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ESTIMATION OF THE FINITE RIGHT ENDPOINT IN THE GUMBEL DOMAIN

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Abstract: A simple estimator for the finite right endpoint of a distribution function in the Gumbel max-domain of attraction is proposed. Large sample properties such as consistency and the asymptotic distribution are derived. A simulation study is presented.

Key words and phrases: Endpoint estimation, extreme value theory, statistical inference.

1. Introduction

Let $X_{n,n} \geq X_{n-1,n} \geq \cdots \geq X_{1,n}$ be the order statistics from the sample $X_1, X_2, \ldots, X_n$ of i.i.d. random variables with common (unknown) distribution function F. Let x^F denote the right endpoint of F. We assume that the distribution function F has a finite right endpoint, $x^F := \sup\{x : F(x) < 1\} \in \mathbb{R}$.

The fundamental result for extreme value theory is due in various degrees of generality to Fisher and Tippett (1928), Gnedenko (1943), de Haan (1970) and Balkema and de Haan (1974). The extreme value theorem (or extremal types theorem) restricts the class of all possible limiting distribution functions to only three types. Thus, if there exist constants $a_n > 0$, $b_n \in \mathbb{R}$ such that

$$\lim_{n\to\infty} F^n(a_n\, x + b_n) = G(x), \tag{1.1}$$

for all x, G non-degenerate, then G must be one of

$$\begin{aligned}\Psi_\alpha(x) &= \exp\{-(-x)^\alpha\},\ x<0, \quad \alpha>0,\\ \Lambda(x) &= \exp\{-\exp(-x)\}, \quad x\in\mathbb{R},\\ \Phi_\alpha(x) &= \exp\{-x^{-\alpha}\},\ x>0, \quad \alpha>0.\end{aligned}$$

Redefining the constants $a_n > 0$ and $b_n \in \mathbb{R}$, these can be nested in a one-parameter family of distributions, the Generalized Extreme Value (GEV) distribution with distribution function

$$G_\gamma(x) := \exp\{-(1+\gamma x)^{-1/\gamma}\},\ 1+\gamma x > 0,\ \gamma \in \mathbb{R}.$$

We consider F in the (max-)domain of attraction of G_γ and use the notation $F \in \mathcal{D}_M(G_\gamma)$. For $\gamma < 0$, $\gamma = 0$, and $\gamma > 0$, the GEV distribution function reduces to the Weibull, Gumbel, and Fréchet distribution functions, respectively. An equivalent extreme value condition allows the limit relation in (1.1) to run over the real line (cf., Theorem 1.1.6 de Haan and Ferreira (2006)): $F \in \mathcal{D}_M(G_\gamma)$ if and only if

$$\lim_{t\to\infty} t\big(1 - F(a(t)\,x + b(t)\big) = (1+\gamma\,x)^{-1/\gamma}, \tag{1.2}$$

for all x such that $1+\gamma x > 0$, $a(t) := a_{[t]}$ and $b(t) := b_{[t]}$, with $[t]$ denoting the integer part of t. The extreme value index γ determines various degrees of tail heaviness. If $F \in \mathcal{D}_M(G_\gamma)$ with $\gamma > 0$, the distribution function F has a power-law decaying tail with infinite right endpoint, while $\gamma < 0$ refers to short tails with a finite right endpoint. The Gumbel domain of attraction $\mathcal{D}_M(G_0)$ encloses a great variety of distributions, ranging from light-tailed distributions, such as the Normal, the exponential, to moderately heavy distributions, such as the Lognormal. All the named distributions have an infinite right endpoint, but a finite endpoint is also possible in the Gumbel domain. We give several examples in Section 2. Light-tailed distributions with finite endpoint, but not so light that they are included in the Gumbel domain, have been in demand as feasible distributions underlying real life phenomena. An example is the extreme value analysis by Einmahl and Magnus (2008) of the best marks in Athletics, aimed at assessing the ultimate records for several events. For instance, Table 3 in Einmahl and Magnus (2008) has several missing values for the estimates of the endpoint which are due to an estimated extreme value index γ near zero. An attempt to fill these blank spaces with an appropriate framework for inference in the Gumbel domain has been provided by Fraga Alves, de Haan, and Neves (2013), although from the view point of application to the Long Jump data set used in Einmahl and Magnus (2008). The tentative estimator proposed by Fraga Alves, de Haan, and Neves (2013) is virtually the same as the one we introduce. The novelty here is in the development of a simple closed-from expression for the estimator.

The problem of estimating the right endpoint x^F of a distribution function lying in the Gumbel extremal domain of attraction is tackled by the semi-parametric statistic

$$X_{n,n} + X_{n-k,n} - \frac{1}{\log 2}\sum_{i=0}^{k-1} \log\Big(\frac{k+i+1}{k+i}\Big) X_{n-k-i,n}$$

or, in a more compact form, by

$$\hat{x}^F := X_{n,n} + \sum_{i=0}^{k-1} a_{i,k}\Big(X_{n-k,n} - X_{n-k-i,n}\Big), \tag{1.3}$$

where $a_{i,k} := -(\log 2)^{-1}\big(\log(k+i) - \log(k+i+1)\big) > 0$, such that $\sum_{i=0}^{k-1} a_{i,k} = 1$. Here and throughout k is in fact a sequence of positive integers going to infinity as $n \to \infty$ but at a much slower rate than n. Thus, we are defining $\hat{x}^F$ as a functional of the top observations of the original sample, relying on an intermediate sequence $k = k_n$, with $k_n \to \infty$, $k_n = o(n)$, as $n \to \infty$. From the non-negativeness of the weighted spacings in (1.3), we see that $\hat{x}^F$ is greater than the maximum $X_{n,n}$ with probability one. This is an advantage in comparison with the usual semi-parametric estimators for the right endpoint of a distribution function in the Weibull domain of attraction. We refer to Hall (1982), Falk (1995), Hall and Wang (1999), and to de Haan and Ferreira (2006) and references therein. To the best of our knowledge, none of these estimators have ensured the extrapolation beyond the sample range. There have been, however, some developments of endpoint estimators connected with $\gamma < 0$ in the sense of bias reduction and/or correction. Li and Peng (2009), Li, Peng, and Xu (2011) and Cai, de Haan, and Zhou (2013) are a few of the works. Recently, Girard, Guillou, and Stupfler (2012) devised an endpoint estimator from the high-order moments pertaining to a distribution with $\gamma < 0$; Li and Peng (2012) proposed a bootstrap estimator for the endpoint evolving from the one of Hall (1982) in case $\gamma \in (-1/2, 0)$. The present paper addresses the class of distribution functions belonging to the Gumbel domain of attraction, for which no corresponding has yet been provided. The appropriate framework is developed in Section 2.

The remainder of the paper is as follows. The rationale behind the proposal of the new estimator for the right endpoint is expounded in Section 3. Consistency and the asymptotic distribution of the estimator, are worked out in Section 4, taking advantage of a form of separability between the maximum and the sum of higher order statistics. In order to perform asymptotics, we require some basic conditions in the context of the theory of regular variation. These are laid out in Section 2. In Section 5 we gather some simulation results. Section 6 is devoted to some applications and conclusions.

2. Framework

Let F be a distribution function (d.f.) with right endpoint $x^F := \sup\{x : F(x) < 1\}$. For now we assume $x^F \leq \infty$.

Suppose F satisfies the extreme value condition

$$\lim_{x \uparrow x^F} \frac{1 - F(t + x\, f(t))}{1 - F(t)} = (1 + \gamma\, x)^{-1/\gamma}, \tag{2.1}$$

for all $x \in \mathbb{R}$ such that $1 + \gamma\, x > 0$, with a suitable positive function f (this is equivalent to (1.2), see Theorem 1.1.6 of de Haan and Ferreira (2006)). For $\gamma = 0$ the limit in (2.1) reads as e^{-x}.

Let U be the (generalized) inverse function of $1/(1-F)$. If F satisfies (2.1) with $\gamma = 0$ then we can assume there exists a positive function a_0 such that, for all $x > 0$,

$$\lim_{t\to\infty} \frac{U(tx) - U(t)}{a_0(t)} = \log x. \tag{2.2}$$

Hence U belongs to the class Π (see Definition B.2.4 of de Haan and Ferreira (2006)) and a_0 is a measurable function such that $\lim_{t\to\infty} a_0(tx)/a_0(t) = 1$ for all $x > 0$. Then a_0 is a slowly varying function and we write $a_0 \in RV_0$ (see Theorem B.2.7 of de Haan and Ferreira (2006)). The functions a_0 and f (of (2.2) and (2.1)) are related to each other by $a_0 = f \circ U$ (see Theorem B.2.21 of de Haan and Ferreira (2006)). We use the notation $U \in \Pi(a_0)$ to emphasize the auxiliary function a_0. We assume the following:

(A1) $U \in \Pi(a_0)$.
(A2) $U(t) = U(t_0) + \int_{t_0}^{t} a(s)\, \frac{ds}{s} + o\big(a(t)\big)$, for some $t_0 \geq 1$, with a positive function $a \in RV_0$ satisfying $a(t) \sim a_0(t)$ as $t \to \infty$.
(B) $x^F := U(\infty) = \lim_{t\to\infty} U(t)$ exists finite.

Under (A1), Proposition B.2.15(3) of de Haan and Ferreira (2006) guarantees the existence of a twice differentiable function $\overline{f}$, with $-\overline{f}'' \in RV_{-2}$, such that $U(t) = \overline{f}(t) + o\big(a_0(t)\big)$. Let $\overline{f}(t) = \overline{f}(t_0) + \int_{t_0}^{t} \overline{f}'(s)\, ds$ be this function. Hence, $U(t) = U(t_0) + \overline{f}(t) - \overline{f}(t_0) + o\big(a(t)\big)$, with $a(t) \sim a_0(t)$, and where we set $\overline{f}'(t) = a(t)/t$. This is (A2). Conversely, (A2) implies (A1) by Proposition B.2.15(5) of de Haan and Ferreira (2006) with $g(s) = a(s)/s \in RV_{-1}$ therein. Under conditions (A1) and (B), we have

$$U(\infty) - U(t) = \int_t^{\infty} \frac{a(s)}{s} ds + o\big(a(t)\big), \quad t \to \infty, \tag{2.3}$$

$$\lim_{t\to\infty} \frac{\int_{tx}^{\infty} U(s)/s ds - \int_t^{\infty} U(s)/s ds}{\int_t^{\infty} a(s)/s ds} = \log x, \tag{2.4}$$

for all $x > 0$. Hence $\int_t^{\infty} U(s)\, ds/s$ is also Π-varying with the auxiliary function

$$q(t) := \int_t^{\infty} a(s)\, \frac{ds}{s} = \int_1^{\infty} a(st)\, \frac{ds}{s} = \int_0^{1} a\Big(\frac{t}{s}\Big)\, \frac{ds}{s}. \tag{2.5}$$

In the usual notation, the above is $\int_t^{\infty} U(s)\, ds/s \in \Pi(q)$. Then q is slowly varying while (2.3) entails that $q(t) \to 0$ as $t \to \infty$ (cf., Lemma C.1 from Appendix C).

Some examples of distributions belonging to the Gumbel domain of attraction with finite right endpoint, where (2.3) holds, are listed below.

Example 1. A random variable X is Negative Fréchet with parameter $\beta > 0$ if it has distribution function $F(x) = 1 - \exp\{-(x^F - x)^{-\beta}\}$, $x \le x^F$, $\beta > 0$. The associated tail quantile function is $U(t) = F^{\leftarrow}(1 - 1/t) = x^F - (\log t)^{-1/\beta}$, $t \ge 1$ (the arrow stands for the generalized inverse). Then $U \in \Pi(a_0)$ with $a_0(t) = (1/\beta)(\log t)^{-1/\beta - 1} \to 0$, as $t \to \infty$. The auxiliary function in (2.4) is $q(t) = (\log t)^{-1/\beta}$, $\beta > 0$.

Example 2. Let $F(x) = 1 - \exp\{-\tan(x/\beta)\}$, $0 \le x < \beta\pi/2$, $\beta > 0$. Then $U(t) = \beta \arctan(\log t)$, $t \ge 1$. Hence U satisfies (2.3) with $a(t) = 1/\big(\log^2 t + \beta^{-2}\big)$ and $U \in \Pi(a)$ where $U(\infty) = \beta\pi/2 = x^F$.

Example 3. Let $F(x) = 1 - \exp\{(\pi/2)^{-\beta} - \big(\arcsin(1 - x/\beta)\big)^{-\beta}\}$, $0 \le x < \beta$, $\beta > 0$. Then $U(t) = \beta\big\{1 - \sin\big(\big[(2/\pi)^\beta + \log t\big]^{-1/\beta}\big)\big\}$, $t \ge 1$, and (2.3) holds with $a(t) = (\log t)^{-(1/\beta + 1)} \cos((\log t)^{-1/\beta})$, $U \in \Pi(a)$ and $U(\infty) = \beta = x^F$.

3. Statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the underlying distribution function F with finite right endpoint x^F. Let $X_{1,n} \le X_{2,n} \le \ldots \le X_{n,n}$ be the corresponding order statistics. We introduce the estimator $\hat{q}(n/k)$ for the auxiliary function q as in (2.5), evaluated at $t = n/k$. This estimator has the property that, as $n \to \infty$, $k = k(n) \to \infty$, and $k(n)/n \to 0$ (provided some suitable, mild restrictions on the second order refinement of $\int_t^\infty U(s)/s\, ds$ hold),

$$\frac{q\big(n/k\big)}{a\big(n/k\big)} \left(\frac{\hat{q}\big(n/k\big)}{q\big(n/k\big)} - 1 \right) \underset{n \to \infty}{\overset{d}{\longrightarrow}} N,$$

where N is a non-degenerate random variable. Several estimators for the right endpoint $x^F = U(\infty) < \infty$ can be devised from (2.3), in the sense that these might evolve from a suitable estimator $\hat{q}(n/k)$ for $q(n/k)$, as

$$\hat{x}^F = \widehat{U}\big(\frac{n}{k}\big) + \hat{q}\big(\frac{n}{k}\big) = X_{n-k,n} + \hat{q}\big(\frac{n}{k}\big). \tag{3.1}$$

Here $\hat{x}^F$ carries analogous large sample properties to $\hat{q}(n/k)$. In particular, the consistency of $\hat{x}^F$ is essentially ensured by the consistency of $\hat{q}(n/k)$. Theorem 1 in Section 4 accounts for this.

We evaluate relation (2.4) at $x = 1/2$, together with $q(t)$ at $t = n/k$ (see last equality in (2.5)), and write, for large enough n,

$$\int_0^1 \Big(U\big(\frac{n}{2ks}\big) - U\big(\frac{n}{ks}\big) \Big) \frac{ds}{s} \approx q\big(\frac{n}{k}\big)(-\log 2).$$

Estimation of $q(n/k)$ arises naturally from the empirical counterparts $\hat{U}\big(n/(\theta ks)\big) = X_{n-[\theta ks],n}$, $s \in (0,1]$, $\theta = 1, 2$, so

$$\hat{q}\big(\frac{n}{k}\big) := -\frac{1}{\log 2}\int_0^1 \big(X_{n-[2ks],n} - X_{n-[ks],n}\big)\,\frac{ds}{s}. \tag{3.2}$$

Simple calculations yield the alternative expression

$$\hat{q}\big(\frac{n}{k}\big) = X_{n,n} + \frac{1}{\log 2}\sum_{i=0}^{k-1}\log\Big(\frac{k+i}{k+i+1}\Big)X_{n-k-i,n}. \tag{3.3}$$

Combining (3.1) with (3.3) leads to the right endpoint estimator

$$\hat{x}^F := X_{n-k,n} + X_{n,n} + \frac{1}{\log 2}\sum_{i=0}^{k-1}\log\Big(\frac{k+i}{k+i+1}\Big)X_{n-k-i,n}. \tag{3.4}$$

After rearranging components, it is possible to express $\hat{x}^F$ as the maximum $X_{n,n}$ added to some weighted mean of non-negative summands as

$$\hat{x}^F = X_{n,n} + \sum_{i=0}^{k-1} a_{i,k}\Big(X_{n-k,n} - X_{n-k-i,n}\Big),$$

with $a_{i,k} := -(\log 2)^{-1}\big(\log(k+i) - \log(k+i+1)\big) > 0$, $i = 1, 2, \ldots, k \in \mathbb{N}$, such that $\sum_{i=0}^{k-1} a_{i,k} = 1$.

Remark 1. The proposed estimator for the right endpoint returns values always larger than $x_{n,n}$. This constitutes a major advantage in comparison to the available semi-parametric estimators for the endpoint in the case of Weibull domain of attraction, for which the extrapolation beyond the sample range is not guaranteed. This inadequacy of the existing estimators often leads to some disappointing results in practical applications, with estimates-yields that may be lower than the observed maximum from the data.

4. Asymptotic Results

Our reasoning here is that the large sample properties of the estimator $\hat{x}^F$ are essentially governed by the asymptotic properties of the estimator $\hat{q}(n/k)$. This shows up in Theorem 1 with respect to consistency. The consistency of $\hat{q}(n/k)$ is tackled in Appendix A, with $\hat{q}(n/k)$ defined in (3.2) (see also (3.3)), for an intermediate sequence $k = k_n$. Similarly, the limiting distribution of $\hat{q}(n/k)$ in Theorem 2 renders the asymptotic distribution of $\hat{x}^F$ via Proposition 1. These proofs regarding $\hat{q}(n/k)$ are postponed to Appendix A.

Theorem 1. *Let $X_1, X_2, \ldots$ be i.i.d. random variables with tail quantile function U satisfying (A1) and (B). Suppose $k = k_n$ is a sequence of positive integers such that $k_n \to \infty$, $k_n/n \to 0$, as $n \to \infty$, and $\hat{q}(n/k)/q(n/k) \xrightarrow{p} 1$. Then $\hat{x}^F := X_{n-k,n} + \hat{q}(n/k)$ is a consistent estimator for $x^F < \infty$, $\hat{x}^F \underset{n\to\infty}{\xrightarrow{p}} x^F$.*

Proof. Let $U_{k+1,n}$ be the $(k+1)$-th ascending order statistic from the random sample $(U_1, U_2, \ldots, U_n)$ of n uniformly distributed random variables on the unit interval. Then $X_{n-k,n} \stackrel{d}{=} U\big(1/U_{k+1,n}\big)$, where $U(\cdot)$ is the underlying tail quantile function and $\stackrel{d}{=}$ stands for equality in distribution. Write

$$\begin{aligned} x^F - \hat{x}^F &\stackrel{d}{=} \Big(U(\infty) - U(\frac{n}{k}) - q(\frac{n}{k})\Big) - \Big(U\big(\frac{1}{U_{k+1,n}}\big) - U(\frac{n}{k})\Big) - q(\frac{n}{k})\Big(\frac{\hat{q}(n/k)}{q(n/k)} - 1\Big) \\ &= I - II - III, \end{aligned}$$

where $I := U(\infty) - U(n/k) - q(n/k) = o\big(a(n/k)\big)$, from (2.3). We have

$$II := U\big(\frac{1}{U_{k+1,n}}\big) - U(\frac{n}{k}) = o_p\Big(a(\frac{n}{k})\Big)$$

because $U \in \Pi(a)$ and Smirnov's Lemma ensures $k/(n\,U_{k+1,n}) \xrightarrow[n\to\infty]{P} 1$ (see Lemma 2.2.3 in de Haan and Ferreira (2006)). Moreover,

$$III := q(n/k)\Big(\frac{\hat{q}\big(n/k\big)}{q\big(n/k\big)} - 1\Big) = o_p(1)$$

by Proposition 2 and the fact that (2.3) implies $q(n/k) = o(1)$.

The limiting distribution of $\hat{q}(n/k)$ (and, later on, the asymptotic distribution of $\hat{x}^F$) is attained under a suitable second order refinement of (2.2): suppose there exist functions a, positive, and A tending to zero as $t \to \infty$, such that for all $x > 0$,

$$\lim_{t\to\infty} \frac{(U(tx) - U(t))/a(t) - \log x}{A(t)} = \frac{1}{2}(\log x)^2\,. \tag{4.1}$$

Remark 2. (4.1) follows directly from Theorem B.3.6, Remark B.3.7, and Corollary 2.3.5 of de Haan and Ferreira (2006) because the former states that, in our setup of $\gamma = 0$ and $x^F < \infty$, the only case allowed is the case of the second order parameter ρ equal to zero. The second order auxiliary function A converges to zero, not changing sign for t near infinity, and for every $x > 0$, $A(tx)/A(t) \to 1, t \to \infty$ (notation: $|A| \in RV_0$).

Example 4. Consider the Negative Fréchet distribution function $F(x) = 1 - \exp\{-(x^F - x)^{-\beta}\}$, $x \le x^F$, $\beta > 0$. The auxiliary function here is $q(t) = (\log t)^{-1/\beta}$, $\beta > 0$, while straightforward calculations yield $A_0(t) = -(1 + 1/\beta)$ $(\log t)^{-1}$, which implies that $-a_0(t)/q(t) = A_0(t)/(1+\beta)$, for t near infinity.

Theorem 2. *Assume* (A1), (B) *and* (4.1) *hold. Let $k = k_n$ be such that, as $n \to \infty$, $k_n \to \infty$, $k_n/n \to 0$, $a(n)/a(n/k_n) \to 1$, and $\sqrt{k_n}\, A(n/k_n) = O(1)$. If*

$$\lim_{n\to\infty} \frac{1}{A(n/k)}\Big(\int_{\frac{1}{2k}}^{1} \frac{U\big(n/ks\big) - U\big(n/2ks\big)}{q(n/k)}\,\frac{ds}{s} - \log 2\Big) = \lambda \in \mathbb{R}\,, \tag{4.2}$$

then

$$\frac{q(n/k)}{a(n/k)}\Big(\frac{\hat{q}(n/k)}{q(n/k)}-1\Big)\underset{n\to\infty}{\overset{d}{\longrightarrow}}\Lambda-\frac{\log 2}{2}-\frac{\lambda}{\log 2}, \tag{4.3}$$

where Λ *has the Gumbel distribution* $\exp\{-e^{-x}\}$ *for all* $x\in\mathbb{R}$.

Here (4.2) concerns a second order refinement of (2.4),

$$\lim_{t\to\infty}\frac{\Big(\int_{tx}^{\infty}U(s)/sds-\int_{t}^{\infty}U(s)/sds\Big)/q(t)-\log x}{Q(t)}=\frac{1}{2}(\log x)^2, \tag{4.4}$$

taken in the point $x=2$ for large enough $t=n/k$. Thus (4.2) stems from the theory of extended regular variation. We refer to Appendix B of de Haan and Ferreira (2006) for a good catalog of results in the theory of extended regular variation.

The assumption that $a(n/k)/a(n)\to 1$ as $n\to\infty$ is more restrictive in terms of screening for an adequate value k to determine the number of top order statistics on which to base our inference. For example, for the Negative Fréchet with $k_n=n^p$, $p\in(0,1)$,

$$\frac{a(n)}{a(n/k_n)}=\Big(1-\frac{\log k_n}{\log n}\Big)^{1/\beta+1}=(1-p)^{1/\beta+1},$$

which is approximately 1 if and only if p approaches zero. A more appropriate choice is $k_n=(\log n)^r$, $r\in(0,2]$, for which

$$\frac{a(n)}{a(n/k_n)}=\Big(1-\frac{\log k_n}{\log n}\Big)^{1/\beta+1}=\Big(1-\frac{r}{\log n}+\frac{\log\log n}{\log n}\Big)^{1/\beta+1}\underset{n\to\infty}{\longrightarrow}1.$$

The upper bound $r\le 2$ is imposed in order to comply with the assumption $\sqrt{k_n}A(n/k_n)=O(1)$.

We believe that such a choice for $k=k_n$, with $\log(k_n)=o(\log n)$, is a feasible one for most models satisfying (2.3). We bring forward the fact here that a mis-specification of k_n, in the sense that $a(n/k_n)/a(n)$ converges to a constant different than 1, has a direct impact on the asymptotic variance of the normalized relative error presented in Theorem 2 rather than upon the asymptotic bias.

Remark 3. The assumption $\log k=o(\log n)$ is a common one in the theoretical analysis of estimators for Weibull-type tails, which form a rich subclass of the Gumbel max-domain of attraction, albeit with $x^F=\infty$, see Goegebeur, Beirlant, and De Wet (2010) and Gardes, Girard, and Guillou (2011).

Example 5. The Negative Fréchet distribution has a tail quantile function given by $U(t)=x^F-(\log t)^{-1/\beta}$, $t\ge 1$, $0<\beta<1$, satisfying (4.4), with $Q(t)=-(\beta\,\log t)^{-1}$.

Proposition 1. *Under the conditions of Theorem* 2,

$$\left|\frac{1}{a(n/k)}\big(\hat{x}^F - x^F\big) - \frac{q(n/k)}{a(n/k)}\Big(\frac{\hat{q}(n/k)}{q(n/k)} - 1\Big)\right| \xrightarrow[n\to\infty]{P} 0.$$

Proof. We write

$$\begin{aligned}\frac{\hat{x}^F - x^F}{a(n/k)} - \frac{q(n/k)}{a(n/k)}\Big(\frac{\hat{q}(n/k)}{q(n/k)} - 1\Big) &= \frac{\hat{x}^F - \hat{q}(n/k)}{a(n/k)} - \frac{x^F - q(n/k)}{a(n/k)} \\ &= \frac{X_{n-k,n} - U(n/k)}{a(n/k)} - \frac{U(\infty) - U(n/k) - q(n/k)}{a(n/k)}.\end{aligned}$$

Under (4.1), Theorem 2.4.1 of de Haan and Ferreira (2006) ensures that $\big(X_{n-k,n} - U(n/k)\big)/a(n/k) = O_p\big(1/\sqrt{k}\big) = o_p(1)$, and the rest follows from (2.3).

The next result gives an alternative formulation of the results of Theorem 2 and Proposition 1 aimed at providing confidence bands for $\hat{x}^F$.

Theorem 3. *Let $X_1, X_2, \ldots$ be i.i.d. random variables with tail quantile function U satisfying* (4.1)*. Let $\hat{a}(n/k)$ be a consistent estimator for $a(n/k)$. Suppose $k = k_n$ is a sequence of positive integers such that, as $n \to \infty$, $k_n \to \infty$, $k_n/n \to 0$, $a(n)/a(n/k_n) \to 1$, and $\sqrt{k_n}\, A(n/k_n) = O(1)$. If*

$$\lim_{n\to\infty} \frac{1}{A(n/k)}\left(\int_{\frac{1}{2k}}^{1} \frac{U\big(n/ks\big) - U\big(n/2ks\big)}{q(n/k)}\,\frac{ds}{s} - \log 2\right) = \lambda \in \mathbb{R},$$

then

$$\frac{1}{\hat{a}(n/k)}\big(\hat{x}^F - x^F\big) \xrightarrow[n\to\infty]{d} \Lambda - \frac{\log 2}{2} - \frac{\lambda}{\log 2}.$$

Proof. The result follows from Theorem 2 and Proposition 1, applying Slutsky's theorem.

There are several possibilities for estimating the auxiliary function $a(n/k)$. An obvious choice is the Maximum Likelihood Estimator (MLE), pretending that the exceedances over a certain high (random) threshold follow a Generalized Pareto distribution (cf., Section 3.4 of de Haan and Ferreira (2006):

$$\hat{a}(n/k) = \hat{\sigma}^{MLE} := \frac{1}{k}\sum_{i=0}^{k-1}\big(X_{n-i,n} - X_{n-k,n}\big).$$

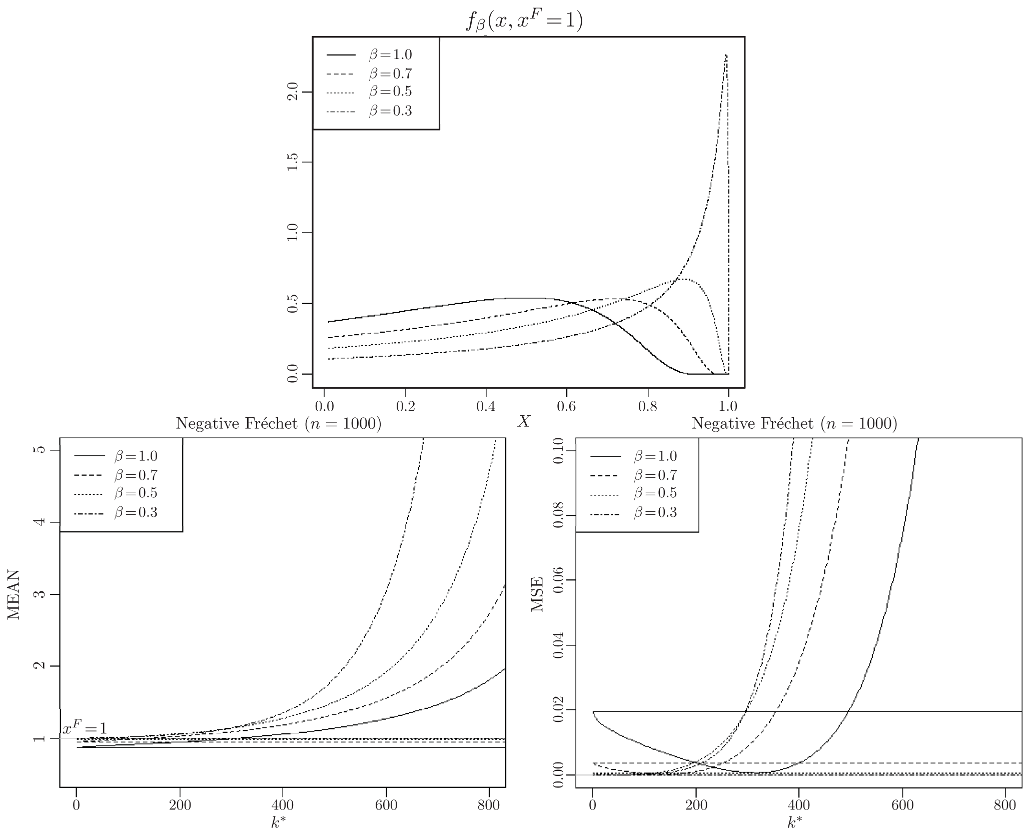

Figure 1. *1st row:* Right tails for probability density functions of Negative Fréchet Model with right endpoint $x^F = 1$ and $\beta = 1$ (*solid line*), $\beta = 0.7$ (*dashed line*), $\beta = 0.5$ (*dotted line*), $\beta = 0.3$ (*dotdash line*). *2nd row:* Mean estimate (*left*) and empirical Mean Squared Error (*right*) of $\hat{x}^F$ defined in (3.4), for the referred models with sample size $n = 1,000$. All plots are depicted against the number $k^* = 2k$ of top observations used in the estimator. The naive maximum estimator, $\tilde{x}^F := X_{n,n}$, is also depicted (*horizontal lines*).

5. Simulations

The Negative Fréchet distribution function $F(x) = 1 - \exp\{-(x^F - x)^{-\beta}\}$, $x \le x^F$, $\beta > 0$, has the tail quantile function $U(t) = x^F - (\log t)^{-1/\beta}$, $t \ge 1$, with $a(t) = \beta^{-1}(\log t)^{-1/\beta - 1}$, $\beta > 0$. The range of β offers various tail shapes, as shown in the graphics drawn in Figure 1 (*1st row*).

We simulated 1,000 samples of size $n = 100$, 1,000, 10,000, from the Negative Fréchet with right endpoint $x^F = 1$ for $\beta = 0.3, 0.5, 0.7, 1$. The results for $n = 1,000$ are depicted in Figure 1 (*2nd row*).

The common approach to selecting the number k of top order statistics used in the estimation (or $k^* = 2k$ in the present case) is to look for a region where the plots are relatively stable. Given the consistency property of the adopted estimator, one should in principle be away from small values of k that are usually associated with a large variance, and not so far off in the tail as to induce bias due to large k. An appropriate choice for an intermediate $k = k_n$ is $k_n = (\log n)^r$, with $r \in (0, 2]$. If we are using $n = 1000$, and set $r = 2$, the maximum allowed for r, we obtain $k \approx 48$ and thus $k^* \approx 96$. With k^* around 100, the plots in Figure 1 look quite stable in a vicinity of the target value $x^F = 1$ represented by the solid horizontal black line. Hence, the slow convergence imposed by $k_n = (\log n)^2$ seems to have little effect on the finite sample behavior of the estimator $\hat{x}^F$. This is particular true in case $0 < \beta < 1$.

The graphs in Figure 1 (*2nd row*) suggest better estimation under the Negative Fréchet model if the parameter β is less than 1, which corresponds to the case where the inherent second order conditions are satisfied. If $\beta \geq 1$, the Negative Fréchet distribution satisfies the first order condition but not the second order. Moreover, the general pattern for the mean estimate of $\hat{x}^F$ involves a moderated bias with k^* in the upper part of the sample, and a quickly increasing bias with k^* around 40% of the sample size.

Similar simulations have been carried out for the other two models in Examples 2 and 3, leading also to favorable results. For any model with finite right endpoint, the sample path of $\hat{x}^F$ departs from the sample maximum, always returning values beyond the sample.

We should highlight that our estimator $\hat{x}^F$ yields better results than the maximum estimator $\tilde{x}^F := X_{n,n}$, which always underestimates the true value x^F. The relative performance of both $\tilde{x}^F$ and $\hat{x}^F$ can also be easily observed if we compare the MSE graphics in Figure 1 (*2nd row, right*): for the top part the sample, depending on the β value, the estimator $\hat{x}^F$ always outperforms the maximum $\tilde{x}^F$, presenting the new estimator a lower mean squared error than the naive maximum estimator.

6. Case Study and Conclusion

This section is dedicated to the estimation of the finite right endpoint in the Gumbel maximum domain of attraction, which embraces light-tailed distributions with finite endpoint. Here extreme value analysis demands the estimation of the right endpoint, although the underlying distribution tail is not so light as to be included in Weibull domain of attraction. Fraga Alves, de Haan, and Neves (2013), working with athletics records data, filled the gap, highlighted in Einmahl and Magnus (2008), on assessing the ultimate records for several athletic events. We here consider an application to statistical extreme value analysis of Anchorage International Airport (ANC) Taxiway Centerline Deviations for Boeing 747

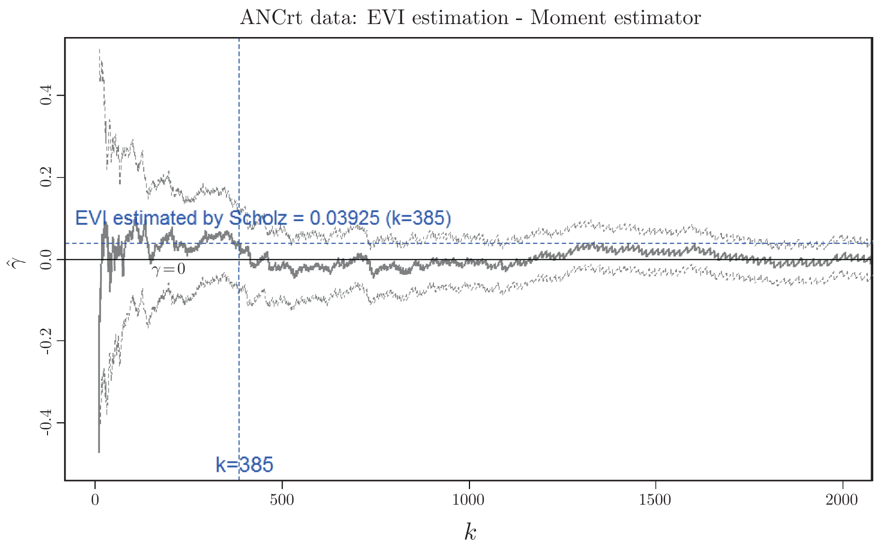

Figure 2. ANCrt data: EVI estimation with Moment estimator and 95% confidence bandwidths, plotted against k.

Aircraft, see Scholz (2003). The goal was to provide a basis for understanding the extreme behavior of centerline deviations of the Boeing-747.

That report addressed the risk of an aircraft deviating at a fixed location along the taxiway beyond a certain threshold distance from the taxiway centerline. The B-747 taxiway deviation data were collected from 9/24/2000 to 9/27/2001 at ANC; during this period, 9,767 deviations were recorded at ANC with a range of [-8.225, 8.863] feet, in both directions of the taxiways. Based on the extreme value limiting assumption, positive deviations (ANCrt data with sample size $n = 4,900$) were extrapolated using the $k = 385$ most extreme deviations at ANC, the chosen k value of top observations to EVI estimation, namely $\hat{\gamma} = 0.03925$, Scholz (2003).

Figure 2 depicts the sample path of $\hat{\gamma}$, the EVI estimate, using the Moment estimator of Dekkers, Einmahl, and de Haan (1989), along with the 95% confidence bandwidths, for the ANCrt data. It is easily checked that the straight line corresponding $\gamma = 0$ is inside the confidence bandwidths for a very large upper part of the sample, (in the graphic $k \leq 2,000$). Consequently, the Gumbel domain of attraction cannot be discarded. Moreover, the testing procedures for detecting a finite right endpoint (cf., Neves and Pereira (2010)) suggest the presence of a distribution with finite right endpoint underlying the ANC deviation data.

Figure 3 depicts the sample path of our endpoint estimator $\hat{x}^F$ against k^*. In the range of $650 \leq k^* \leq 2,300$ the graph is quite stable. If we rely on that

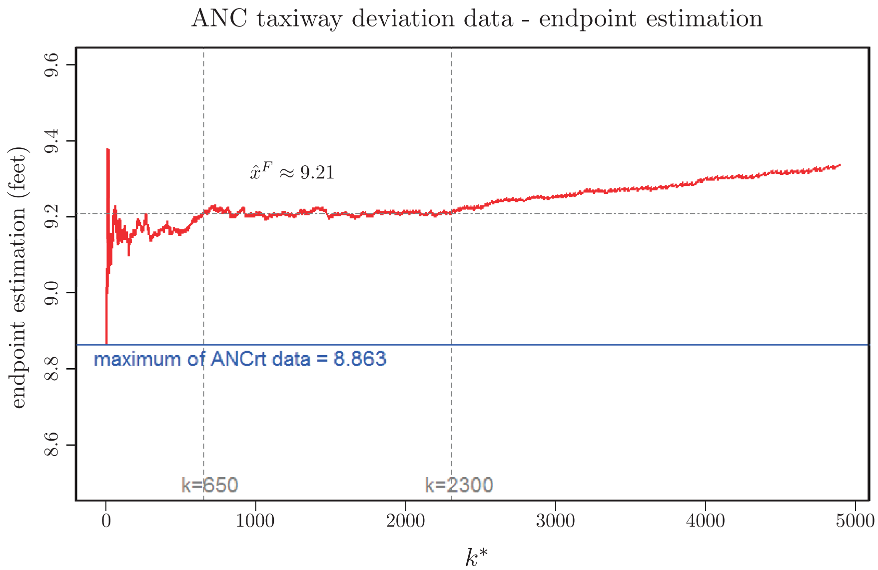

Figure 3. endpoint estimation.

region we suggest for an endpoint estimate a value of approximately 9.21 ft, this for the period 9/24/2000 to 9/27/2001.

We conclude that the proposed estimator $\hat{x}^F$ performs reasonably well for parent distributions in the Gumbel domain with finite right endpoint x^F.

The robustness of the endpoint estimator (1.3), under Weibull domain of attraction, is a topic of further research, but beyond the present scope.

Acknowledgements

We are grateful to Professor Laurens de Haan for introducing the appropriate characterization of distributions with finite right endpoint in the Gumbel domain, at the origin of the proposed estimator. We thank Fritz Scholz (University of Washington), George Legarreta (FAA), and Jerry Robinson (Boeing) for sharing the taxiway deviation data at the Anchorage International Airport. We thank the anonymous referees for their comments and suggestions.

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Appendix A: Consistency of $\hat{q}(n/k)$

Let $U_1, \ldots, U_n$ be independent and identically distributed uniform random variables on the unit interval and let $U_{1,n} \leq U_{2,n} \leq \ldots \leq U_{n,n}$ be their order statistics. Since $k = k_n$ is an intermediate sequence $k_n = o(n)$ as $n \to \infty$, we can define a sequence of Brownian motions $\{W_n(s)\}_{s\geq 0}$ such that, for each $\varepsilon > 0$,

$$\sup_{\frac{1}{\theta k}\leq s\leq 1} s^{\frac{3}{2}+\varepsilon}\left|\sqrt{\theta k}\left(\frac{\theta k}{nU_{[\theta ks]+1,n}}-\frac{1}{s}\right)-\frac{1}{s^2}W_n(s)\right| = o_p(1), \tag{A.1}$$

for all $\theta \geq 1$ (cf., Lemma 2.4.10 of de Haan and Ferreira (2006), with $\gamma = 1$).

Let $X_1, X_2, \ldots$ be i.i.d random variables with distribution function $F \in \mathcal{D}(G_0)$, with finite right endpoint x^F, such that (2.3) holds. Note that $U(1/U_i) \stackrel{d}{=} X_i$, $i = 1, 2, \ldots$. In view of relation (2.3), the following holds:

$$\frac{U(tx)-U(t)}{a(t)} = \int_{1/x}^{1} \frac{a\left(\frac{t}{s}\right)}{a(t)}\,\frac{ds}{s} + \frac{a(tx)}{a(t)}\,o(1) + o(1), \quad (t \to \infty)$$

for all $x > 0$. Given that $a \in RV_0$, we obtain for sufficiently large n that

$$\frac{X_{n-[\theta ks],n} - U\left(\frac{n}{\theta k}\right)}{a\left(\frac{n}{\theta k}\right)} \stackrel{d}{=} \frac{U\left(\frac{n}{\theta k}\,\frac{\theta k}{nU_{[\theta ks]+1,n}}\right) - U\left(\frac{n}{\theta k}\right)}{a\left(\frac{n}{\theta k}\right)} \approx \int_{\frac{nU_{[\theta ks]+1,n}}{\theta k}}^{1} \frac{a\left(\frac{n}{\theta k}\frac{1}{x}\right)}{a\left(\frac{n}{\theta k}\right)}\,\frac{dx}{x}.$$

The uniform inequalities in Lemma C.1.1(ii) tell us that, for any $\varepsilon > 0$,

$$\frac{a\big((n/\theta k)(1/s)\big)}{a\big(n/\theta k\big)} = 1 \pm \varepsilon s^{-\varepsilon}, 0 < s \leq 1.$$

Since $U_{[\theta ks]+1,n} \in [0, 1]$ and for every $s \in (0, 1]$,

$$\frac{n\,U_{[\theta ks]+1,n}}{\theta k} \leq \frac{n\,U_{[\theta k]+1,n}}{\theta k} \xrightarrow[n\to\infty]{P} 1,$$

we get the upper bound

$$\begin{aligned} &\frac{X_{n-[\theta ks],n} - U\big(n/\theta k\big)}{a\big(n/\theta k\big)} \\ &\quad\leq -\log s - \log\left(1 + \left(\frac{nU_{[\theta ks]+1,n}}{\theta ks} - 1\right)\right) + \left(\left(\frac{nU_{[\theta ks]+1,n}}{\theta k}\right)^{-\varepsilon} - 1\right) \\ &\quad= -\log s - \frac{1}{s}\left(\frac{nU_{[\theta ks]+1,n}}{\theta k} - s\right)\big(1 + o_p(1)\big) + (s^{-\varepsilon} - 1)\big(1 + o_p(1)\big), \end{aligned}$$

with the $o_p(1)$-term tending to zero uniformly for $s \in [(\theta k)^{-1}, 1]$. A similar lower bound is also possible. We can apply Cramér's δ-method to relation (A.1) to obtain

$$\frac{X_{n-[\theta ks],n} - U\left(\frac{n}{\theta k}\right)}{a\left(\frac{n}{\theta k}\right)} = -\log s + \frac{1}{\sqrt{\theta k}}\big(s^{-1}W_n(s) + o_p(s^{-1/2-\varepsilon})\big) \pm (s^{-\varepsilon} - 1)\big(1 + o_p(1)\big), \tag{A.2}$$

as $n \to \infty$, uniformly for $(\theta k)^{-1} \le s \le 1$, $\theta \ge 1$. We consider the normalized difference between a sample intermediate quantile and corresponding theoretical quantile

$$R_\theta(s) := \frac{X_{n-[\theta ks],n} - U\big(\frac{n}{\theta ks}\big)}{a\big(\frac{n}{\theta ks}\big)} \tag{A.3}$$
$$= \frac{X_{n-[\theta ks],n} - U\big(\frac{n}{\theta k}\big)}{a\big(\frac{n}{\theta k}\big)} + \left(\frac{a\big(\frac{n}{\theta k}\big)}{a\big(\frac{n}{\theta ks}\big)} - 1\right) \frac{X_{n-[\theta ks],n} - U\big(\frac{n}{\theta k}\big)}{a\big(\frac{n}{\theta k}\big)} + \frac{U\big(\frac{n}{\theta k}\big) - U\big(\frac{n}{\theta ks}\big)}{a\big(\frac{n}{\theta ks}\big)}.$$

Bearing on (A.2) combined with the uniform inequalities in Lemma C.1(1), we thus get for any $\varepsilon > 0$,

$$\begin{aligned} R_\theta(s) &= -\log s + \frac{1}{\sqrt{\theta k}}\Big(\frac{W_n(s)}{s} + s^{-1/2-\varepsilon} o_p(1)\Big) \\ &\quad \pm (s^{-\varepsilon} - 1)\big(1 + o_p(1)\big) \pm \varepsilon s^{-\varepsilon}(-\log s) + \log s \pm \varepsilon s^{-\varepsilon} \\ &= \frac{1}{\sqrt{\theta k}} \frac{W_n(s)}{s} \pm (s^{-\varepsilon} - 1)\big(1 + o_p(1)\big) \mp \varepsilon s^{-\varepsilon} \log s, \end{aligned} \tag{A.4}$$

for $s \in [(\theta k)^{-1}, 1]$, all $\theta \ge 1$. Thus the distribution of deviations between high (large) sample quantiles and their theoretical counterparts is attainable, with a vanishing bias, by means of a different normalization than on the left hand-side of (A.2). The weak convergence of $\hat{q}(n/k)$ is supported on the latter.

Proposition 2. *Let $X_1, X_2, \dots$ be i.i.d. random variables with tail quantile function U satisfying (2.3). Suppose $k = k_n$ is a sequence of positive integers such that $k_n \to \infty$, $k_n/n \to 0$, as $n \to \infty$. Then*

$$\frac{\hat{q}\big(n/k\big)}{q\big(n/k\big)} \xrightarrow[n\to\infty]{p} 1.$$

Proof. We begin by noting that

$$\frac{\hat{q}\big(n/k\big)}{q\big(n/k\big)} = -\frac{1}{\log 2}\int_0^1 \frac{\widehat{U}\big(n/2ks\big) - \widehat{U}\big(n/ks\big)}{q\big(n/k\big)} \frac{ds}{s}$$
$$= -\frac{1}{\log 2}\left\{\int_{\frac{1}{2k}}^1 \frac{X_{n-[2ks],n} - U\big(\frac{n}{2ks}\big)}{q\big(\frac{n}{k}\big)} \frac{ds}{s} - \int_{\frac{1}{k}}^1 \frac{X_{n-[ks],n} - U\big(\frac{n}{ks}\big)}{q\big(\frac{n}{k}\big)} \frac{ds}{s}\right. \tag{A.5}$$
$$\left. - \int_{\frac{1}{2k}}^{\frac{1}{k}} \frac{X_{n,n} - U\big(\frac{n}{ks}\big)}{q\big(\frac{n}{k}\big)} \frac{ds}{s} + \int_{\frac{1}{2k}}^1 \frac{U\big(\frac{n}{2ks}\big) - U\big(\frac{n}{ks}\big)}{q\big(\frac{n}{k}\big)} \frac{ds}{s}\right\}. \tag{A.6}$$

We write (see Eq. (A.3) with $\theta = 2$)

$$\int_{\frac{1}{2k}}^1 \frac{X_{n-[2ks],n} - U\big(\frac{n}{2ks}\big)}{q\big(\frac{n}{k}\big)} \frac{ds}{s} - \int_{\frac{1}{k}}^1 \frac{X_{n-[ks],n} - U\big(\frac{n}{ks}\big)}{q\big(\frac{n}{k}\big)} \frac{ds}{s}$$

$$= \int_{\frac{1}{2}}^{1} \frac{X_{n-[2ks],n} - U\left(\frac{n}{2ks}\right)}{q\left(\frac{n}{k}\right)} \frac{ds}{s} =: I_1(k,n), \tag{A.7}$$

where

$$I_1(k,n) = \frac{a(n/k)}{q\left(\frac{n}{k}\right)} \Big\{ \int_{\frac{1}{2}}^{1} R_2(s) \frac{ds}{s} + \int_{\frac{1}{2}}^{1} \Big(\frac{a\left(\frac{n}{2ks}\right)}{a(n/k)} - 1 \Big) R_2(s) \frac{ds}{s} \Big\}. \tag{A.8}$$

From Lemma C.2.,

$$\begin{aligned} I_1(k,n) &= \int_{\frac{1}{2}}^{1} \frac{X_{n-[2ks],n} - U\left(\frac{n}{2ks}\right)}{q\left(\frac{n}{k}\right)} \frac{ds}{s} \\ &\leq \frac{a(n/k)}{q\left(\frac{n}{k}\right)} \Big| \int_{\frac{1}{2}}^{1} R_2(s) \frac{ds}{s} \Big| + \Big(\frac{a(n/k)}{q(n/k)} \Big)^2 \Big| \int_{\frac{1}{2}}^{1} R_2(s) \log(2s) \frac{ds}{s} \Big| \\ &\leq \frac{a(n/k)}{q\left(\frac{n}{k}\right)} \Big(1 + \frac{a(n/k)}{q(n/k)} \log 2 \Big) \Big| \int_{\frac{1}{2}}^{1} R_2(s) \frac{ds}{s} \Big|, \end{aligned} \tag{A.9}$$

with high probability, for sufficiently large n. We can provide a similar lower bound.

Owing to (A.4), for any positive ε,

$$\Big| \int_{\frac{1}{2}}^{1} R_2(s) \frac{ds}{s} \Big| \leq \Big| \frac{1}{\sqrt{2k}} \int_{\frac{1}{2}}^{1} s^{-2} W_n(s) ds \Big| + \int_{\frac{1}{2}}^{1} \big(s^{-\varepsilon} - 1 \big) \frac{ds}{s} \big(1 + o_p(1) \big)$$
$$- \varepsilon \int_{\frac{1}{2}}^{1} s^{-\varepsilon} \log s \frac{ds}{s}.$$

Since $\varepsilon > 0$ is arbitrary,

$$0 < \int_{\frac{1}{2}}^{1} \big(s^{-\varepsilon} - 1 \big) \frac{ds}{s} = \frac{2^{\varepsilon} - 1}{\varepsilon} - \log 2 \underset{\varepsilon \downarrow 0}{\longrightarrow} 0,$$

meaning that

$$\int_{\frac{1}{2}}^{1} \big(s^{-\varepsilon} - 1 \big) \frac{ds}{s}$$

can be discarded. A similar line of reasoning applies to

$$\varepsilon \int_{\frac{1}{2}}^{1} s^{-\varepsilon} \log\big(\frac{1}{s}\big) \frac{ds}{s} = 2^{\varepsilon} \log 2 - \frac{2^{\varepsilon} - 1}{\varepsilon} \underset{\varepsilon \downarrow 0}{\longrightarrow} 0.$$

Let

$$Y_n := \frac{1}{\sqrt{2k}} \int_{\frac{1}{2}}^{1} W_n(s) \frac{ds}{s^2},$$

be a sequence of normal random variables with zero mean, and

$$Var(Y_n) = \frac{1 - \log 2}{k} \underset{n \to \infty}{\longrightarrow} 0.$$

Thus $\{Y_n\}_{n\geq 1}$ is a sequence of degenerate random variables, eventually, and the two integrals in (A.5) vanish with probability tending to one as $n\to\infty$. As $a(n/k)/q(n/k)=o(1)$, one has

$$I_1(k,n)=o_p(1)\ \Big(=o_p\Big(\frac{a(n/k)}{q(n/k)}\Big)=O_p\Big(\frac{a(n/k)}{\sqrt{k}q(n/k)}\Big)\Big).$$

For the first integral in (A.6),

$$\begin{aligned} I_2(k,n) &:= \int_{\frac{1}{2k}}^{\frac{1}{k}} \frac{X_{n,n}-U\big(\frac{n}{ks}\big)}{q\big(\frac{n}{k}\big)}\,\frac{ds}{s} \\ &= \int_{1/2}^{1} \frac{X_{n,n}-U\big(\frac{n}{s}\big)}{q\big(\frac{n}{k}\big)}\,\frac{ds}{s} \\ &\overset{d}{=} \frac{a(n)}{q(n/k)}\Big\{\frac{U\big(\frac{1}{U_{1,n}}\big)-U(n)}{a(n)}\log 2-\int_{\frac{1}{2}}^{1}\frac{U\big(\frac{n}{s}\big)-U(n)}{a(n)}\,\frac{ds}{s}\Big\} \\ &= \frac{a(n)}{q(n/k)}\Big\{-\log(n\,U_{1,n})\log 2+\int_{\frac{1}{2}}^{1}\log s\,\frac{ds}{s}\Big\}\big(1+o_p(1)\big) \\ &= \frac{a(n)}{q(n/k)}\log 2\Big(-\log(n\,U_{1,n})-\frac{1}{2}\log 2\Big)\big(1+o_p(1)\big). \end{aligned}$$

Then the probability integral transformation yields

$$-\log(n\,U_{1,n})\overset{d}{=}E_{n,n}-\log n, \tag{A.10}$$

where $E_{n,n}$ is the maximum of n i.i.d. standard exponential random variables. Hence, the random variable (A.10) converges in distribution to a Gumbel random variable with distribution function $\exp\{-e^{-x}\}$, $x\in\mathbb{R}$. Moreover, $a(n)/q(n/k)\to 0$, as $n\to\infty$, because $a(n/k)/q(n/k)=o(1)$ (see Lemma C.1(2) in Appendix C), where the auxiliary positive function a satisfies $a(t)\to 0$, as $t\to\infty$, by assumption. Therefore

$$I_2(k,n)=o_p(1)\ \Big(=O_p\big(\frac{a(n)}{q(n/k)}\big)\Big). \tag{A.11}$$

We show that the last integral in (A.6) is bounded. We establish the upper bound,

$$\int_{\frac{1}{2k}}^{1}\frac{U\big(\frac{n}{ks}\big)-U\big(\frac{n}{2ks}\big)}{q(n/k)}\,\frac{ds}{s}\leq\int_{0}^{1}\frac{U\big(\frac{n}{ks}\big)-U\big(\frac{n}{2ks}\big)}{q(n/k)}\,\frac{ds}{s}, \tag{A.12}$$

and the lower bound,

$$\int_{\frac{1}{2k}}^{1}\frac{U\big(n/ks\big)-U\big(n/2ks\big)}{q(n/k)}\,\frac{ds}{s}$$

$$= \int_0^{1-\frac{1}{2k}} \frac{U\big(n/(ks+1/2)\big) - U\big(n/(2ks+1)\big)}{q\big(n/k\big)} \frac{ds}{s+1/2k}$$
$$\geq \int_0^1 \frac{U\big(\frac{n}{ks+1/2}\big) - U\big(\frac{n}{2ks+1}\big)}{q\big(\frac{n}{k}\big)} \frac{ds}{s+\frac{1}{2k}} - 2\int_{1-\frac{1}{2k}}^1 \frac{U\big(\frac{n}{ks+1/2}\big) - U\big(\frac{n}{2ks+1}\big)}{q\big(\frac{n}{k}\big)} \frac{ds}{s+\frac{1}{2k}}.$$

Making $t = n/k$ run on the real line towards infinity, then $\Pi-$variation (2.4) follows

$$\lim_{t\to\infty} \frac{\int_0^1 U\big(\frac{tx}{s}\big)\frac{ds}{s} - \int_0^1 U\big(\frac{t}{s}\big)\frac{ds}{s}}{q(t)} = \log x, \quad x > 0, \tag{A.13}$$

and clearly entails the limit for the upper bound in (A.12):

$$\frac{\int_0^1 U\big(\frac{n}{ks}\big)\frac{ds}{s} - \int_0^1 U\big(\frac{n}{2ks}\big)\frac{ds}{s}}{q(n/k)} = -\frac{\int_0^1 U\big(\frac{n}{2ks}\big)\frac{ds}{s} - \int_0^1 U\big(\frac{n}{ks}\big)\frac{ds}{s}}{q(n/k)} \underset{n\to\infty}{\longrightarrow} \log 2.$$

Regarding the lower bound, we write

$$\int_{\frac{1}{2k}}^1 \frac{U\big(\frac{n}{ks}\big) - U\big(\frac{n}{2ks}\big)}{q(n/k)} \frac{ds}{s} \geq \int_0^1 \frac{U\Big(\frac{n}{k\big(s+\frac{1}{2k}\big)}\Big) - U\Big(\frac{n}{2k\big(s+\frac{1}{2k}\big)}\Big)}{q\big(\frac{n}{k}\big)} \frac{ds}{s+\frac{1}{2k}} \tag{A.14}$$
$$- 2\frac{q\big(\frac{n}{2k}\big)}{q\big(\frac{n}{k}\big)} \int_{1-\frac{1}{2k}}^1 \frac{U\Big(\frac{n}{k\big(s+\frac{1}{2k}\big)}\Big) - U\Big(\frac{n}{2k\big(s+\frac{1}{2k}\big)}\Big)}{} \frac{ds}{s+\frac{1}{2k}}, \tag{A.15}$$

and note that, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$,

$$\Big|\frac{1}{s+1/(2k)} - \frac{1}{s}\Big| < \varepsilon. \tag{A.16}$$

In turn,

$$\int_0^1 \frac{U\Big(\frac{n}{k\big(s+\frac{1}{2k}\big)}\Big) - U\Big(\frac{n}{2k\big(s+\frac{1}{2k}\big)}\Big)}{q\big(\frac{n}{k}\big)} \frac{ds}{s+\frac{1}{2k}} > \int_0^1 \frac{U\Big(\frac{n}{k\big(s+\frac{1}{2k}\big)}\Big) - U\Big(\frac{n}{2k\big(s+\frac{1}{2k}\big)}\Big)}{q\big(\frac{n}{k}\big)} \Big(\frac{1}{s} - \varepsilon\Big) ds.$$

For the first part of the right-hand side here we use (A.13), while the second part is dealt by Theorem B.2.19 of de Haan and Ferreira (2006) involving the fact that $U \in \Pi(a)$:

$$\int_0^1 \frac{U\Big(\frac{n}{k\big(s+\frac{1}{2k}\big)}\Big) - U\Big(\frac{n}{2k\big(s+\frac{1}{2k}\big)}\Big)}{q\big(\frac{n}{k}\big)} \frac{ds}{s} + \varepsilon \frac{a\big(\frac{n}{k}\big)}{q\big(\frac{n}{k}\big)} \int_0^1 \frac{U\Big(\frac{n}{2k\big(s+\frac{1}{2k}\big)}\Big) - U\Big(\frac{n}{k\big(s+\frac{1}{2k}\big)}\Big)}{a\big(\frac{n}{k}\big)} ds$$
$$= \log 2\big(1+o(1)\big) - \varepsilon \frac{a\big(n/k\big)}{q\big(n/k\big)} \log 2\big(1+o(1)\big) \underset{n\to\infty}{\longrightarrow} \log 2.$$

For the latter, we recall that $a(n/k)=o\big(q(n/k)\big)$. Now we write $\delta=1/(2k)>0$ everywhere in (A.15). Furthermore, we assume that there exists $n_0\in\mathbb{N}$ such that, for $n\geq n_0$, the term $n\delta$ is large enough and the integral in (A.15) can rephrased as

$$I_\delta^* := \frac{\int_{1-\delta}^1\Big(U\big(\frac{2}{s+\delta}n\delta\big)-U\big(\frac{1}{s+\delta}n\delta\big)\Big)\frac{ds}{s+\delta}}{\int_{n\delta}^1 a(s)\,\frac{ds}{s}}. \tag{A.17}$$

For every fixed $\delta>0$, from the Π-variation of U with for the numerator of I_δ^* properly rescaled by $a(n\delta)$ (cf., Theorem B.2.19 in de Haan and Ferreira (2006)):

$$\frac{\int_{1-\delta}^1\Big(U\big(\frac{2}{s+\delta}n\delta\big)-U\big(\frac{1}{s+\delta}n\delta\big)\Big)\frac{ds}{s+\delta}}{a(n\delta)}\underset{n\to\infty}{\longrightarrow}\int_{1-\delta}^1\log 2\,\frac{ds}{s+\delta}=\log(1+\delta)\log 2.$$

For arbitrary small δ, the latter approaches zero. We then apply Cauchy's rule to obtain $\lim_{\delta\to 0}I_\delta^*$ (we recall that $\delta\to 0$ implies $n\to\infty$). Apply Eq. (2.11) of Chiang (2000) to the numerator of I_δ^* to get

$$\lim_{\delta\to 0}I_\delta^*=\lim_{\delta\to 0}\frac{\int_{1-\delta}^1\Big(U'\big(\frac{2}{s+\delta}n\delta\big)\frac{2s}{(s+\delta)^3}-U'\big(\frac{1}{s+\delta}n\delta\big)\frac{s}{(s+\delta)^3}\Big)\,ds}{-\frac{a(n\delta)}{n\delta}}$$
$$+\lim_{\delta\to 0}\Big\{\delta\int_{1-\delta}^1\frac{U\big(\frac{2n\delta}{s+\delta}\big)-U\big(\frac{n\delta}{s+\delta}\big)}{a(n\delta)}\,\frac{ds}{(s+\delta)^2}-\delta\frac{U(2n\delta)-U(n\delta)}{a(n\delta)}\Big\}.$$

Since $U'(t)=a(t)/t$, the limit becomes equal to the the limit of

$$-\int_{1-\delta}^1\Big(\frac{a\big(2n\delta/(s+\delta)\big)}{a(n\delta)}-\frac{a\big(n\delta/(s+\delta)\big)}{a(n\delta)}\Big)\frac{s\,ds}{(s+\delta)^2}$$
$$+\delta\Big(\int_{1-\delta}^1\frac{U\big(2n\delta/(s+\delta)\big)-U\big(n\delta/(s+\delta)\big)}{a(n\delta)}\,\frac{ds}{(s+\delta)^2}-\frac{U(2n\delta)-U(n\delta)}{a(n\delta)}\Big).$$

We can now take any arbitrary small δ (making $n\to\infty$) in order to apply the uniform convergence of $a\in RV_0$ and $U\in\Pi(a)$ so that the above integrals are ensured finite and then equal to zero by definition. Hence, all the terms are negligible as δ converges to zero meaning that $\lim_{\delta\to 0}I_\delta^*$ becomes null. Therefore,

$$\int_{\frac{1}{2k}}^1\frac{U\big(n/2ks\big)-U\big(n/ks\big)}{q\big(n/k\big)}\,\frac{ds}{s}\underset{n\to\infty}{\longrightarrow}-\log 2.$$

Consistency of $\hat{q}(n/k)$ follows by noting that $q(n/k)\sim q\big(n/(2k)\big)$.

Appendix B: Asymptotic distribution of $\hat{q}(n/k)$

In order to establish the asymptotic distribution of the proposed estimator for $q(n/k)$ we need insight about the distributional representation obtained from

(A.2). Specifically, if the tail quantile function satisfies (4.1), then Theorem 2.4.2 of de Haan and Ferreira (2006) ascertains that, for each $\varepsilon > 0$,

$$\sup_{\frac{1}{\theta k}\leq s\leq 1} s^{1/2+\varepsilon}\left|\sqrt{\theta k}\left(\frac{X_{n-[\theta ks],n}-U\left(\frac{n}{\theta k}\right)}{a_0\left(\frac{n}{\theta k}\right)}+\log s\right)-\frac{W_n(s)}{s}-\sqrt{\theta k}A_0\left(\frac{n}{\theta k}\right)\frac{1}{2}(\log s)^2\right| \underset{n\to\infty}{\overset{p}{\longrightarrow}} 0, \tag{B.1}$$

provided $k=k_n\to\infty$, $k_n/n=o(n)$ and $\sqrt{k_n}A_0(n/k_n)=O(1)$.

We have the following result (cf., (2.4.7) of de Haan and Ferreira (2006)).

Proposition B.1. Given (2.3), suppose (4.1) holds. Let $k=k_n\to\infty$, $k_n/n=o(n)$ and $\sqrt{k_n}A(n/k_n)\to\lambda\in\mathbb{R}$, as $n\to\infty$. Then, for $\theta\geq 1$ and for each $\varepsilon>0$ sufficiently small,

$$\sup_{\frac{1}{\theta k}\leq s\leq 1} s^{1/2+\varepsilon}\left|\sqrt{\theta k}\frac{X_{n-[\theta ks],n}-U\left(n/\theta ks\right)}{a\left(n/\theta ks\right)}-\frac{W_n(s)}{s}\right|=o_p(1).$$

Proof. As with the equality right after (A.3), we have that

$$R_\theta(s):=\frac{X_{n-[\theta ks],n}-U\left(\frac{n}{\theta ks}\right)}{a\left(\frac{n}{\theta ks}\right)}=\frac{a_0\left(\frac{n}{\theta k}\right)}{a\left(\frac{n}{\theta ks}\right)}\left\{\frac{X_{n-[\theta ks],n}-U\left(\frac{n}{\theta k}\right)}{a_0\left(\frac{n}{\theta k}\right)}-\frac{U\left(\frac{n}{\theta ks}\right)-U\left(\frac{n}{\theta k}\right)}{a_0\left(\frac{n}{\theta k}\right)}\right\}.$$

Noting that

$$\frac{a_0(t)}{a(t/s)}=\frac{a_0(t)}{a(t)}\frac{a(t)}{a(t/s)},$$

for all $s>0$, Lemma C.2 combined with Remark C.1. yields the expansion

$$\frac{a_0(t)}{a\left(\frac{t}{s}\right)}=\frac{a_0(t)}{a(t)}\left(1-\frac{a(t)}{q(t)}\log s+o\left(\frac{a(t)}{q(t)}\right)\right)=\frac{a_0(t)}{a(t)}\left(1+A(t)\log s+o\left(A(t)\right)\right), \tag{B.2}$$

for all $s>0$. Here $|A|\in RV_0$ and $a_0(t)/a(t)=1+o\left(A(t)\right)$.
Having set $1/(\theta k)\leq s\leq 1$, we thus have from (B.1), the uniform bounds in (C.2), and the second equality in (B.2), that

$$\begin{aligned}\sqrt{\theta k}\,R_\theta(s)=&\frac{W_n(s)}{s}+A\left(\frac{n}{\theta k}\right)\frac{\log s}{s}W_n(s)\mp\varepsilon s^{-\varepsilon}\sqrt{\theta k}\,A\left(\frac{n}{\theta k}\right)\\&\pm\varepsilon s^{-\varepsilon}\log s\sqrt{\theta k}\,A^2\left(\frac{n}{\theta k}\right)+o_p(s^{-\frac{1}{2}-\varepsilon})+o_p\left(s^{-\frac{1}{2}-\varepsilon}\log s\,A\left(\frac{n}{\theta k}\right)\right),\end{aligned}$$

uniformly in s. Hence, the assumption that $\sqrt{k}A(n/k)=O(1)$ entails that $\log(1/s)A\left(n/(\theta k)\right)\to 0$, whereas $\varepsilon s^{-\varepsilon}\sqrt{\theta k}A\left(n/(\theta k)\right)$ virtually becomes $o(s^{-1/2-\varepsilon})$ for each $\varepsilon>0$ arbitrarily small and uniformly in $s\in[(\theta k)^{-1},1]$. The o_p-terms are uniform in $s \in [1/(\theta k), 1]$. Hence the following representation is valid for $\varepsilon \in (0,1)$,

$$\sqrt{\theta k}\, R_\theta(s) = \frac{W_n(s)}{s} + o_p(s^{-1/2-\varepsilon}).$$

Proof of the Theorem 2. We have

$$\begin{aligned}
&\frac{q(n/k)}{a(n/k)}\Big(\frac{\hat{q}(n/k)}{q(n/k)} - 1\Big) \\
&\quad = -\frac{1}{a(n/k)}\frac{1}{\log 2}\Big\{\int_{\frac{1}{2}}^{1}\Big(X_{n-[2ks],n} - U\big(\frac{n}{2ks}\big)\Big)\,\frac{ds}{s} - \int_{\frac{1}{2k}}^{\frac{1}{k}}\Big(X_{n,n} - U\big(\frac{n}{ks}\big)\Big)\,\frac{ds}{s} \\
&\qquad -q\big(\frac{n}{k}\big)\Big(\int_{\frac{1}{2k}}^{1}\frac{U\big(n/ks\big) - U\big(n/2ks\big)}{q\big(\frac{n}{k}\big)}\,\frac{ds}{s} - \log 2\Big)\Big\} \\
&\quad = -\frac{1}{\log 2}\big\{J_1(k,n) - J_2(k,n)\big\} + \frac{q\big(n/k\big)}{a\big(n/k\big)}\frac{1}{\log 2}J_3(k,n). \qquad \text{(B.3)}
\end{aligned}$$

By mimicking the steps from (A.7) to (A.8), we obtain for the first integral above that

$$\begin{aligned}
\sqrt{2k}\, J_1(k,n) &:= \sqrt{2k}\int_{\frac{1}{2}}^{1}\frac{X_{n-[2ks],n} - U\big(n/2ks\big)}{a\big(n/k\big)}\,\frac{ds}{s} \\
&= \int_{\frac{1}{2}}^{1}\sqrt{2k}\, R_2(s)\,\frac{ds}{s} + \int_{\frac{1}{2}}^{1}\Big(\frac{a\big(n/2ks\big)}{a\big(n/k\big)} - 1\Big)\sqrt{2k}\, R_2(s)\,\frac{ds}{s}\,.
\end{aligned}$$

Hence, Proposition B.1, while assuming that $\sqrt{k}\, a(n/k)/q(n/k) = O(1)$ (cf., Remark C.1) and application of the uniform bounds in (C.3) with $a_0(t) := a(t)\big(1 + o(A(t)\big)$ and $A_0(t) := A(t)$, imply that for each $\varepsilon > 0$,

$$\begin{aligned}
\sqrt{2k}\, J_1(k,n) = &\int_{\frac{1}{2}}^{1} W_n(s)\big(1 - \log(2s)\big)\,\frac{ds}{s^2} \\
&+ o_p(1)\int_{\frac{1}{2}}^{1}\log\big(\frac{1}{2s}\big)\big(\frac{1}{s}\big)^{3/2+\varepsilon}\,ds + o_p\Big(A_0\big(\frac{n}{k}\big)\Big)\,.
\end{aligned}$$

Since the integral $\int_{1/2}^{1} W_n(s)(1 - \log(2s))s^{-2}\,ds$ converges to a sum of independent normal random variables, this allows us to conclude that the first random component in (B.3) is negligible with high probability because

$$J_1(k,n) = O_p\big(\frac{1}{\sqrt{k}}\big)\,.$$

We now have that

$$J_2(k,n) := \int_{\frac{1}{2k}}^{\frac{1}{k}}\frac{X_{n,n} - U\big(\frac{n}{ks}\big)}{a\big(\frac{n}{k}\big)}\,\frac{ds}{s}$$

$$= \frac{a(n)}{a(n/k)}\Big\{-\log 2 \log\big(nU_{1,n}\big) + \frac{a_0(n)}{a(n)} \int_{\frac{1}{2}}^{1} \log s \frac{ds}{s} + A_0(n) \int_{\frac{1}{2}}^{1} \Big(\frac{(\log s)^2}{2} \pm \varepsilon s^{-\varepsilon}\Big) \frac{ds}{s}\Big\}.$$

We have that $a_0(n)/a(n) - 1 = o\big(A(n)\big)$ and $A_0(n) = A(n)$; hence

$$\frac{a(n/k)}{a(n)} \frac{1}{\log 2} J_2(k,n) = -\log\big(nU_{1,n}\big) - \frac{\log 2}{2} + \frac{1}{\log 2} A(n) \int_{\frac{1}{2}}^{1} \Big(\frac{(\log s)^2}{2} \pm \varepsilon s^{-\varepsilon}\Big) \frac{ds}{s} + o\big(A(n)\big)$$
$$= -\log\big(nU_{1,n}\big) - \frac{\log 2}{2} + o(1).$$

Furthermore, assuming that $k = k(n)$ is such that $a(n)/a(n/k) \to 1$, then

$$\frac{a(n/k)}{a(n)} \frac{1}{\log 2} J_2(k,n) \xrightarrow[n\to\infty]{d} \Lambda - \frac{\log 2}{2},$$

where Λ denotes a Gumbel random variable with distribution function $\exp\{-e^{-x}\}$, $x \in \mathbb{R}$ (cf., (A.10)). If $a(n)/a(n/k)$ converges to a constant different than 1, then a change in the scale is performed. The following also holds given (C.3) and that $\sqrt{k_n}\, A(n/k_n) = O(1)$:

$$\frac{1}{\log 2} J_2(k,n) \xrightarrow[n\to\infty]{d} \Lambda - \frac{\log 2}{2}.$$

We turn to the bias term $J_3(k,n)$. By assumption,

$$\frac{J_3(k,n)}{A(n/k)} = \frac{1}{A(n/k)} \Big(\int_{\frac{1}{2k}}^{1} \frac{U\big(n/ks\big) - U\big(n/2ks\big)}{q\big(n/k\big)} \frac{ds}{s} - \log 2\Big) \xrightarrow[n\to\infty]{} \lambda,$$

as $n \to \infty$. Therefore, since $A(n/k) \sim -a(n/k)/q(n/k)$ (cf., Remark C.1), the deterministic term $J_3(k,n)$ renders a contribution to the asymptotic bias of

$$\frac{q(n/k)}{a(n/k)} \frac{1}{\log 2} J_3(k,n) \xrightarrow[n\to\infty]{} -\frac{\lambda}{\log 2}.$$

Appendix C: Auxiliary Results

Lemma C.1.
1. *Suppose* $U \in \Pi(a)$. *Then,*

(i) *there exists a positive function* a *satisfying* $a(t) \sim a_0(t)$, *as* $t \to \infty$, *such that for any* $\varepsilon > 0$ *there exists* $t_0 = t_0(\varepsilon)$ *such that, for* t, $st \geq t_0$, $s \in (0,1]$,

$$\left| \frac{U(st) - U(t)}{a(t)} - \log s \right| \leq \varepsilon \max(s^{\varepsilon}, s^{-\varepsilon});$$

(ii) $a \in RV_0$ *and for any* $\varepsilon > 0$ *there exists* $t_0 = t_0(\varepsilon)$ *such that, for* t, $st \geq t_0$, $s \in (0,1]$,

$$\left| \frac{a(st)}{a(t)} - 1 \right| \leq \varepsilon \max(s^{\varepsilon}, s^{-\varepsilon}).$$

1. *Suppose* $a > 0$ *is a slowly varying function, integrable over finite intervals of* $\mathbb{R}^+$ *such that*

$$\int_t^{\infty} a(s) \frac{ds}{s} < \infty$$

for every $t > 0$. *Then* $a(t) \to 0$, *as* $t \to \infty$, *and*

$$\lim_{t\to\infty} \int_t^{\infty} \frac{a(s)}{a(t)} \frac{ds}{s} = \infty.$$

Proof. Part 1.(i) of the Lemma comes from de Haan and Ferreira (2006) (cf., Proposition B.2.17), while (ii) is a result from Drees (1998) (cf., Proposition B.1.10 of de Haan and Ferreira (2006)). The second part follows from Karamata's theorem for regularly varying functions (cf., Theorem B.1.5 of de Haan and Ferreira (2006)).

Lemma C.2. *Under* (A1) *and* (B),

$$\lim_{t\to\infty} \frac{a(tx)/a(t) - 1}{a(t)/q(t)} = -\log x, \quad x > 0.$$

Proof. The underlying assumption that $U \in \Pi(a)$ entails

$$\begin{aligned}\frac{q(t)}{a(t)}\Big(\frac{a(tx)}{a(t)} - 1\Big) &= \frac{q(t)}{U(tx) - U(t)} \, \frac{U(tx) - U(t)}{a(t)} \Big(\frac{a(tx)}{a(t)} - 1\Big) \\ &= \frac{q(t)}{U(tx) - U(t)} \, \log x \Big(\frac{a(tx)}{a(t)} - 1\Big)\big(1 + o(1)\big). \quad (t \to \infty) \quad \text{(C.1)}\end{aligned}$$

Furthermore, according to (2.3),

$$\frac{q(t)}{U(tx) - U(t)} = \frac{\int_t^{\infty} a(s) \frac{ds}{s}}{\int_t^{tx} a(s) \frac{ds}{s}\big(1 + o(1)\big)} = 1 + \frac{\int_{tx}^{\infty} a(s) \frac{ds}{s}}{\int_t^{tx} a(s) \frac{ds}{s}}\big(1 + o(1)\big).$$

By taking the limit of the latter term when $t \to \infty$, we get from Cauchy's rule together with the fundamental theorem of integral calculus that

$$\lim_{t\to\infty} \frac{\int_{tx}^{\infty} a(s) \frac{ds}{s}}{\int_t^{tx} a(s) \frac{ds}{s}} = \lim_{t\to\infty} \frac{-a(tx)}{a(tx) - a(t)} = -\lim_{t\to\infty} \Big(\frac{a(tx)}{a(t)} - 1\Big)^{-1}.$$

After (C.1), we get

$$\frac{q(t)}{a(t)}\left(\frac{a(tx)}{a(t)}-1\right) = \log x\left(1+\frac{\int_{tx}^{\infty} a(s)\,\frac{ds}{s}}{\int_{t}^{tx} a(s)\,\frac{ds}{s}}\right)\left(\frac{a(tx)}{a(t)}-1\right)\big(1+o(1)\big)$$
$$= -\log x + \log x\left(\frac{a(tx)}{a(t)}-1\right)\big(1+o(1)\big). \quad (t\to\infty)$$

In addition to the second order condition (4.1), Theorem 2.3.6 of de Haan and Ferreira (2006) ascertains the existence of functions a_0 and A_0 satisfying, as $t\to\infty$, $A_0(t)\sim A(t)$ and $a_0(t)/a(t)-1=o\big(A(t)\big)$, with the property that for any $\varepsilon>0$, there exists $t_0=t_0(\varepsilon)$ such that for all t, $tx\geq t_0$,

$$\left|\frac{(U(tx)-U(t))/a_0(t)-\log x}{A_0(t)}-\frac{1}{2}(\log x)^2\right| \leq \varepsilon\max(x^{\varepsilon},x^{-\varepsilon}), \tag{C.2}$$

$$\left|\frac{a_0(tx)/a_0(t)-1}{A_0(t)}-\log x\right| \leq \varepsilon\max(x^{\varepsilon},x^{-\varepsilon}). \tag{C.3}$$

Remark C.1. Relation (C.3), combined with Lemma C.2, ascertains that $-a_0(t)/q(t)=cA_0(t)$, with $c\neq 0$ because $\rho=\gamma=0$ (cf., Eq. (B.3.4) and Remark B.3.5 in de Haan and Ferreira (2006)). Hence the assumption in this paper that the function q can be redefined in order that $-a/q\sim A$ is satisfied.

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