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AN ITERATIVE CONSTRUCTION OF CONFIDENCE INTERVALS FOR A PROPORTION

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Abstract: Under the criterion of the set inclusion, two-sided admissible $1-\alpha$ confidence intervals for the probability of success for a binomial random variable are constructed using a new iterative method that is based on a direct analysis of coverage probability. A refined Clopper-Pearson interval is derived and compared with the Blyth-Still-Casella interval, and is recommended for statistical practice due to its performance and accessibility. A generalization is provided to the case of a discrete sample space with a single parameter distribution. Some details and an R-code that computes the refined Clopper-Pearson interval are given in Supplementary Materials.

Key words and phrases: Binomial distribution, coverage probability, set inclusion.

1. Introduction

Inference on a single proportion p is an important problem in statistical practice, and confidence intervals for p that are reliable, optimal, and accessible are needed. Let X be a binomial random variable with n trials and a probability of success p, denoted by $X \sim Bin(n,p)$. Let $f(x;n,p)$ be the probability mass function for X. The sample space, $\{0,\ldots,n\}$, is denoted by S. We are interested in constructing two-sided confidence interval $C(X)=[L(X),U(X)]$ for p, that is short and computationally simple, from a class of intervals $\mathcal{B}$ satisfying

$$Cover_C(p)=\sum_{x=0}^{n} I_{[L(x),U(x)]}(p) f(x;n,p) \geq 1-\alpha, \forall p \in [0,1]; \tag{1.1}$$

$$L(x) \leq L(x'), \forall\ x \leq x' \in S; \tag{1.2}$$

$$U(x) = 1-L(n-x), \forall\ x \in S \tag{1.3}$$

for $\alpha \in (0,1)$. Thus, the coverage probability function of $C(X)$ should be at least $1-\alpha$, and large values of X should go with large values of p.

As all intervals in $\mathcal{B}$ are of level $1-\alpha$, shorter ones are preferred. For two intervals $C_1(X)$ and $C_2(X)$ in $\mathcal{B}$, $C_1(X)$ is no worse than $C_2(X)$ if

$$C_1(x) \text{ is a subset of } C_2(x) \text{ for any } x \in S. \tag{1.4}$$

This is the set inclusion criterion introduced in Wang (2006) to evaluate the precision of an interval. If

$$C_S(X) = \cap_{\ \forall\ C(X)\in\mathcal{B}} C(X), \tag{1.5}$$

the intersection of all intervals in $\mathcal{B}$, also belongs to $\mathcal{B}$, then $C_S(X)$ is the smallest interval in $\mathcal{B}$. It automatically guarantees the minimum expected length and the minimum false coverage probability, two commonly used criteria for precision. Unfortunately, $C_S(X)$ belongs to $\mathcal{B}$ only when n or $1-\alpha$ is small. See Wang (2006) for a sufficient and necessary condition on when C_S belongs to $\mathcal{B}$. We construct admissible intervals in $\mathcal{B}$: $C_A(X) \in \mathcal{B}$ that satisfies:

$$\text{if an interval } C\in\mathcal{B} \text{ is a subset of } C_A \text{ for any } x\in S, \text{ then } C(X)=C_A(X). \tag{1.6}$$

We also compare intervals utilizing such criteria as average expected length, total length and the "standard deviation" of expected length in Section 2.4.

There are many confidence intervals for p. Vollset (1993), Newcombe (1998), and Pires and Amado (2008) summarized more than twenty intervals that are either approximate or exact, but none stands out in practice. Here we mention a few.

The best known $1-\alpha$ interval for p is the Wald interval,

$$\left[\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right],$$

where $\hat{p} = X/n$ and $z_{\alpha/2}$ is the upper $\alpha/2$th percentile of a standard normal distribution. Based on the Central Limit Theorem, it is still in wide use. However, it violates conditions (1.1) and (1.2), see Brown, Cai, and DasGupta (2001). In fact, the Central Limit Theorem concerns the convergence of the distribution of $\hat{p}$ to a normal distribution at a single value of p, but does not guarantee the coverage probability at least $1-\alpha$ for all values of p in $[0,1]$. The infimum coverage probability of four other widely used approximate $1-\alpha$ intervals, the Wilson score interval (1927), the Agresti and Coull interval (1998), the Jeffreys prior interval (Brown, Cai, and DasGupta (2001), and Pires and Amado (2008)) and the likelihood-based interval (Newcombe (1998)), converges to some value strictly less than $1-\alpha$ for $\alpha = 0.1$, 0.05, and 0.01 as n goes large, as shown in Huwang (1995) and Wang and Zhang (2013). For example, when $1-\alpha = 0.9$ and $n = 5{,}000$, the infimum coverage probabilities of these five intervals are 0, 0.80, 0.88, 0.70, and 0.69, respectively. This raises a question of consistency for large sample intervals. Among them, the Agresti-Coull interval has the largest infimum coverage probability.

The first exact interval in $\mathcal{B}$ is the $1-\alpha$ Clopper and Pearson interval (1934). It is well known to be conservative. But Wang (2006) pointed out that the $1-2\alpha$

Clopper-Pearson interval is the smallest in $\mathcal{B}$ if and only if its lower confidence limit at $x = n$ is no larger than its upper confidence limit at $x = 0$. Sterne (1954) constructed confidence set for p by inverting tests. Following this, Crow (1956) and Blyth and Still (1983) inverted a family of test acceptance regions for $H_0 : p = p_0$ for all $p_0 \in [0, 1]$ and obtained admissible confidence intervals in $\mathcal{B}$. The Blyth-Still interval is so far the "best" exact small sample interval for p. However, the intervals reported in Blyth and Still (1983) are 2-digit, which results in an infimum coverage probability below the claimed confidence level. StatXact 9, a commercial software for exact inferences, computes C^{BSC}, the Blyth-Still interval with Casella refinement (1986) up to 4-digit. This will be the competitor with our results. The Blyth-Still interval has a problem of implementation and is not necessarily unique.

In this paper, we present a direct interval construction based on analysis of coverage probability. Roughly speaking, we squeeze any given $1 - \alpha$ interval (lift up the lower limit and press down the upper limit) at all sample points one by one in a predetermined order, and stop squeezing when the infimum coverage probability touches $1-\alpha$. See Remark 1. This yields admissible intervals. Section 2 discusses interval construction for a proportion. For statistical practice, we recommend interval C^I in Section 2.3 due to a well defined construction, an overall performance on expected length, and the availability of R-code from the author for implementation, as discussed in Section 2.4. Section 3 generalizes to the case of a discrete random variable. A brief summary is given in Section 4. All proofs are given in Appendix.

2. Admissible Confidence Intervals for p

To obtain an interval $C(X) = [L(X), U(X)]$ in $\mathcal{B}$, $2n + 2$ unknowns, $L(0)$, $U(0), \ldots, L(n{+}1), U(n{+}1)$, are to be determined. Due to (1.3), we reduce to only $n + 1$ unknowns, for example $L(0), \ldots, L(n + 1)$, or some other $n + 1$ equivalent unknowns. It is still difficult to solve these $n + 1$ unknowns simultaneously. For a given value x of X, $L(x)$ and $U(x)$ are the only two unknowns and it is better to solve these two at a time by making $U(x) - L(x)$ as small as possible, which leads to an iterative method. This method generates $1 - \alpha$ admissible confidence interval in $\mathcal{B}$. We call it *the squeezing-one-at-a-time method*. In this section, we discuss the construction of $C(X)$ under any order on S, and then show how to improve any given interval in $\mathcal{B}$ to be admissible.

2.1. The interval construction with a special order on S

For simplicity, we describe an iterative construction of an interval in $\mathcal{B}$ in the order of $x = 0, 1, \ldots, [n/2]$. Let $C^*(X) = [L^*(X), U^*(X)]$ be the interval generated in this subsection following Steps 1-3 described below. We determine the

$n+1$ unknowns in the order of $L^*(0)$ and $U^*(0)$ as a pair, then $L^*(1)$ and $U^*(1)$, $\ldots$, and then $L^*([n/2])$ and $U^*([n/2])$. For the other $x-$values, the confidence limits are obtained automatically following (1.3). The interval construction based on any order is discussed in the next subsection.

Step 1. Set $L^*(0) = 0$ (then $U^*(n) = 1$) due to (1), see Lemma 1 in Wang (2006), and solve for $U^*(0)$. For a given number $c \in [0, 1]$, let

$$L_c(x) = \begin{cases} 0 & \text{if } x < n, \\ 1 - c & \text{if } x = n, \end{cases} \tag{2.1}$$

and $U_c(x) = 1 - L_c(n - x)$. Then $c = U_c(0)$, $L_c(x)$ meets (1.2), and $C_c(X) \stackrel{def}{=} [L_c(X), U_c(X)]$ satisfies (1.3). Let

$$c^* = \inf \ D_1, \text{ where } D_1 = \{c \in [0, 1] : C_c(X) \in \mathcal{B}\}. \tag{2.2}$$

To see whether $C_c(X)$ belongs to $\mathcal{B}$, compute $Cover_{C_c}(p)$ to see that it is no less than $1 - \alpha$ for all $p \in [0, 1]$. In summary, $[L^*(0), U^*(0)] = [0, c^*]$ is determined in this step, as well as $[L^*(n), U^*(n)] = [1 - c^*, 1]$, due to (1.3).

Lemma 1. *For c^* and D_1 given in* (2.2), *c^* is unique and $C_{c^*}(X) \in \mathcal{B}$.*

Step 2. Suppose $[L^*(x), U^*(x)]$ is determined up to $x < x_0$ for some $x_0 < n/2$. For $0 \le a \le b \le 1$, let

$$L_{a,b}(x) = \begin{cases} L^*(x) & \text{if } x < x_0, \\ a & \text{if } x_0 \le x < n - x_0, \\ 1 - b & \text{if } x = n - x_0, \\ 1 - U^*(n - x) & \text{if } x > n - x_0, \end{cases} \tag{2.3}$$

and $U_{a,b}(x) = 1 - L_{a,b}(n - x)$. Then $U_{a,b}(x_0) = b$ and $C_{a,b}(X) \stackrel{def}{=} [L_{a,b}(X), U_{a,b}(X)]$ satisfies (1.3). Consider those a and b's if $C_{a,b}(X)$ also satisfies (1.2). Pick a^* and b^* so that

$$b^* - a^* = \inf_{(a,b) \in D_2} (b - a), \text{ where } D_2 = \{(a, b) : C_{a,b}(X) \in \mathcal{B}\}. \tag{2.4}$$

Then we obtain $[L^*(x_0), U^*(x_0)] = [a^*, b^*]$, as well as $[L^*(n - x_0), U^*(n - x_0)] = [1 - b^*, 1 - a^*]$.

Lemma 2. *For each $1 \le x_0 < n/2$, there exists a pair of (a^*, b^*) satisfying* (2.4), *and $(a^*, b^*) \in D_2$.*

Lemma 3. *For each $1 \le x_0 < n/2$, the pair (a^*, b^*) given in Lemma 2 is unique.*

Step 3. Suppose $[L^*(x), U^*(x)]$ is determined up to $x < n/2$.

Step 3-1. If n is an odd number, the interval construction is complete.

Step 3-2. If n is an even number, the interval construction is complete if $L^*(n/2)$ is determined since $U^*(n/2) = 1 - L^*(n/2)$. For $d \in [0, 1/2]$, let

$$L_d(x) = \begin{cases} L^*(x) & \text{if } x < \frac{n}{2}, \\ d & \text{if } x = \frac{n}{2}, \\ 1 - U^*(n-x) & \text{if } x > \frac{n}{2}, \end{cases} \tag{2.5}$$

and $U_d(x) = 1 - L_d(n-x)$. Then $U_d(n/2) = 1 - d$ and $C_d(X) \stackrel{def}{=} [L_d(X), U_d(X)]$ satisfies (1.3). Consider those d's if C_d also satisfies (1.2). Let

$$d^* = \sup\ D_3, \text{ where } D_3 = \{d \in [0, \frac{1}{2}] : C_d(X) \in \mathcal{B}_B\}. \tag{2.6}$$

Then $[L^*(n/2), U^*(n/2)] = [d^*, 1 - d^*]$ and the interval construction is complete.

Theorem 1. *An interval $C^*(X) = [L^*(X), U^*(X)]$ generated by Steps* 1–3 *belongs to $\mathcal{B}$, and is unique.*

Remark 1. The iterative method squeezes undetermined intervals one at a time. In each step, say $x = x_0$, we pick the shortest $C^*(x_0)$ and make the other undetermined intervals as large as possible. Interval (2.3), for example, is the largest interval in $\mathcal{B}$ that is equal to $C^*(x)$ for $x < x_0$ or $x > n - x_0$ and $[a, b]$ for $x = x_0$. The same holds for interval (2.1).

Remark 2. The uniqueness of $C^*(X)$ is important, as no subjective selection is needed.

Theorem 2. *Interval $C^*(X) = [L^*(X), U^*(X)]$ generated by Steps* 1–3 *is admissible in $\mathcal{B}$ under the set inclusion criterion* (1.4).

Remark 3. In interval construction, the infimum coverage probability needs to be computed quite often. For a real number A and a function $g(x)$, let $g(A_-)$ denote the limit of $g(x)$ when x approaches A from the left. Wang (2007) pointed out that for any interval $C(X) = [L(X), U(X)]$ in $\mathcal{B}$ the infimum coverage probability is one of the $Cover_C(L(x)_-)$'s for $1 \le x \le n$. Therefore, in our numerical calculation, $Cover_C(p)$ is computed at $p = L(x) - \delta$ for a small $\delta = 0.00000001$ if $L(x) > 0$, and the minimum of these quantities (at most n of them) is a precise approximation of the infimum coverage probability of $C(X)$.

Example 1. For $X \sim Bin(8, p)$, we construct a 95% admissible confidence interval $C^*(X) = [L^*(X), U^*(X)]$ in $\mathcal{B}$.

Step 1. Following (2.1), $C_c(X) = [L_c(X), U_c(X)]$ has the form

x	0	1	2	3	4	5	6	7	8
$L_c(x)$	0	0	0	0	0	0	0	0	$1-c$
$U_c(x)$	c	1	1	1	1	1	1	1	1

with one unknown c. Interval C_c has a coverage probability

$$Cover_{C_c}(p) = f(0;8,p)I_{[0,c]}(p) + \sum_{i=1}^{7} f(i;7,p) + f(8;8,p)I_{[1-c,1]}(p).$$

For each given c of values 0, 0.0001,. . . , 1, following Remark 3, the infimum coverage probability of $C_c(X)$, achieved at $p = L(8) - \delta$, is computed. If this quantity is at least 0.95, then record c. Find the smallest c among those recorded, which is equal to c^*. Here $c^* = 0.3126$, and $[L^*(0), U^*(0)] = [0, 0.3126]$ and $[L^*(8), U^*(8)] = [0.6874, 1]$.

Step 2. Now $x_0 = 1$. Following (2.3), $C_{a,b}(X) = [L_{a,b}(X), U_{a,b}(X)]$ has the form

x	0	1	2	3	4	5	6	7	8
$L_{a,b}(x)$	0	a	a	a	a	a	a	$1-b$	0.6874
$U_{a,b}(x)$	0.3126	b	$1-a$	$1-a$	$1-a$	$1-a$	$1-a$	$1-a$	1

with unknowns a and b. For $a \le 1-a$, $a \in [0, 0.5]$, while for $b \ge a$ and $a \le 1-b$, $b \in [a, 1-a]$. For each a from $0, 0.0001, \ldots, 0.5$ and each b from $a, a+0.0001, \ldots, 1-a$, compute coverage probabilities for $C_{a,b}(X)$ at $p = a - \delta, 1-b-\delta$, and $0.6874 - \delta$. If the smallest among the three is at least 0.95, then record the values of a, b, and $b-a$. Find the smallest $b-a$ and the corresponding a and b to be $b^* - a^*$, a^*, and b^*. Here $a^* = 0.0063$, $b^* = 0.6874$, $[L^*(1), U^*(1)] = [0.0063, 0.6874]$, and $[L^*(7), U^*(7)] = [0.3126, 0.9937]$.

Step 2'. Repeat Step 2 twice for $x_0 = 2$ and $x_0 = 3$ and obtain $C^*(2)$ and $C^*(6)$, and $C^*(3)$ and $C^*(5)$.

Step 3-2. Since $n = 8$ is an even number, following (2.5), $C_d(X) = [L_d(X), U_d(X)]$ has the form

x	0	1	2	3	4	5	6	7	8
$L_d(x)$	0	..	..	0.1111	d	0.2892	..	..	0.6874
$U_d(x)$	0.3126	..	..	0.7108	$1-d$	0.8889	..	..	1

with unknown d. Note $d \in [0.1111, 0.2892]$. For each d from $0.1111, 0.1112, \ldots, 0.2892$, compute the coverage probability for $C_d(X)$ at $p = L_d(x) - \delta$ for $x = 1, \ldots, 8$. If their smallest is at least 0.95, then record this value of d. Find the smallest $1-2d$ and the corresponding d to be $1-2d^*$ and d^*. Here $d^* = 0.1929$,

Table 1. 95% confidence intervals $C^* = [L^*, U^*]$, C^P, and C^I in Examples 1−3, and C^{BSC} for p when $n = 8$.

x	$L^*(x)$	$U^*(x)$	L^{BSC}	U^{BSC}	L^P	U^P	L^I	U^I
0	0	0.3126	0	0.3491	0	0.4003	0	0.3695
1	0.0063	0.6874	0.0063	0.5000	0.0063	0.5000	0.0063	0.5000
2	0.0463	0.6874	0.0463	0.6509	0.0463	0.5997	0.0463	0.6305
3	0.1111	0.7108	0.1111	0.7108	0.1111	0.7108	0.1111	0.7108
4	0.1929	0.8071	0.1929	0.8071	0.1929	0.8071	0.1929	0.8071
5	0.2892	0.8889	0.2892	0.8889	0.2892	0.8889	0.2892	0.8889
6	0.3126	0.9537	0.3491	0.9537	0.4003	0.9537	0.3695	0.9537
7	0.3126	0.9937	0.5000	0.9937	0.5000	0.9937	0.5000	0.9937
8	0.6874	1	0.6509	1	0.5997	1	0.6305	1

and $[L^*(4), U^*(4)] = [0.1929, 0.8071]$. The interval construction is complete, and interval $C^*(X)$ is given in Table 1. This interval has the shortest possible length at $x = 0$ and $x = 8$, but $L^*(6) = L^*(7)$. See more discussion in Section 2.4.

2.2. The interval construction with any given order on S

We have presented an iterative interval construction for the order $X = 0, 1, \ldots, [n/2]$. There are interval constructions based on different orders, each resulting in an admissible interval. Thus, let $\{i_0, \ldots, i_{[n/2]}\}$ be a permutation of $\{0, \ldots, [n/2]\}$(denoted by S_h), and let $C^P(X) = [L^P(X), U^P(X)]$ be the interval constructed following the order $L^P(i_0), U^P(i_0)$; $L^P(i_1), U^P(i_1)$; $\ldots$; $L^P(i_{[n/2]})$, $U^P(i_{[n/2]})$. Then $C^P(X)$ on the other $x-$values is obtained following (1.3).

Suppose $C^P(X)$ is already determined for $L^P(i_0), U^P(i_0)$, $L^P(i_1), U^P(i_1)$, $\ldots$, $L^P(i_{k-1})$, $U^P(i_{k-1})$ for some integer $0 \le k \le [n/2]$. Now we obtain $L^P(i_k)$ and $U^P(i_k)$. For some $a \le b$, consider the largest interval $C^P_{a,b}(X) = [L^P_{a,b}(X), U^P_{a,b}(X)] \in \mathcal{B}$ satisfying

$$L^P_{a,b}(x) = \begin{cases} L^P(x) & \text{if } x \in \{i_0, \ldots, i_{k-1}\}, \\ a & \text{if } x = i_k, \\ 1 - b & \text{if } x = n - i_k, \\ 1 - U^P(n-x) & \text{if } x \in n - \{i_0, \ldots, i_{k-1}\}. \end{cases} \tag{2.7}$$

In fact, $C^P_{a,b}(X)$ is just equal to the union of all intervals in $\mathcal{B}$ satisfying (2.7) and the intervals given in (2.1) and (2.3) are special cases of $L^P_{a,b}(x)$. The closed form of $L^P_{a,b}(x)$ for the other $x-$values is complicated and is omitted. We give $L^P_{a,b}(X)$ in the next example. Take a^P and b^P so that

$$b^P - a^P = \inf_{(a,b) \in D_P} (b - a), \text{ for } D_P = \{(a,b) : C^P_{a,b}(X) \in \mathcal{B}\}. \tag{2.8}$$

Similar to Lemmas 1 and 2, we can show $(a^P, b^P) \in D_P$. Thus define

$$L^P(i_k) = a^P \text{ and } U^P(i_k) = b^P,$$

and the construction of $C^P(X)$ is complete. As in Theorems 1 and 2, $C^P(X)$ belongs to $\mathcal{B}$, is unique and admissible.

Example 2. (Example 1 continued) We construct $C^P(X)$ following the order $X = 2, 0, 1, 3, 4$ for illustration.

Step 1. Following (2.7), $C^P_{a,b}(X) = [L^P_{a,b}(X), U^P_{a,b}(X)]$ has the form

x	0	1	2	3	4	5	6	7	8
$L^P_{a,b}(x)$	0	0	a	a	a	a	$1-b$	$1-b$	$1-b$
$U^P_{a,b}(x)$	b	b	b	$1-a$	$1-a$	$1-a$	$1-a$	1	1

with unknowns a and b. We find $a^P = 0.0463$ and $b^P = 0.5997$, and then $[L^P(2), U^P(2)] = [0.0463, 0.5997]$ and $[L^P(6), U^P(6)] = [0.4003, 0.9537]$.

Step 2. Following (2.7), $C^P_{0,b}(X) = [L^P_{0,b}(X), U^P_{0,b}(X)]$ has the form

x	0	1	2	3	4	5	6	7	8
$L^P_{0,b}(x)$	0	0	0.0463	0.0463	0.0463	0.0463	0.4003	0.4003	$1-b$
$U^P_{0,b}(X)$	b	0.5997	0.5997	0.9537	0.9537	0.9537	0.9537	1	1

with unknown b. We find $b^P = 0.4003$, and then $[L^P(0), U^P(0)] = [0, 0.4003]$ and $[L^P(8), U^P(8)] = [0.5997, 1]$.

More details of the construction are given in the Supplementary Materials. All $C^P(x)$'s are reported in Table 1. This interval has a short length at $x = 2$, where we started construction.

2.3. Improving any existing non-admissible interval

Our interval constructions can be described as follows. Starting from the largest and trivial $1-\alpha$ interval $C^T(X) = [L^T(X), U^T(X)] \equiv [0, 1]$, we squeeze each $[L^T(x), U^T(x)]$ in a predetermined order on S within class $\mathcal{B}$. This approach can be applied to squeeze any given $C(X) \in \mathcal{B}$ to yield an admissible interval. The $1-\alpha$ Clopper-Pearson interval, denoted by $C^{CP}(X) = [L^{CP}(X), U^{CP}(X)]$, is conservative; however, it is intuitively attractive and easy to derive. We improve $C^{CP}(X)$ to an admissible interval, denoted by $C^I(X) = [L^I(X), U^I(X)]$. Note that the length of $C^{CP}(x)$ is maximized at $x = [n/2]$, we construct $C^I(X)$ by squeezing each $C^{CP}(x)$ in the order $x = [n/2], \ldots, 0$ in steps similar to those in Section 2.1 but in the opposite order $x = 0, \ldots, [n/2]$, and define C^I at the rest of $x-$values by (1.3).

Step 1-1(I). If n is an odd number, go to Step 2(I).

Step 1-2(I). If n is an even number, $U^I(n/2) = 1 - L^I(n/2)$. For $d \in [0, 1/2]$, let

$$L_d^I(x) = \begin{cases} L^{CP}(x) & \text{if } x \neq \frac{n}{2}, \\ d & \text{if } x = \frac{n}{2}, \end{cases} \tag{2.9}$$

and $U_d^I(x) = 1 - L_d^I(x)$. Then $U_d^I(n/2) = 1 - d$, and $C_d^I(X) \stackrel{def}{=} [L_d^I(X), U_d^I(X)]$ satisfies (1.3). Consider those d's for which C_d^I also satisfies (1.2). Let

$$d^I = \sup \ D_1^I, \ \text{ where } \ D_1^I = \{d \in [0, \frac{1}{2}] : C_d^I(X) \in \mathcal{B}_B\}. \tag{2.10}$$

Take $[L^I(n/2), U^I(n/2)] = [d^I, 1 - d^I]$, a subset of $[L^{CP}(n/2), U^{CP}(n/2)]$.

Step 2(I). For any $0 < x_0 < n/2$, suppose $C^I(x)$ has been defined for $x_0 < x \leq [n/2]$, so $L^I(x)$ is defined for $x_0 < x < n - x_0$ due to (1.3). If n is odd and $x_0 = [n/2]$, then C^I has not been defined for any x yet. For two numbers $0 \leq a \leq b \leq 1$, let

$$L_{a,b}^I(x) = \begin{cases} L^{CP}(x) & \text{if } x < x_0 \text{ or } x > n - x_0, \\ a & \text{if } x = x_0, \\ L^I(x) & \text{if } x_0 < x < n - x_0, \\ 1 - b & \text{if } x = n - x_0, \end{cases} \tag{2.11}$$

and $U_{a,b}^I(x) = 1 - L_{a,b}^I(n - x)$. Then $U_{a,b}^I(x_0) = b$ and $C_{a,b}^I(X) \stackrel{def}{=} [L_{a,b}^I(X), U_{a,b}^I(X)]$ satisfies (1.3). Consider those a and b's for which $C_{a,b}^I(X)$ satisfies (1.2). Pick a^I and b^I so that

$$b^I - a^I = \inf_{(a,b) \in D_2^I} (b - a), \ \text{ for } \ D_2^I = \{(a, b) : C_{a,b}^I(X) \in \mathcal{B}\}. \tag{2.12}$$

Then take $[L^I(x_0), U^I(x_0)] = [a^I, b^I]$, which is a subset of $[L^{CP}(x_0), U^{CP}(x_0)]$, as well as $[L^I(n - x_0), U^I(n - x_0)] = [1 - b^I, 1 - a^I]$,

Step 3(I). Now we construct $[L^I(0), U^I(0)] = [0, U^I(0)]$ (and $[L^I(n), U^I(n)] = [1 - U^I(0), 1]$). For $c \in [0, 1]$, let

$$L_c^I(x) = \begin{cases} L^{CP}(0) = 0 & \text{if } x = 0, \\ L^I(x) & \text{if } 0 < x < n, \\ 1 - c & \text{if } x = n, \end{cases} \tag{2.13}$$

and $U_c^I(x) = 1 - L_c^I(n - x)$. Then $c = U_c^I(0)$ and $C_c^I(X) \stackrel{def}{=} [L_c^I(X), U_c^I(X)]$ satisfies (1.3). Consider those c's for which $C_c^I(X)$ satisfying (1.2). Let

$$c^I = \inf \ D_3^I, \ \text{ where } \ D_3^I = \{c \in [0, 1] : C_c^I(X) \in \mathcal{B}\}. \tag{2.14}$$

Take $[L^I(0), U^I(0)] = [0, c^I]$ and $[L^I(n), U^I(n)] = [1 - c^I, 1]$, due to (1.3). The construction of $C^I(X)$ is complete.

Interval $C^I(X)$ is a subset of $C^{CP}(X)$, and is unique and admissible. The proof is similar to those of Theorems 1 and 2, and is omitted.

Example 3. (Example 1 continued) Here we improve the 95% Clopper-Pearson interval $C^{CP}(X) = [L^{CP}(X), U^{CP}(X)]$ given as

x	0	1	2	3	4	5	6	7	8
$L^{CP}(x)$	0	0.0032	0.0319	0.0852	0.1570	0.2449	0.3491	0.4735	0.6305
$U^{CP}(x)$	0.3695	0.5265	0.6509	0.7551	0.8430	0.9148	0.9681	0.9968	1

Shrinking $C^{CP}(X)$ in the order $x = 4, 3, 2, 1, 0$, $C^I(X)$ is computed, and reported in Table 1. It has a smallest total length among the four reported intervals.

2.4. Comparison

We have C^*, C^P, C^I, and a competitor C^{BSC}. All four are admissible and of level $1 - \alpha$. We take, for interval $C(X)$, $Length_C(X) = U(X) - L(X)$.

Figure 1 shows the expected length $E_p(Length_C)$ of the four intervals when $n = 8$ and $1 - \alpha = 0.95$. Similar figures are expected for other sample sizes and levels. The intervals C^* (too wide at $x = 1$) and C^P (too wide at $x = 0$) are not for future competition, while $C^P(X)$, including C^*, has the shortest length at the starting point of interval construction among all intervals in $\mathcal{B}$. However, a locally best interval does not necessarily imply an overall good performance. For instance, C^* in Example 1 has the shortest length at $x = 0$ but is wide at $x = 1$. The interval $C^I(X)$, derived by squeezing C^{CP}, seems a reasonable choice–being a subset of C^{CP}, C^I can never be too wide at any sample point, and should have an overall good performance.

Our numerical study supports this claim. To make comparisons fair, interval limits are accurate to 4 digits after the decimal point and infimum coverage is as close as possible to, and strictly no less than, the nominal level. A slight modification is then needed for C^{BSC} from StatXact 9. For example, in the setting of Figure 1, the lower limit of C^{BSC} at $x = 2$ is 0.04639 by StatXact 9, which yields an infimum coverage probability 0.9499986, a little less than 0.95. If we change it to 0.0463 (the best fit up to 4 decimal accuracy to have a correct coverage), there results an infimum coverage probability of 0.9500012. Figure 2 displays the expected length of the 95% intervals C^I and C^{BSC} for $n = 20$ and $n = 30$. As expected, the expected length of C^I is smaller for p close to 0.5 and

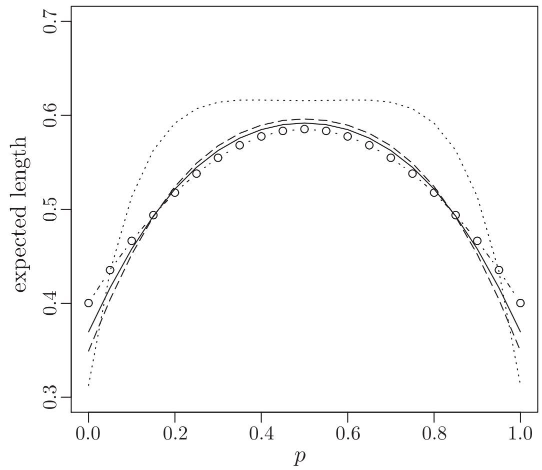

Figure 1. Expected length of the 95% confidence intervals C^*(dot), $C^{BSC}(dash)$, C^P(dash-circle) and C^I(solid), in Examples 1, 2, and 3 when $n = 8$.

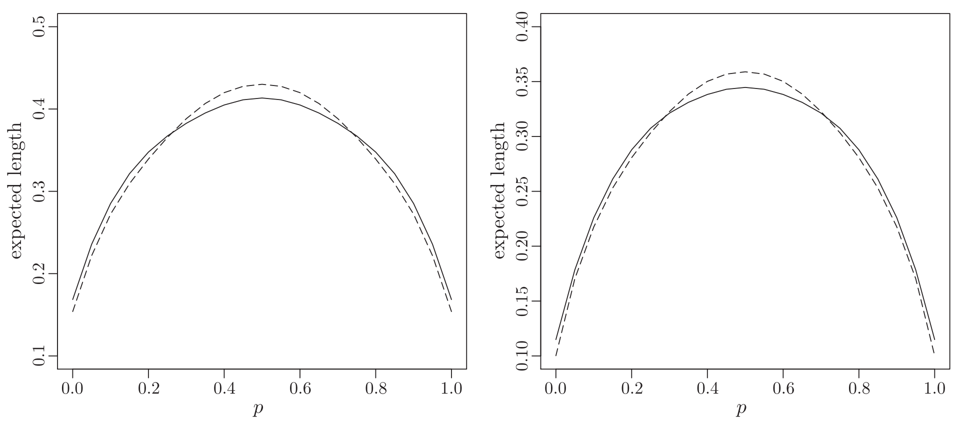

Figure 2. Expected length of the 95% confidence intervals $C^{BSC}(dash)$ and C^I(solid), when $n = 20$ and $n = 30$.

is larger for p close to 0 or 1 because we start squeezing C^{CP} at $x = [n/2]$. The average expected length (the area under the expected length curve),

$$AELength = \int_0^1 E_p Length_C(X)dp \tag{2.15}$$

is the same, at least numerically, for the two intervals, as well as the total length: $Tlength = \sum_{x=0}^n Length_C(x)$. However, C^I has a smaller "standard deviation" of expected length,

$$SDElength = \Big(\int_0^1 [E_p(Length_C(X)) - AElength]^2 dp \Big)^{1/2},$$

Table 2. Comparison of two intervals C^I and C^{BSC} at different levels and sample sizes.

n	Tlength	AElength	SDElength	Tlength	AElength	SDElength
		99% C^I			99% C^{BSC}	
8	5.6936	0.6326222	0.080732	5.6936	0.6326222	0.080732
20	9.2126	0.4386952	0.0808369	9.2126	0.4386952	0.08865476
30	11.2358	0.3624452	0.07634151	11.236	0.3624516	0.08147061
		95% C^I			95% C^{BSC}	
8	4.7084	0.5231556	0.065211	4.7084	0.5231556	0.071989
20	7.242	0.3448571	0.06644264	7.2424	0.3448762	0.07733478
30	8.7738	0.2830258	0.06307469	8.7744	0.2830452	0.07058812
		90% C^I			90% C^{BSC}	
8	4.064	0.4515556	0.05779491	4.064	0.4515556	0.07201108
20	6.2188	0.2961333	0.06433938	6.2154	0.2959714	0.06943668
30	7.4956	0.2417935	0.05295100	7.49566	0.2417955	0.05891413

than C^{BSC}, and is then more stable. See Table 2 for more details.

As pointed out by a referee, when a given prior distribution $\pi(p)$ about p exists, an average probability mass function

$$h(x) = \int_0^1 f(x; n, p)\pi(p)dp$$

with respect to this prior can be computed for any $x \in S$. Then an order on S is obtained by ranking $h(x)$ from the largest to the smallest, and we squeeze interval C^{CP} according to this order. Instead of comparing $AELength$ in (2.15), we compare the weighted average expected length,

$$WAELength = \int_0^1 E_p Length_C(X)\pi(p)dp. \qquad (2.16)$$

The resultant interval C^π would have a smaller $WAELength$ than C^{BSC}. For example, when $\pi(p)$ is a symmetric beta distribution $beta(p; \alpha, \beta)$ with $\alpha = \beta > 1$, C^π is equal to C^I because the order by $h(x)$ here is the same as the order under which C^I is constructed. Therefore, C^π has a shorter $WAELength$ since such beta distributions put more weight on those p's close to 0.5, and C^I has a shorter $E_pLength$ at these p's, as shown in Figures 1 and 2. Table 3 reports $WAELength$ for two intervals.

Blyth and Still (1983, p.110) proposed five complicated rules on the selection of test acceptance regions for interval construction, and these rules cannot be met simultaneously. For example, their rules a) (minimize the difference between two tail probabilities) and d) (make the slope of its power function as close to zero

Table 3. A comparison of $WAELength$ for the intervals C^I and C^{BSC} at different levels and sample sizes, using beta distributions.

n	beta(p;2,2)	beta(p;5,5)	beta(p;10,10)	beta(p;2,2)	beta(p;5,5)	beta(p;10,10)
		99% C^I			99% C^{BSC}	
8	0.6687248	0.6980417	0.7095757	0.6687248	0.6980417	0.7095757
20	0.4746483	0.5008912	0.5101356	0.4781796	0.507495	0.518108
30	0.3963984	0.421501	0.4308902	0.3988055	0.426874	0.4377858
		95% C^I			95% C^{BSC}	
8	0.5522812	0.5748420	0.5832294	0.5552485	0.5792979	0.5878416
20	0.3743549	0.3958376	0.4067811	0.3794085	0.403898	0.4178052
30	0.3109593	0.3306100	0.3373951	0.3145195	0.3387862	0.3483518
		90% C^I			90% C^{BSC}	
8	0.4773721	0.4975143	0.5052138	0.4837479	0.50938	0.5190704
20	0.3247083	0.3452646	0.3522806	0.3269854	0.3511658	0.3603209
30	0.2651966	0.2814203	0.2868806	0.2680767	0.288445	0.2964686

as possible at the null hypothesis value p_0) cannot be achieved at the same time since the binomial distribution is not symmetric for any $p \neq 0.5$. Major statistical softwares, including SAS and Minitab, do not provide the Blyth-Still interval.

The interval C^I is obtained following two well-defined steps: construct the Clopper-Pearson interval C^{CP}; squeeze $C^{CP}(x)$'s in the order $x = [n/2], \ldots, 0$, and is unique. Figure 3 provides a comparison between C^{CP} and C^I when $n = 20$ and $1 - \alpha = 0.95$. The improvement of C^I over C^{CP} is substantial: the infimum coverage probabilities are 0.95001 and 0.95797, respectively, the total length of C^I is 94.20876% of that for C^{CP}, and a uniformly shorter expected length of C^I is shown in Figure 3 (b). Most importantly, C^I can be computed quickly using R-code available from the author. For example, when $n = 200$, all 201 $C^I(x)$'s are calculated in 3.8 minutes on a HP laptop Intel(R) Core(TM) i5-2520M CPU @2.50GHz RAM 8GB. A faster computation is expected if R-code is written more efficiently.

If the smallest confidence interval exists in $\mathcal{B}$, then C^*, C^P, C^I, C^{BSC} and any admissible interval are identical and are equal to the smallest interval. Otherwise, the minimum complete class can be obtained by collecting all improved intervals as shown in Section 2.3. Casella (1986) proposed a method that also refines any interval $C(X)$ in $\mathcal{B}$. His method lifts the lower limits of the $n+1$ intervals of $C(X)$ one at a time in the order $x = n, n-1, \ldots, 0$, while we minimize the length of these intervals in any order on S. Our method can produce an interval that has a good performance at any predetermined sample point, and is more flexible.

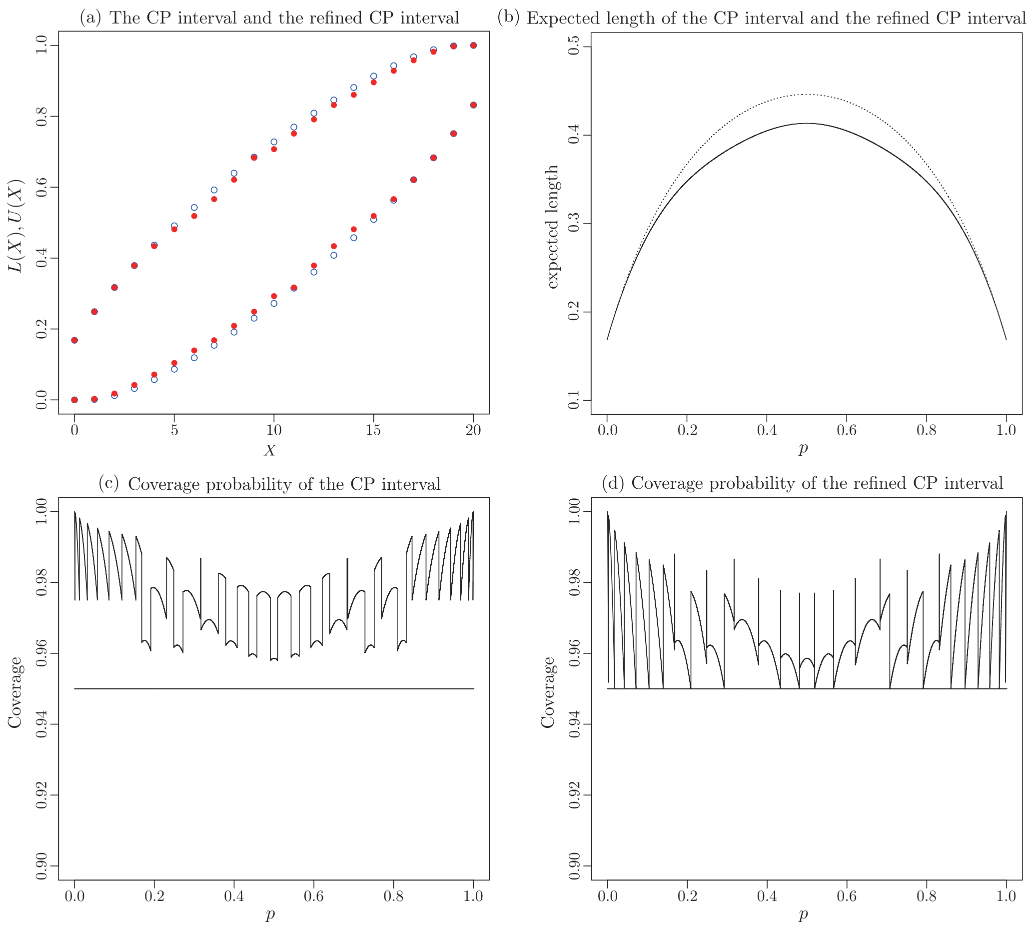

Figure 3. A comparison of the intervals C^{CP} and C^{I}, their coverage probabilities and expected lengths, when $n = 20$ and $1 - \alpha = 0.95$. In (a), C^{CP} (circle), C^{I} (solid circle); in (b), the expected lengths of C^{CP} (dot) and C^{I} (solid).

3. Interval Construction on a Discrete Sample Space

The proposed iterative interval construction can also be generalized to the case of a discrete sample space with a single parameter distribution family. In fact, the construction may be simpler than the case of a binomial distribution because condition (1.3) is not needed.

Suppose a discrete statistic X is observed with a cdf $F(x;\theta)$, for a parameter $\theta \in [A, B]$ with A finite and B finite or infinite. Let X take at most countable values $\{x_i\}_{i=1}^{n}$ with $x_i < x_{i+1}$. Let $p_G(x;\theta)$ be the pmf of X, and assume the distribution family $F(x;\theta)$ is stochastically nondecreasing in θ. This covers Poisson, geometric, and negative binomial distributions. We search for admissible intervals in a class $\mathcal{B}_G$ of $1-\alpha$ confidence intervals of form $C(X)=[L(X),U(X)]$ satisfying

$$L(x_i)\le L(x_{i+1}),\ \ U(x_i)\le U(x_{i+1}),\forall\ i. \tag{3.1}$$

We provide an interval construction in the order $X=x_1,x_2,\ldots,x_n$, though one can construct the interval in any predetermined order. Let $[L^G(X),U^G(X)]$ denote the interval to be constructed.

Step 1-G. Set $L^G(x_1)=A$ and solve for $U^G(x_1)$. Consider an interval $C_c^G(X)=[L_c^G(X),U_c^G(X)]$ with

$$[L_c^G(X),U_c^G(X)]=\begin{cases}[A,c] & \text{if } X=x_1,\\ [A,B] & \text{otherwise,}\end{cases} \tag{3.2}$$

for some constant $c\in[A,B]$. Take $c=c^G$ so that

$$c^G=\inf\ \ D_1^G,\ \text{ where } D_1^G=\{c:\ \ C_c^G(X)\in\mathcal{B}_G\}. \tag{3.3}$$

Then $[L^G(x_1),U^G(x_1)]=[A,c^*]$.

Step 2-G. Suppose $[L^G(x),U^G(x)]$ is determined up to $x<x_i$ for some $i\ge 2$. Consider an interval $C_{a,b,x_i}(X)=[L_{a,b}(X),U_{a,b}(X)]$ with

$$[L_{a,b}(x),U_{a,b}(X)]=\begin{cases}[L^G(x),U^G(x)] & \text{if } x<x_i,\\ [a,b] & \text{if } x=x_i,\\ [a,B] & \text{otherwise,}\end{cases} \tag{3.4}$$

for some a and b. Pick a_i^G and b_i^G so that

$$b_i^G-a_i^G=\inf_{(a,b)\in D_{2;G}^{x_i}}(b-a),\ \text{ where } D_{2;G}^{x_i}=\{(a,b):C_{a,b,x_i}(X)\in\mathcal{B}_G\}. \tag{3.5}$$

Then $[L^G(x_i),U^G(x_i)]=[a_i^G,b_i^G]$. By induction, $L^G(x_i)$ is assigned a value for all x_i since X is discrete, as well as $U^G(x_i)$. The construction is complete.

Lemma 4. *For any $1-\alpha>0$ and for any confidence interval $C(X)=[L(X),U(X)]\in\mathcal{B}_G$, $L(x_{i+1})\le U(x_i)$ for any consecutive values x_i and x_{i+1} of X.*

Theorem 3. *For $1-\alpha>0$, the interval $C^G(X)=[L^G(X),U^G(X)]$ generated by Steps 1-G and 2-G is in $\mathcal{B}_G$.*

Since we squeeze intervals one at a time, the interval $C^G(X)$ is admissible and is locally best at $X=x_1$. Each pair (a_i^G,b_i^G) is unique. The proofs here are similar to those in Section 2, and are omitted.

One can start interval construction at any point $x = x_j$ and follow any predetermined order on the x_i's modifying Steps 1-G and 2-G. The resultant intervals are admissible, and typically different. If so, the smallest confidence interval does not exist. If x_j is the first element in the permutation, then $C^G(x_j) \overset{denoted}{=} [L^G(x_j), U^G(x_j)]$ not only has the minimum length, but is a subset of $C(x_j)$ for any interval $C(X) \in \mathcal{B}_G$. Here

$$L^G(x_j) = \begin{cases} \sup\{\theta : F(x_{j-1};\theta) \geq 1-\alpha\} & \text{if } F(x_{j-1};A) \geq 1-\alpha, \\ A & \text{otherwise,} \end{cases} \tag{3.6}$$

$$U^G(x_j) = \begin{cases} \inf\{\theta : F(x_j;\theta) \leq \alpha\} & \text{if } F(x_j;B) \leq \alpha, \\ B & \text{otherwise.} \end{cases} \tag{3.7}$$

That $C^G(x_j) \subset C(x_j)$ follows from two facts:

(a) $L^G(x_j)$ and $U^G(x_j)$ are the largest lower confidence limit and the smallest upper confidence limit, respectively, among all $C(X)$'s in $\mathcal{B}_G$ at $X = x_j$. See Bol'shev (1965).

(b) The interval $[L_j(X), U_j(X)]$ belongs to $\mathcal{B}_G$, where

$$L_j(x) = AI_{[x_1,x_{j-1}]}(x) + L^G(x_j)I_{[x_j,x_n]}(x),$$
$$U_j(x) = U^G(x_j)I_{[x_1,x_j]}(x) + BI_{[x_{j+1},x_n]}(x),$$

for any x.

Example 4. For a Poisson random variable X with mean $\lambda \in [0, +\infty) = [A, B)$, estimate λ with 95% confidence level when $x = 2$.

If we use an interval constructed starting at $x = 2$, then

$$[L^G(2), U^G(2)] = [0.3553, 6.2957] \tag{3.8}$$

follows from (3.6) and (3.7), and no other interval values are needed.

If we use an interval, say $[L^G_0(X), U^G_0(X)]$, constructed in the order $x = 0, 1, 2, \ldots$, then the interval construction stops at $x = 2$.

To construct the interval at $x = 0$, following Step 1-G or (3.6) and (3.7), we obtain

$$[L^G_0(0), U^G_0(0)] = [0, 2.9958].$$

To construct the interval at $x = 1$, following Step 2-G, interval $C_{a,b,1}(X)$ has the form

x	0	1	2	$\cdots$	n	$\cdots$
$L_{a,b,1}(x)$	0	a	a	a	a	a
$U_{a,b,1}(x)$	2.9958	b	$+\infty$	$+\infty$	$+\infty$	$+\infty$

with unknowns $a \leq 2.9958 \leq b$. Compute the coverage probability function for this interval for $\lambda \geq 0$. If the smallest is at least 0.95, then record the values of a and b. Find the smallest $b-a$, and the corresponding a and b to be $L_0^G(1)$ and $U_0^G(1)$, respectively. Here $[L_0^G(1), U_0^G(1)] = [0.0512, 4.7439]$.

To construct the interval at $x = 2$, following (3.4) again, we obtain $[L_0^G(2), U_0^G(2)] = [0.3553, 6.2958]$, which happens to be the same as (3.8).

Besides obtaining the locally best intervals $C^G(X)$, we can also refine any given interval in $\mathcal{B}_G$. For example, the Clopper-Pearson type of interval for θ, can be improved in a way similar to that in Section 2.3. Again, the resultant confidence interval depends on the order of the interval construction.

4. Summary

In this article, an iterative method is provided to construct two-sided $1-\alpha$ confidence intervals for a single proportion p, and is extended to the case of a discrete sample space with a single parameter distribution. The resultant interval is unique and admissible under the criterion of set inclusion, and depends on the order of interval construction. This method can be used to generate the locally best intervals or to refine any given $1-\alpha$ interval to be admissible. Our numerical study shows that C^I proposed in Section 2.3 has the same accuracy (confidence level) and the same precision (average expected length) as the Blyth-Still-Casella interval, but is more stable and easier to implement. It is recommended for statistical practice.

Acknowledgements

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Appendix: Proofs

Proof of Lemma 1. It is trivial that D_1 is not empty if notice $c = 1$ belongs in D_1, so c^* is unique. To show $C_{c^*}(X) \in \mathcal{B}$, i.e., $c^* \in D_1$, let $\{c_i\}_{i=1}^{+\infty}$ be a nonincreasing sequence in D_1 with a limit c^*. Then for each fixed p and the indicator functions of the confidence intervals $C_{c_i}(X)$ and $C_{c^*}(X)$, we have

$$I_{C_{c^*}(X)}(p) = \lim_{i \to +\infty} I_{C_{c_i}(X)}(p)$$

for each value of X. Thus

$$Cover_{C_{c^*}}(p) = E_p[I_{C_{c^*}(X)}(p)] = \lim_{i \to +\infty} E_p[I_{C_{c_i}(X)}(p)] = \lim_{i \to +\infty} Cover_{C_{c_i}}(p) \geq 1-\alpha,$$

where the second equality is due to the Dominated Convergence Theorem. This implies $c^* \in D_1$.

Proof of Lemma 2. First consider the case of $x_0 = 1$.

For the existence of a^* and b^*, first note $C_{c^*}(X) \in \mathcal{B}$ due to Lemma 1 and $C_{0,1}(X) = C_{c^*}(X)$, then $(a,b) = (0,1) \in D_2$. Therefore, D_2 is not empty. Let $\{(a_i, b_i)\}_{i=1}^{+\infty} \subset D_2$ satisfying $\lim_{i\to+\infty}(b_i - a_i) = inf_{(a,b)\in D_2}(b-a)$. Since b_i and a_i are both bounded, there exists a subsequence $\{(a_{i_j}, b_{i_j})\}_{j=1}^{+\infty}$ so that both the limits of a_{i_j} and b_{i_j} exist and both of subsequences are monotone in j. Then let

$$b^* = \lim_{j\to+\infty} b_{i_j} \text{ and } a^* = \lim_{j\to+\infty} a_{i_j}.$$

W.l.o.g., assume,

$$b^* = \lim_{i\to+\infty} b_i \quad \text{and } a^* = \lim_{i\to+\infty} a_i.$$

To show $(a^*, b^*) \in D_2$, there are four possibilities for the monotonicity of a_i and b_i.

(1) a_i nondecreasing and b_i nonincreasing. Then

$$I_{C_{a^*,b^*}(X)}(p) = \lim_{i\to+\infty} I_{C_{a_i,b_i}(X)}(p)$$

for each value of X and each $p \in [0,1]$, and

$$Cover_{C_{a^*,b^*}}(p) = \lim_{i\to+\infty} E_p[I_{C_{a_i,b_i}(X)}(p)] = \lim_{i\to+\infty} Cover_{C_{a_i,b_i}}(p) \geq 1-\alpha,$$

where the first equality is due to the Dominated Convergence Theorem. This implies $(a^*, b^*) \in D_2$.

(2) a_i nondecreasing and b_i nondecreasing. For any $p \neq b^*$,

$$I_{C_{a^*,b^*}(X)}(p) = \lim_{i\to+\infty} I_{C_{a_i,b_i}(X)}(p)$$

for each value of X and each $p \in [0,1]$, and for $p = b^*$,

$$I_{C_{a^*,b^*}(X)}(p) \geq I_{C_{a_i,b_i}(X)}(p)$$

for each value of X, then

$$Cover_{C_{a^*,b^*}}(p) \geq \underline{\lim}_{i\to+\infty} Cover_{C_{a_i,b_i}}(p) \geq 1-\alpha,$$

This implies $(a^*, b^*) \in D_2$.

(3) a_i nonincreasing and b_i nonincreasing. The proof is similar to case (2).

(4) a_i nonincreasing and b_i increasing or a_i decreasing and b_i nondecreasing. This case is impossible because $b_i - a_i \geq b^* - a^*$.

As a summary, $(a^*, b^*) \in D_2$ for $x_0 = 1$. Since $(a^*, b^*) \in D_2$ for $x_0 = 1$, then D_2 is not empty for $x_0 = 2$, and the interval construction does not stop at $x_0 = 1$.

Secondly, for the case of $x_0 > 1$ but $x_0 < n/2$, the proof is similar to the case of $x_0 = 1$ and is omitted.

Proof of Lemma 3. Suppose the claim is not true. Let

$$x_s = min\{x_0 : (a^*, b^*) \text{ is not uniquely determined for the given } x_0\}.$$

Then x_s exists and $x_s > 0$ due to Lemma 1. Let (a_s^*, b_s^*) and (a'_s, b'_s) be two different pairs satisfying (2.4) for $x_0 = x_s$ and $a_s^* < a'_s$(then $b_s^* < b'_s$). Let $Cover_s^*(p)$ and $Cover'_s(p)$ be the coverage probability functions for intervals $C_{a_s^*, b_s^*}(X)$ and $C_{a'_s, b'_s}(X)$, respectively. So they both are no less than $1 - \alpha$ due to Lemma 2. There are two cases for $b_s^* - a_s^*$: $= 0$ or > 0.

If $b_s^* - a_s^* = 0$. Claim $U^*(x_s - 1) = b_s^*$. Suppose not, then $U^*(x_s - 1) < b_s^*$. For a $p_0 \in (U^*(x_s - 1), a_s^*)$, then $p_0 \notin C_{a_s^*, b_s^*}(x)$ for any $x \in S$. Therefore,

$$Cover_s^*(p_0) = 0,$$

a contradiction with the fact that $C_{a_s^*, b_s^*}(X)$ is of level $1 - \alpha$. Thus $U^*(x_s - 1) = b_s^*$. On the other hand, since $b'_s - a'_s = b_s^* - a_s^* = 0$, the same argument can be applied on $C_{a', b'}(X)$. Then $U^*(x_s - 1) = b'_s$ as well. Note $b_s^* < b'_s$, hence, $U^*(x_s - 1)$ is not uniquely determined, which contradicts with the definition of x_s. So $b_s^* - a_s^* > 0$.

For a number $a \in (a_s^*, min\{a', b_s^*\})$, consider a confidence interval $C_{a, b_s^*}(X)$. i.e.,

$$L_{a, b_s^*}(x) = \begin{cases} L^*(x) & \text{if } x < x_s, \\ a & \text{if } x_s \le x < n - x_s, \\ 1 - b_s^* & \text{if } x = n - x_s, \\ 1 - U^*(n - x) & \text{if } x > n - x_s. \end{cases} \tag{A.1}$$

It is clear that

$$C_{a_s^*, b_s^*}(x) = C_{a, b_s^*}(x) = C_{a'_s, b'_s}(x), \text{ for } x < x_s \text{ and } x > n - x_s. \tag{A.2}$$

We will show $C_{a, b_s^*}(X) \in \mathcal{B}$ below. Since $b_s^* - a < b_s^* - a_s^*$, this contradicts with the fact that $b_s^* - a_s^*$ is the infimum of D_2.

To show $C_{a, b_s^*}(X) \in \mathcal{B}$, we only need to proof $Cover_{C_{a, b_s^*}}(p) \ge 1 - \alpha$ for all $p \in [0, 1]$. For any $p \notin [a_s^*, a) \cup (1 - a, 1 - a_s^*]$, $Cover_{C_{a, b_s^*}}(p) = Cover_s^*(p) \ge 1 - \alpha$ because

$$\{x : p \in C_{a, b_s^*}(x)\} = \{x : p \in C_{a_s^*, b_s^*}(x)\}$$

due to the first equality of (A.2). For any $p \in [a_s^*, a) \cup (1-a, 1-a_s^*]$, since

$$\{x : p \in C_{a,b_s^*}(x)\} = \{x : p \in C_{a_s',b_s'}(x)\}$$

due to the second equality of (A.2), we have $Cover_{C_{a,b_s^*}}(p) = Cover_s'(p) \geq 1-\alpha$. Therefore, $C_{a,b_s^*}(X) \in \mathcal{B}$, and a contradiction is constructed.

Proof of Theorem 1. If n is an odd number, then the interval construction is actually complete in Step 2, and the theorem follows Lemmas 2 and 3.

If n is an even number, i.e., the interval construction on $[L^*(x), U^*(x)]$ for $x = [n/2]$ follows Step 3-2. The uniqueness of d^* defined in (2.6) is trivial, and the proof of $C_{d^*}(X)$ belonging to $\mathcal{B}$ is similar to Lemma 1.

Proof of Theorem 2. For any interval $C(X) = [L(X), U(X)] \in \mathcal{B}$ satisfying $C(x) \subset C^*(x)$ for all $x \in S$, suppose $C(X) \neq C^*(X)$. Let

$$x_0 = min\{x : C(x) \neq C^*(x)\}.$$

So $x_0 \leq n/2$, and

$$U(x_0) - L(x_0) < U^*(x_0) - L^*(x_0). \tag{A.3}$$

If n is an even number and $1 \leq x_0 < n/2$ or if n is an odd number and $1 \leq x_0$, let $C'(X) = [L'(X), U'(X)]$ with

$$L'(x) = \begin{cases} L(x) = L^*(x) & \text{if } x < x_0, \\ L(x_0) & \text{if } x_0 \leq x < n - x_0, \\ 1 - U(x_0) & \text{if } x = n - x_0, \\ 1 - U(n-x) = 1 - U^*(n-x) & \text{if } x > n - x_0, \end{cases} \tag{A.4}$$

which always contains $C(X)$. Therefore, $C'(X) = C_{a,b}(X)$ with $a = L(x_0)$ and $b = U(x_0)$, and this pair (a, b) belongs to D_2 given in (2.4) for $x = x_0$. Hence $U(x_0) - L(x_0) \geq U^*(x_0) - L^*(x_0)$ due to (2.4), which contradicts with (A.3).

If $x_0 = 0$ or if n is an even number and $x_0 = n/2$, we introduce an interval $C'(X)$ as in (2.1) or (2.5), respectively, and construct a contradiction for (A.3), and the proof is similar to the previous case and omitted.

Proof of Lemma 4. Suppose the claim is not true, i.e., there exist two consecutive points x_{i_0} and x_{i_0+1} with $L(x_{i_0+1}) > U(x_{i_0})$. Consider the coverage probability of this interval at any value $\theta_0 \in (U(x_{i_0}), L(x_{i_0+1}))$. Since $U(x) \leq U(x_{i_0})$ for all $x \leq x_{i_0}$ and $L(x) \geq L(x_{i_0+1})$ for any $x \geq x_{i_0+1}$, θ_0 does not belongs to $C(X)$ for any value of X. Thus

$$Cover_C(\theta_0) = E_{\theta_0} I_{C(X)}(\theta_0) = 0,$$

a contradiction with $C(X)$ of being level $1-\alpha$.

Proof of Theorem 3. If $C_{a_i^G,b_i^G,x_i}(X)\in\mathcal{B}_G$ for any i, note, for each $\theta\in[A,B]$, the indicator function $I_{C_{a_i^G,b_i^G,x_i}}(\theta)$ is nonincreasing and converging to $I_{C^G}(\theta)$ as i increases to n(it may be $+\infty$), then,

$$Cover_{C^G}(\theta)=\lim_{i\to n}Cover_{C_{a_i^G,b_i^G,x_i}}(\theta)\geq 1-\alpha,$$

where the first equality is due to Fatou's Lemma. The theorem is established.

Now we prove $C_{a_i^G,b_i^G,x_i}(X)\in\mathcal{B}_G$ for any i.

If $[A,B]$ is finite, then the proof of $C_{a_i^G,b_i^G,x_i}(X)\in\mathcal{B}_G$ is the same as that of Lemmas 1 and 2. So we focus on the case of $B=+\infty$.

Let $\{c_j\}_{j=1}^{+\infty}$ be a nonincreasing sequence in D_1^G with a limit c^G. Then, for each fixed θ, we have

$$I_{C_{c^G;G}(x)}(\theta)=\lim_{j\to+\infty}I_{C_{c_j;G}(x)}(\theta)$$

for each value x of X. Thus

$$Cover_{C_{c^G;G}}(\theta)=\lim_{j\to+\infty}E_\theta[I_{C_{c_j;G}(X)}(\theta)]=\lim_{j\to+\infty}Cover_{C_{c_j;G}}(\theta)\geq 1-\alpha,$$

where the first equality is due to the Dominated Convergence Theorem. This implies $c^G\in D_1^G$.

So far we have shown $C_{c^G;G}(X)\in\mathcal{B}_G$. Therefore, $D_{2;G}^{x_2}$ is not empty because $C_{A,B,x_2}(X)$ belongs to $\mathcal{B}_G$, which is due to the facts that $C_{A,B,x_2}(X)=C_{c^G;G}(X)$ and $C_{c^G;G}(X)\in\mathcal{B}_G$.

If $\inf_{(a,b)\in D_{2;G}^{x_2}}(b-a)=+\infty$, then $b_2^G=+\infty=B$, $D_{2;G}^{x_2}=\{(a,+\infty):a\geq A, C_{a,B,x_2}(X)\in\mathcal{B}_G\}$ and $a_2^G=sup\{a\geq A:C_{a,B,x_2}(X)\in\mathcal{B}_G\}$. Repeat the proof in the paragraph before the last paragraph on a nondecreasing sequence $\{a_j\}_{j=1}^{+\infty}$ in $D_{2;G}^{x_2}$ having a limit a_2^G, and conclude $(a_2^G,b_2^G)=(a_2^*,+\infty)\in D_{2;G}^{x_2}$.

If $\inf_{(a,b)\in D_{2;G}^{x_2}}(b-a)<+\infty$, as in Lemma 2, let $\{a_j\}_{j=1}^{+\infty}$ and $\{b_j\}_{j=1}^{+\infty}$ be the monotone sequences so that $(a_j,b_j)\in D_{2;G}^{x_2}$ and $\lim_{j\to+\infty}(b_j-a_j)=\inf_{(a,b)\in D_{2;G}^{x_2}}(b-a)$. By Lemma 4, $\{a_j\}_{j=1}^{+\infty}$ is bounded by c^G, then $\{b_j\}_{j=1}^{+\infty}$ is also bounded. Repeat the proof of Lemma 2 and conclude $(a_2^G,b_2^G)\in D_{2;G}^{x_2}$, $a_2^G=\lim_{j\to+\infty}a_j$ and $b_2^G=\lim_{j\to+\infty}b_j$.

Therefore, we have shown that $C_{a_2^G,b_2^G,x_2}(X)\in\mathcal{B}_G$.

Repeat the last four paragraphs, we can show that $C_{a_i^G,b_i^G,x_i}(X)\in\mathcal{B}_G$ by induction on i. Therefore, $C_{a_i^G,b_i^G,x_i}(X)\in\mathcal{B}_G$ for any $i\geq 2$. The proof is complete.

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