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A NOTE ON THE COVARIANCE STRUCTURE OF A CONTINUOUS-TIME ARMA PROCESS

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Abstract: We have derived some matrix equations for speedy computation of the conditional covariance kernel of a discrete-time process obtained from irregularly sampling an underlying continuous-time ARMA process. These results are applicable to both stationary and non-stationary ARMA processes. We have also demonstrated that these matrix results can be useful in shedding new insights on the covariance structure of a continuous-time ARMA process.

Key words and phrases: Irregularly sampled data, Kalman filter, stochastic differential equations.

1. Introduction

Owing to the sampling procedure or the presence of missing data, many time series data, say $\{Y_{t_i}\}_{i=0,\dots,N}$, are sampled with unequal time intervals. In many cases, the data are obtained from irregularly sampling an underlying continuous-time process. That is, there exists a continuous-time process $\{X_t, t \in \mathbf{R}\}$ such that $Y_{t_i} = X_{t_i}$. More generally, Y_{t_i} can be some functional of the underlying continuous-time process, measured perhaps with observation error. The underlying continuous-time process is often modeled by some linear stochastic differential equations, for example, by a continuous-time autoregressive moving average (CARMA) model. This linear specification results in a tractable likelihood for the observed discrete-time data. Hence this method has been routinely used in analyzing discrete-time sampled time series. See, e.g., Harvey (1989), Bergstrom (1990), Tong (1990) and Jones (1981, 1993). In some cases, there need not be any underlying continuous-time process, and its existence is merely a means to provide a convenient but useful analysis. In other cases, the continuous-time process may be the object of the study; see Bergstrom (1990).

This note is mainly concerned with the derivation of matrix equations for computing the conditional covariance kernel arising from irregularly sampling an underlying continuous-time ARMA process. It should be pointed out that these results apply to both stationary and non-stationary processes. Under suitable regularity conditions, Shoji and Ozaki (1998) derived matrix equations which the conditional covariance kernel has to satisfy. However, for the case of irregularly sampling, their method requires solving as many distinct matrix equations as there are distinct sampling intervals. Here, we obtain the main result by showing that all the conditional covariance matrices are simple transformations of the solution of a matrix equation, thereby speeding up the computations needed in the statistical inference of a continuous-time ARMA process based on irregularly sampled discrete-time data. Also, we simplify the matrix equations so that they are more readily solvable. Furthermore, the results derived here may be helpful in studying the covariance structure of a stationary or non-stationary CARMA process.

The organization of this note is as follows. The CARMA processes are briefly reviewed and the main results are stated in Section 2. In Section 3, we specialize the main results to the case of asymptotically stationary CARMA processes. In particular, we obtain a new method of moment estimator of the instantaneous variance parameter of a CARMA model. We also discuss some new insights on the covariance structure of the derivatives of the underlying CARMA process. Section 4 contains some conclusions.

2. Main Results

We recall the definition of linear CARMA(p,q) processes (with $0 \le q < p$). See, e.g., Brockwell (1993) and Brockwell and Stramer (1995) for further discussions. A CARMA(p,q) process is defined as the solution of the pth order differential equation:

$$Y_t^{(p)} - \alpha_p Y_t^{(p-1)} - \cdots - \alpha_1 Y_t - \alpha_0 = \sigma[W_t^{(1)} + \beta_1 W_t^{(2)} + \cdots + \beta_q W_t^{(q+1)}], \quad (1)$$

where the superscript $^{(j)}$ denotes j-fold differentiation with respect to t; $\{W_t, t \ge 0\}$ is the standard Brownian motion, and $\alpha_0, \ldots, \alpha_p$, $\beta_1, \ldots, \beta_q$ and σ are constants. We assume that $\sigma > 0$, $\alpha_1 \neq 0$ and $\beta_q \neq 0$ and define $\beta_j := 0$ for $j > q$. The derivatives $W_t^{(j)}$, $j > 0$ do not exist in the usual sense; hence we interpret (1) as being equivalent to the *observation* and *state* equations:

$$\begin{aligned} Y_t &= \boldsymbol{\beta}' \boldsymbol{X}_t, \qquad t \ge 0, \\ d\boldsymbol{X}_t &= (A\boldsymbol{X}_t dt + \alpha_0 \boldsymbol{l})dt + \sigma \boldsymbol{l} dW_t, \end{aligned} \quad (2)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_p \end{bmatrix}, \quad \boldsymbol{X}_t = \begin{bmatrix} X_t \\ X_t^{(1)} \\ \vdots \\ X_t^{(p-2)} \\ X_t^{(p-1)} \end{bmatrix}, \quad \boldsymbol{l} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \boldsymbol{\beta} = \begin{bmatrix} 1 \\ \beta_1 \\ \vdots \\ \beta_{p-2} \\ \beta_{p-1} \end{bmatrix},$$

and the superscript $'$ denotes the transpose of a vector. Equation (2) is an Ito differential equation for the state vector $\boldsymbol{X}_t$. We assume that $\boldsymbol{X}_0$ is independent of $\{W_t, t \geq 0\}$ and $\boldsymbol{X}_0$ is determined by initial conditions that could be random or deterministic. In case $\beta_j = 0$, $j \geq 1$, the state vector $\boldsymbol{X}_t$ becomes the vector of derivatives of the continuous-time AR(p) process $\{Y_t\}$.

The process $\{Y_t, t \geq 0\}$ is said to be a CARMA(p, q) process with parameter $\boldsymbol{\theta} = (\alpha_0, \ldots, \alpha_p, \beta_1, \ldots, \beta_q, \sigma)$ if $Y_t = \boldsymbol{\beta}' \boldsymbol{X}_t$, where $\{\boldsymbol{X}_t\}$ is a solution of (2). The solution of (2) can be written as

$$\boldsymbol{X}_t = e^{At}\boldsymbol{X}_0 + \alpha_0 \int_0^t e^{A(t-u)}\boldsymbol{l}du + \sigma \int_0^t e^{A(t-u)}\boldsymbol{l}dW_u,$$

where $e^{At} = I + \sum_{n=1}^{\infty}[(At)^n(n!)^{-1}]$, and I is the identity matrix.

Let the mean vector of $\{\boldsymbol{X}_t\}$ be denoted by $\boldsymbol{m}_t$. It satisfies the equation:

$$\boldsymbol{m}_t = \frac{\alpha_0}{\alpha_1}(e^{At} - I)H + e^{At}\boldsymbol{m}_0,$$

where $H = [1, 0, \ldots, 0]'$. The covariance kernel of $\{\boldsymbol{X}_t\}$, denoted by $\gamma_{s,t}$, is

$$\begin{aligned}\gamma_{s,t} &= E[(\boldsymbol{X}_s - \boldsymbol{m}_s)(\boldsymbol{X}_t - \boldsymbol{m}_t)'] \\ &= e^{As}V_0e^{A't} + \sigma^2 \int_0^{t\wedge s} e^{A(s-u)}\boldsymbol{l}\boldsymbol{l}'e^{A'(t-u)}du \\ &= \begin{cases} e^{A(s-t)}V_t, \ 0 \leq t \leq s < \infty, \\ V_s e^{A'(t-s)}, 0 \leq s \leq t < \infty, \end{cases}\end{aligned}$$

where $t \wedge s = \min(t, s)$ and

$$\begin{aligned}V_t &= \gamma_{t,t} \\ &= e^{At}V_0e^{A't} + \sigma^2 \int_0^t e^{A(t-u)}\boldsymbol{l}\boldsymbol{l}'e^{A'(t-u)}du.\end{aligned}$$

It follows from the above equations on $\boldsymbol{m}_t$ and V_t that the states and the observations, $\boldsymbol{X}_{t_i}$ and Y_{t_i}, at the sampling times $t_0, t_1, \ldots$, satisfy the discrete-time state and observation equations:

$$\begin{aligned}\boldsymbol{X}_{t_{i+1}} &= \boldsymbol{m}_{t_{i+1}} + e^{A(t_{i+1}-t_i)}(\boldsymbol{X}_{t_i} - \boldsymbol{m}_{t_i}) + \boldsymbol{Z}_{t_i}, \ \ i = 0, 1, \ldots, \\ Y_{t_i} &= \boldsymbol{\beta}'\boldsymbol{X}_{t_i}, \qquad\qquad\qquad\qquad\qquad\qquad i = 0, 1, \ldots,\end{aligned} \tag{3}$$

where $\boldsymbol{Z}_{t_i}$ is independent of $\boldsymbol{X}_{t_i}$, and $\{\boldsymbol{Z}_{t_i}, i = 0, 1, \ldots\}$ is an independent sequence of Gaussian random vectors with zero mean and covariance matrices

$$\Sigma_i = E(\boldsymbol{Z}_{t_i}\boldsymbol{Z}_{t_i}') = \sigma^2 \int_{t_i}^{t_{i+1}} e^{A(t_{i+1}-u)}\boldsymbol{l}\boldsymbol{l}'e^{A'(t_{i+1}-u)}du. \tag{4}$$

These equations are needed for applications of the Kalman recursions (see, e.g., Chapter 12 of Brockwell and Davis (1991)). From these recursions we can easily compute $\hat{Y}_{t_i|t_{i-1}} = E(Y_{t_i}|y_{t_j}, j \le i-1)$, and $p_{t_i|t_{i-1}} = \mathrm{var}(Y_{t_i}|y_{t_j}, j \le i-1)$, $i \ge 1$, which are handy for computing minus twice the log-likelihood function,

$$-2l = \sum_{i=0}^{N} \left[\frac{(Y_{t_i} - \hat{Y}_{t_i|t_{i-1}})^2}{p_{t_i|t_{i-1}}} + \log p_{t_i|t_{i-1}} \right] + (N+1)\log(2\pi).$$

If $\{\boldsymbol{X}_t, t \ge 0\}$ is stationary, the initial conditions can be set as $\hat{Y}_{t_0|t_{-1}} = -\alpha_0/\alpha_1$ and $p_{t_0|t_{-1}} = \boldsymbol{\beta}' V \boldsymbol{\beta}$, where $V = \sigma^2 \int_0^\infty e^{Au} \boldsymbol{l}\boldsymbol{l}' e^{A'u} du$ is the stationary variance. Otherwise, we can start the recursions with some diffuse initial conditions.

A non-linear optimization algorithm can be used in conjunction with the expression for $-2l$ to find the maximum likelihood estimator of the parameter $\boldsymbol{\theta}$. The computation of e^{At} is most readily performed by first block-diagonalizing A and then applying a Padé approximation on each block. See, e.g., Ward (1977).

The observation equation (3) can be simplified as follows:

$$\boldsymbol{X}_{t_{i+1}} = \boldsymbol{m} + e^{A(t_{i+1}-t_i)}(\boldsymbol{X}_{t_i} - \boldsymbol{m}) + \boldsymbol{Z}_{t_i},$$

where $\boldsymbol{m} = -\frac{\alpha_0}{\alpha_1} H$. Note that if $\{\boldsymbol{X}_t, t \ge 0\}$ is asymptotically stationary, then $\boldsymbol{m}$ is the stationary mean of $\{\boldsymbol{X}_t, t \ge 0\}$. The covariance matrices Σ_i can be computed via at least two approaches. If A is diagonalizable, the integrand in (4) is a matrix whose elements are linear combinations of exponentials, and hence the integral admits a closed-form solution. More generally, the integral can be computed via a Jordan canonical decomposition of A; see Doob (1944) and Jones (1981) for related discussions. Alternatively, Shoji and Ozaki (1998) applied integration by parts to the right hand side of (4) to get

$$A\Sigma_i + \Sigma_i A' = \sigma^2 (e^{A(t_{i+1}-t_i)} \boldsymbol{l}\boldsymbol{l}' e^{A'(t_{i+1}-t_i)} - \boldsymbol{l}\boldsymbol{l}'). \tag{5}$$

Shoji and Ozaki (1998) showed that if A has no pair of reverse-sign eigenvalues, that is, for each eigenvalue λ of A, $-\lambda$ is not an eigenvalue of A, then there is a unique solution to the matrix equation (5), in which case Σ_i can be obtained by solving (5).

In practice, (5) may have to be solved by some sparse matrix equation techniques for the higher order case. For irregularly sampled data, (5) may need to be solved as many times as there are distinct sampling intervals. The following theorem shows that all the solutions for different sampling intervals are related to a solution of an auxiliary matrix equation, which simplifies solving (5). Moreover, part (b) of the theorem below further reduces the dimension of the matrix equation. A new necessary and sufficient condition for the existence of a unique solution to (5) is also given.

Theorem 2.1.

(a) *If there exists a unique solution* Σ_i *to* (5)*, then there exists a unique solution* V *to the linear equation*

$$AV + VA' = -\sigma^2 \boldsymbol{l}\boldsymbol{l}', \tag{6}$$

and Σ_i *is given by*

$$\Sigma_i = V - e^{A(t_{i+1}-t_i)} V e^{A'(t_{i+1}-t_i)}. \tag{7}$$

(b) *If* V *is the unique solution of* (6)*, then* $V^* = [V_{1,1}, \ldots, V_{p,p}]'$*, the vector of diagonal elements of* V*, is the unique solution to the linear equation*

$$BV^* = -\frac{\sigma^2}{2}\boldsymbol{l}, \tag{8}$$

where $B = [B_{i,j}]_{p\times p}$ *is a* $p \times p$ *matrix with*

$$B_{i,j} = \begin{cases} (-1)^{j-i}\, \alpha_{2j-i}, & 1 \le 2j - i \le p, \\ (-1)^{j-i-1}, & 2j - i = p + 1, \\ 0, & \text{otherwise}. \end{cases} \tag{9}$$

The off-diagonal elements of V *are given as follows, for* $i \neq j$*:*

$$V_{i,j} = \begin{cases} (-1)^{\frac{i-j}{2}} V_{\frac{i+j}{2},\frac{i+j}{2}}, & \text{if } i + j \text{ is even}, \\ 0, & \text{if } i + j \text{ is odd}. \end{cases} \tag{10}$$

(c) *The matrix* B *defined in* (9) *is nonsingular if and only if there exists a unique solution* Σ_i *to* (5)*, i.e.,* $A\Sigma_i + \Sigma_i A' = \sigma^2(e^{A(t_{i+1}-t_i)}\boldsymbol{l}\boldsymbol{l}' e^{A'(t_{i+1}-t_i)} - \boldsymbol{l}\boldsymbol{l}')$.

Proof. (a) Let A, B, C and D be any $p \times p$ matrices and write $A = [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_p]$. Define $\text{vec}(A) = [\boldsymbol{a}_1', \ldots, \boldsymbol{a}_p']'$ and the Kronecker product $A \otimes B$, which is of order $p^2 \times p^2$, as $A \otimes B = [a_{ij}B]$. The following identities (see, e.g., Chapter 7 of Schott (1997) and Chapter 3 of Hale (1969)) will be needed below:

(i) $\text{vec}(ABC) = (C' \otimes A)\text{vec}(B)$,
(ii) $(A \otimes B)(C \otimes D) = AC \otimes BD$,
(iii) $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$,
(iv) $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$,
(v) $Ae^{At} = e^{At}A$ for all t.

Take the vec operator on both sides of (5) and (6) to get

$$[(I\otimes A)+(A\otimes I)]\mathrm{vec}(\Sigma_i)=\sigma^2\mathrm{vec}(e^{A(t_{i+1}-t_i)}\boldsymbol{l}\boldsymbol{l}'e^{A'(t_{i+1}-t_i)}-\boldsymbol{l}\boldsymbol{l}'),\qquad(11)$$

$$[(I\otimes A)+(A\otimes I)]\mathrm{vec}(V)=-\sigma^2\mathrm{vec}(\boldsymbol{l}\boldsymbol{l}').\qquad(12)$$

From (11) and (12), we see that the existence of a unique solution to (11) implies the nonsingularity of the matrix $[(I\otimes A)+(A\otimes I)]$, and hence the existence of a unique solution to equation (12). Now, subtract (12) from (11) to find

$$\begin{aligned}
&[(I\otimes A)+(A\otimes I)]\mathrm{vec}(\Sigma_i-V)\\
&=\sigma^2\mathrm{vec}(e^{A(t_{i+1}-t_i)}\boldsymbol{l}\boldsymbol{l}'e^{A'(t_{i+1}-t_i)})\\
&=\sigma^2(e^{A(t_{i+1}-t_i)}\otimes e^{A(t_{i+1}-t_i)})\mathrm{vec}(\boldsymbol{l}\boldsymbol{l}')\\
&=-(e^{A(t_{i+1}-t_i)}\otimes e^{A(t_{i+1}-t_i)})[(I\otimes A)+(A\otimes I)]\mathrm{vec}(V)\\
&=-\left[(e^{A(t_{i+1}-t_i)}\otimes e^{A(t_{i+1}-t_i)}A)+(e^{A(t_{i+1}-t_i)}A\otimes e^{A(t_{i+1}-t_i)})\right]\mathrm{vec}(V)\\
&=-\left[(e^{A(t_{i+1}-t_i)}\otimes Ae^{A(t_{i+1}-t_i)})+(Ae^{A(t_{i+1}-t_i)}\otimes e^{A(t_{i+1}-t_i)})\right]\mathrm{vec}(V)\\
&=-[(I\otimes A)+(A\otimes I)](e^{A(t_{i+1}-t_i)}\otimes e^{A(t_{i+1}-t_i)})\mathrm{vec}(V)\\
&=-[(I\otimes A)+(A\otimes I)]\mathrm{vec}(e^{A(t_{i+1}-t_i)}Ve^{A'(t_{i+1}-t_i)}).
\end{aligned}$$

By the nonsingularity of $[(I\otimes A)+(A\otimes I)]$, we have $\Sigma_i=V-e^{A(t_{i+1}-t_i)}Ve^{A'(t_{i+1}-t_i)}$.

(b) From (6), let $\Delta=AV+VA'+\sigma^2\boldsymbol{l}\boldsymbol{l}'=0$. The (i,j)th element of Δ equals

$$\begin{aligned}
\Delta_{i,j}&=(AV+VA')_{i,j}+\sigma^2(\boldsymbol{l}\boldsymbol{l}')_{i,j}\\
&=\sum_{k=1}^{p}(A_{i,k}V_{k,j}+V_{i,k}A'_{k,j})+\sigma^2(\boldsymbol{l}\boldsymbol{l}')_{i,j}\\
&=\begin{cases}V_{i+1,j}+V_{i,j+1}, & i\le j<p,\\ V_{i+1,p}+\sum_{k=1}^{p}V_{i,k}\alpha_k, & i<j=p,\\ \sum_{k=1}^{p}(\alpha_kV_{k,p}+V_{p,k}\alpha_k)+\sigma^2, & i=j=p.\end{cases}
\end{aligned}$$

Thus $\Delta=0$ is equivalent to the following system of equations:

$$V_{i+1,j}+V_{i,j+1}=0,\ i\le j<p,\qquad(13)$$

$$V_{i+1,p}+\sum_{k=1}^{p}V_{i,k}\alpha_k=0,\ i<j=p,\qquad(14)$$

$$\sum_{k=1}^{p}(\alpha_kV_{k,p}+V_{p,k}\alpha_k)+\sigma^2=0,\ i=j=p.\qquad(15)$$

From (13) we see that $V_{i,i+1} = -V_{i+1,i}$, for $1 \le i \le p-1$. Hence $V_{i,i+1} = V_{i+1,i} = 0$ owing to the symmetry of V. Below, $[x]$ denotes the largest integer smaller than or equal to x. For $j > 0$,

$$\begin{aligned} V_{i,i+j} &= (-1)V_{i+1,i+j-1} \\ &\;\;\vdots \\ &= (-1)^{[\frac{j}{2}]} V_{i+[\frac{j}{2}],i+j-[\frac{j}{2}]} \\ &= \begin{cases} (-1)^{\frac{j}{2}} V_{i+\frac{j}{2},i+\frac{j}{2}}, & if\ j\ is\ even, \\ 0, & if\ j\ is\ odd. \end{cases} \end{aligned}$$

Equivalently, for $i, j \ge 1$,

$$V_{i,j} = \begin{cases} (-1)^{\frac{i-j}{2}} V_{\frac{i+j}{2},\frac{i+j}{2}}, & if\ i+j\ is\ even, \\ 0, & if\ i+j\ is\ odd. \end{cases}$$

For $i \le p-1$,

$$\begin{aligned} & V_{i+1,p} + \sum_{k=1}^{p} V_{i,k}\alpha_k \\ &= (-1)^{\frac{p-i-1}{2}} V_{\frac{p+i+1}{2},\frac{p+i+1}{2}} \mathbf{1}_{\{p+i+1 \text{ is even}\}} + \sum_{\{r:1\le i+2r\le p\}} V_{i,i+2r}\ \alpha_{i+2r} \\ &= (-1)^{\frac{p-i-1}{2}} V_{j,j} \mathbf{1}_{\{2j-i=p+1\}} + \sum_{\{r:1\le i+2r\le p\}} (-1)^r V_{i+r,i+r}\ \alpha_{i+2r} \\ &= (-1)^{j-i-1} \mathbf{1}_{\{2j-i=p+1\}} V_{j,j} + \sum_{\{j:1\le 2j-i\le p\}} (-1)^{j-i}\alpha_{2j-i} V_{j,j}, \end{aligned}$$

where $\mathbf{1}_A$ is the indicator function of A. So, (14) is equivalent to

$$(-1)^{j-i-1} \mathbf{1}_{\{2j-i=p+1\}} V_{j,j} + \sum_{\{j:1\le 2j-i\le p\}} (-1)^{j-i}\alpha_{2j-i} V_{j,j} = 0. \tag{16}$$

On the other hand,

$$\begin{aligned} \sum_{k=1}^{p} V_{p,k}\alpha_k &= \sum_{\{r:1\le p-2r\le p\}} V_{p,p-2r}\ \alpha_{p-2r} \\ &= \sum_{\{r:1\le p-2r\le p\}} (-1)^r V_{p-r,p-r}\ \alpha_{p-2r} \\ &= \sum_{\{j:1\le 2j-p\le p\}} (-1)^{p-j} \alpha_{2j-p} V_{j,j}. \end{aligned}$$

Thus, (15) is equivalent to

$$\sum_{\{j:1\leq 2j-p\leq p\}} (-1)^{p-j}\alpha_{2j-p}V_{j,j} = -\frac{\sigma^2}{2}. \tag{17}$$

This completes the proof of (b) because the combination of (16) and (17) is equivalent to (8).

(c) The existence of a unique solution to (5) is equivalent to the existence of a unique solution to (6). However, (6) is equivalent to (8) and (10). This completes the proof.

The expressions for the matrix B, for $p = 1, \ldots, 5$, are illustrated as follows:

$$p=1: \begin{bmatrix}\alpha_1\end{bmatrix},\ p=2: \begin{bmatrix}\alpha_1 & 1\\ 0 & \alpha_2\end{bmatrix},\ p=3: \begin{bmatrix}\alpha_1 & -\alpha_3 & 0\\ 0 & \alpha_2 & 1\\ 0 & -\alpha_1 & \alpha_3\end{bmatrix},$$

$$p=4: \begin{bmatrix}\alpha_1 & -\alpha_3 & -1 & 0\\ 0 & \alpha_2 & -\alpha_4 & 0\\ 0 & -\alpha_1 & \alpha_3 & 1\\ 0 & 0 & -\alpha_2 & \alpha_4\end{bmatrix},\ p=5: \begin{bmatrix}\alpha_1 & -\alpha_3 & \alpha_5 & 0 & 0\\ 0 & \alpha_2 & -\alpha_4 & -1 & 0\\ 0 & -\alpha_1 & \alpha_3 & -\alpha_5 & 0\\ 0 & 0 & -\alpha_2 & \alpha_4 & 1\\ 0 & 0 & \alpha_1 & -\alpha_3 & \alpha_5\end{bmatrix}.$$

Also note that the matrix B is nonsingular if and only if its determinant equals zero, which happens only if the parameter vector lies in a set of zero Lebesgue measure.

3. Asymptotically Stationary CARMA Processes

Under the condition that all the eigenvalues of A have negative real parts, $\{\boldsymbol{X}_t\}$ can be shown to be asymptotically stationary. See, e.g., Chapter 5 of Karatzas and Shreve (1991). In the stationary case, the mean vector and the covariance kernel are given by the following formulas:

$$\boldsymbol{m} = -\frac{\alpha_0}{\alpha_1}H,$$
$$\gamma_{s,t} = \begin{cases} e^{A(s-t)}V, & 0\leq t\leq s<\infty,\\ Ve^{A'(t-s)}, & 0\leq s\leq t<\infty,\end{cases}$$

where $V = \sigma^2\int_0^\infty e^{Au}\boldsymbol{l}\boldsymbol{l}'e^{A'u}du$ is the stationary variance matrix of $\{\boldsymbol{X}_t\}$ and is the solution to the linear matrix equation $AV+VA' = -\sigma^2\boldsymbol{l}\boldsymbol{l}'$ (see (6.20) on page 357 of Karatzas and Shreve (1991)). This can be readily solved via part (b) of Theorem 2.1. Notice that if $\{\boldsymbol{X}_t, t\geq 0\}$ is asymptotically stationary, Theorem

2.1 (b) can be used to get a method of moments estimator of σ^2 from the $\alpha_j^{'}$s and the variance of X_t. This is because $V^* = -\sigma^2 B^{-1}\boldsymbol{l}/2$, and so $V_{1,1}$, the variance of X_t, is equal to $-\sigma^2 b^{1p}/2$, where $B^{-1} = [b^{ij}]$. It is also interesting to note from (10) that, for a stationary CARMA model and given t, the ith and jth derivatives of $\boldsymbol{X}_t$ are uncorrelated when i and j differ by an odd number. When i and j differ by an even number the correlation between $X_t^{(i)}$ and $X_t^{(j)}$ is the same as the variance of $X_t^{(i/2+j/2)}$ multiplied by the factor $(-1)^{i/2-j/2}$.

4. Conclusion

We have derived matrix equations for computing the conditional covariance kernel of a continuous-time ARMA process. While (7) appears intuitive for the stationary case, it is interesting that it continues to hold for the non-stationary case with a suitable V. The use of these results for the inference on non-stationary CARMA processes constitutes an interesting future research project.

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