(\omega;\vartheta)}{\partial b_{12}} f(\omega) \right\} d\omega, \\
V_6(\vartheta) &= H_6(\vartheta) + \int_{-\pi}^{\pi} tr \left\{ \frac{\partial f^{-1}(\omega;\vartheta)}{\partial b_{22}} f(\omega) \right\} d\omega.
\end{aligned}$$

Note that the two terms in (S3.7) converge to *zero* as $H_2 \to H_1$. Equally, in the derivatives of $f(\omega;\vartheta)$ with respect to H, b_{11}, b_{12} and b_{22},

$$\begin{aligned}
&\frac{\partial f(\omega;\vartheta)}{\partial \vartheta_i} \\
= \; & 2(1-\cos\omega) \left\{ \frac{\partial}{\partial \vartheta_i} R_0(H) B \right. \qquad \text{(S3.8)} \\
& \left. + \frac{\partial}{\partial \vartheta_i} \left(R_0(H_1) - R_0(H) \right) A \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} A' + \frac{\partial}{\partial \vartheta_i} \left(R_0(H_2) - R_0(H) \right) A \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} A' \right\}, \qquad \text{(S3.9)}
\end{aligned}$$

the terms in (S3.9) coming from the terms in (S3.7) are negligible as $H_2 \to H_1$. Thus, $\frac{\partial f(\omega;\vartheta)}{\partial \vartheta_i}$ converges to $2(1-\cos\omega) \left(\frac{\partial}{\partial \vartheta_i} R_0(H_1) B \right)$, as $H_2 \to H_1$. Therefore, in the following, for simplicity we concentrate on the terms in (S3.6) and (S3.8).

We show that $V_3(\vartheta) = V_4(\vartheta) = V_5(\vartheta) = V_6(\vartheta) = 0$. Denote $R_0 = R_0(H)$ and denote the values of R_0 and B evaluated at the true parameter values as R_0^0 and B^0. Next,

$$\begin{aligned}
\frac{\partial f^{-1}(\omega;\vartheta)}{\partial H} f(\omega) &= -f^{-1}(\omega;\vartheta) \frac{\partial f(\omega;\vartheta)}{\partial H} f^{-1}(\omega;\vartheta) f(\omega) \\
&= -\frac{\partial R_0}{\partial H} R_0^{-1} B^{-1} B^0 \\
&\to -\frac{1}{R_0} \frac{\partial R_0}{\partial H} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\end{aligned}$$

and consequently,

$$\int_{-\pi}^{\pi} tr \left\{ \frac{\partial f^{-1}(\omega;\vartheta)}{\partial H} f(\omega) \right\} d\omega \to -2 \frac{1}{R_0} \frac{\partial R_0}{\partial H} = -2L_1(H).$$

Next,

$$
\begin{aligned}
H_3(\vartheta) &= \frac{\partial}{\partial H}\int_{-\pi}^{\pi}\log|f(\omega;\vartheta)|d\omega \\
&= \frac{\partial}{\partial H}\int_{-\pi}^{\pi}\left(\log\left[2\left(1-\cos\omega\right)\right]+\log R_0^2+\log(\det B)\right)d\omega \\
&= 2\int_{-\pi}^{\pi}\frac{\partial}{\partial H}\log R_0\left(H\right)d\omega = 2L_1\left(H\right).
\end{aligned}
$$

Then,

$$
V_3(\vartheta) \;=\; H_3(\vartheta)+\int_{-\pi}^{\pi} tr\left\{\frac{\partial f^{-1}(\omega;\vartheta)}{\partial H}f(\omega)\right\}d\omega = 0.
$$

Further, it can be shown that

$$
\begin{aligned}
H_4(\vartheta) &= \frac{\partial}{\partial b_{11}}\int_{-\pi}^{\pi}\log|f(\omega;\vartheta)|d\omega = \frac{2b_{22}\pi}{\det B}, \\
H_5(\vartheta) &= \frac{\partial}{\partial b_{12}}\int_{-\pi}^{\pi}\log|f(\omega;\vartheta)|d\omega = -\frac{4b_{12}\pi}{\det B}, \\
H_6(\vartheta) &= \frac{\partial}{\partial b_{22}}\int_{-\pi}^{\pi}\log|f(\omega;\vartheta)|d\omega = \frac{2b_{11}\pi}{\det B}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\frac{\partial f^{-1}(\omega;\vartheta)}{\partial b_{11}}f(\omega) &= -\frac{R_0^0}{R_0}\frac{1}{(\det B)^2}\begin{pmatrix} -b_{12}b_{22}b_{12}^0+b_{22}^2b_{11}^0 & -b_{12}b_{22}b_{22}^0+b_{22}^2b_{12}^0 \\ -b_{12}b_{22}b_{11}^0+b_{12}^2b_{12}^0 & -b_{12}b_{22}b_{12}^0+b_{12}^2b_{22}^0 \end{pmatrix}, \\
tr\left(\frac{\partial f^{-1}(\omega;\vartheta)}{\partial b_{11}}f(\omega)\right) &= -(\det B)^{-2}\left(-b_{12}b_{22}b_{12}^0+b_{22}^2b_{11}^0-b_{12}b_{22}b_{12}^0+b_{12}^2b_{22}^0\right)\to-\frac{b_{22}}{\det B}.
\end{aligned}
$$

Therefore,

$$
V_4(\vartheta) \;=\; H_4(\vartheta)+\int_{-\pi}^{\pi} tr\left\{\frac{\partial f^{-1}(\omega;\vartheta)}{\partial b_{11}}f(\omega)\right\}d\omega = 0.
$$

Finally, the proof for $V_5(\vartheta)=0$ and $V_6(\vartheta)=0$ is similar to the one for $V_4(\vartheta)=0$.

Next, we show that the matrix of the second derivatives $\{V_{ij}\}_{i,j=3,\ldots,6}$ converges to W_{33}, where

$$
\begin{aligned}
&W_{33} \\
&= \begin{pmatrix} 2L_2(H) & (\det B)^{-1}b_{22}L_1(H) & -2(\det B)^{-1}b_{12}L_1(H) & (\det B)^{-1}b_{11}L_1(H) \\ (\det B)^{-1}b_{22}L_1(H) & 2\pi(\det B)^{-2}b_{22}^2 & -4\pi(\det B)^{-2}b_{12}b_{22} & 2\pi(\det B)^{-2}b_{12}^2 \\ -2(\det B)^{-1}b_{12}L_1(H) & -4\pi(\det B)^{-2}b_{12}b_{22} & 4\pi(\det B)^{-2}((\det B)+2b_{12}^2) & -4\pi(\det B)^{-2}b_{11}b_{12} \\ (\det B)^{-1}b_{11}L_1(H) & 2\pi(\det B)^{-2}b_{12}^2 & -4\pi(\det B)^{-2}b_{11}b_{12} & 2\pi(\det B)^{-2}b_{11}^2 \end{pmatrix}.
\end{aligned}
$$

In particular, for V_{33},

$$
V_{33}=\frac{\partial V_3(\vartheta)}{\partial H}=\frac{\partial}{\partial H}\left[H_3(\vartheta)+\int_{-\pi}^{\pi} tr\left\{\frac{\partial f^{-1}(\omega;\vartheta)}{\partial H}f(\omega)\right\}d\omega\right].
$$

First,

$$\begin{aligned}\frac{\partial}{\partial H}H_3(\vartheta) &= 2\frac{\partial}{\partial H}\int_{-\pi}^{\pi}\frac{\partial}{\partial H}\log R_0 d\omega = 2\frac{\partial}{\partial H}\left(\int_{-\pi}^{\pi}\frac{1}{R_0}\frac{\partial}{\partial H}R_0 d\omega\right)\\ &= 2\int_{-\pi}^{\pi}\frac{R_0\frac{\partial^2}{\partial H^2}R_0-\frac{\partial}{\partial H}R_0\frac{\partial}{\partial H}R_0}{R_0^2}d\omega.\end{aligned}$$

Next,

$$\begin{aligned}\frac{\partial}{\partial H}\int_{-\pi}^{\pi}tr\left\{\frac{\partial f^{-1}(\omega;\vartheta)}{\partial H}f(\omega)\right\}d\omega &= -2\int_{-\pi}^{\pi}\frac{\partial}{\partial H}\left(\frac{\frac{\partial}{\partial H}R_0}{R_0^2}\right)R_0^0B^{-1}B^0 d\omega\\ &= -2\int_{-\pi}^{\pi}\frac{R_0^2\frac{\partial^2}{\partial H^2}R_0-2\frac{\partial}{\partial H}R_0R_0\frac{\partial}{\partial H}R_0}{R_0^4}R_0^0B^{-1}B^0 d\omega.\end{aligned}$$

And consequently in the limit,

$$V_{33} \to 2\int_{-\pi}^{\pi}\frac{\left(\frac{\partial}{\partial H}R_0\right)^2}{R_0^2}d\omega = 2L_2(H).$$

Next, for V_{34},

$$\frac{\partial}{\partial b_{11}}H_3(\vartheta) = 0,$$

and

$$\begin{aligned}\frac{\partial}{\partial b_{11}}\int_{-\pi}^{\pi}tr\left\{\frac{\partial f^{-1}(\omega;\vartheta)}{\partial H}f(\omega)\right\}d\omega &= -\int_{-\pi}^{\pi}tr\frac{\partial}{\partial b_{11}}\frac{1}{R_0}\frac{\partial R_0}{\partial H}\frac{1}{2R_0}2R_0^0B^{-1}B^0 d\omega\\ &= -\int_{-\pi}^{\pi}\frac{1}{R_0}\frac{\partial R_0}{\partial H}d\omega\frac{R_0^0}{R_0}tr\frac{\partial}{\partial b_{11}}B^{-1}B^0\\ &= b_{22}\frac{1}{(\det B)}L_1(H),\end{aligned}$$

where we use that

$$\begin{aligned}\frac{\partial}{\partial b_{11}}B^{-1}B_0 &= -B^{-1}\frac{\partial}{\partial b_{11}}BB^{-1}B_0 = -B^{-1}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}B^{-1}B_0 = \frac{1}{(\det B)}\begin{pmatrix}b_{22} & 0\\ -b_{12} & 0\end{pmatrix}B^{-1}B^0\\ &\to \frac{1}{(\det B)}\begin{pmatrix}b_{22} & 0\\ -b_{12} & 0\end{pmatrix}.\end{aligned}$$

Therefore,

$$V_{34} = \frac{\partial V_3(\vartheta)}{\partial b_{11}} = \frac{\partial}{\partial b_{11}}\left[H_3(\vartheta)+\int_{-\pi}^{\pi}tr\left\{\frac{\partial f^{-1}(\omega;\vartheta)}{\partial H}f(\omega)\right\}d\omega\right] \to b_{22}\frac{1}{(\det B)}L_1(H).$$

Finally, for V_{44},

$$\begin{aligned}
\frac{\partial V_4(\vartheta)}{\partial b_{11}} &= \frac{\partial}{\partial b_{11}}\left[\frac{2\pi b_{22}}{(\det B)} - \frac{2\pi}{(\det B)^2}\left\{-b_{12}b_{22}b_{12}^0 + b_{22}^2 b_{11}^0 - b_{12}b_{22}b_{12}^0 + b_{12}^2 b_{22}^0\right\}\right] \\
&= -\frac{2\pi b_{22}^2}{(\det B)^2} - 2\pi\frac{-\left(-b_{12}b_{22}b_{12}^0 + b_{22}^2 b_{11}^0 - b_{12}b_{22}b_{12}^0 + b_{12}^2 b_{22}^0\right)2(\det B)b_{22}}{(\det B)^4} \\
&\to \frac{2\pi}{(\det B)^2}b_{22}\left\{-b_{22} + \frac{2b_{22}(-b_{12}b_{12}^0 + b_{22}b_{11}^0 - b_{12}b_{12}^0 + b_{12}^2)}{\det B}\right\} \\
&\to \frac{2\pi}{(\det B)^2}b_{22}\left(-b_{22} + 2b_{22}\right) = \frac{2\pi}{(\det B)^2}b_{22}^2.
\end{aligned}$$

All other terms of V_{ij} follow from similar arguments.

Finally, it can be shown that the matrix W_{33} is positive. For this, note that, by the Cauchy-Schwarz inequality, we have $2\pi L_2(H) > L_1^2(H)$, and $\det B = (\det A)^2 > 0$. Therefore,

$$\begin{aligned}
& \det 2L_2(H) \\
= \; & 2L_2(H) > 0, \\
& \det\begin{bmatrix} 2L_2(H) & (\det B)^{-1}b_{22}L_1(H) \\ (\det B)^{-1}b_{22}L_1(H) & 2\pi(\det B)^{-2}b_{22}^2 \end{bmatrix} \\
= \; & \frac{b_{22}^2(4\pi L_2(H) - L_1^2(H))}{(\det B)^2} > 0, \\
& \det\begin{bmatrix} 2L_2(H) & (\det B)^{-1}b_{22}L_1(H) & -2(\det B)^{-1}b_{12}L_1(H) \\ (\det B)^{-1}b_{22}L_1(H) & 2\pi(\det B)^{-2}b_{22}^2 & -4\pi(\det B)^{-2}b_{12}b_{22} \\ -2(\det B)^{-1}b_{12}L_1(H) & -4\pi(\det B)^{-2}b_{12}b_{22} & 4\pi(\det B)^{-2}(\det B + 2b_{12}^2) \end{bmatrix} \\
= \; & \frac{4\pi b_{22}^2(4\pi L_2(H) - L_1^2(H))}{(\det B)^3} > 0,
\end{aligned}$$

$$\begin{aligned}
& \det\begin{bmatrix} 2L_2(H) & (\det B)^{-1}b_{22}L_1(H) & -2(\det B)^{-1}b_{12}L_1(H) & (\det B)^{-1}b_{11}L_1(H) \\ (\det B)^{-1}b_{22}L_1(H) & 2\pi(\det B)^{-2}b_{22}^2 & -4\pi(\det B)^{-2}b_{12}b_{22} & 2\pi(\det B)^{-2}b_{12}^2 \\ -2(\det B)^{-1}b_{12}L_1(H) & -4\pi(\det B)^{-2}b_{12}b_{22} & 4\pi(\det B)^{-2}((\det B) + 2b_{12}^2) & -4\pi(\det B)^{-2}b_{11}b_{12} \\ (\det B)^{-1}b_{11}L_1(H) & 2\pi(\det B)^{-2}b_{12}^2 & -4\pi(\det B)^{-2}b_{11}b_{12} & 2\pi(\det B)^{-2}b_{11}^2 \end{bmatrix} \\
= \; & \frac{16\pi^2(2\pi L_2(H) - L_1^2(H))}{(\det B)^3} > 0.
\end{aligned}$$

This completes the proof of Claim 2.

References

[1] Chan, K.S. and Tsai, H. (2008). Inference of seasonal long-memory aggregate time series. Technical Report No 391, Department of Statistics and Actuarial Science, The University of Iowa; downloadable from `http://www.stat.uiowa.edu/techrep/`