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EXACT RUN LENGTH DISTRIBUTIONS FOR ONE-SIDED EXPONENTIAL CUSUM SCHEMES

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Abstract: Exact expressions are derived for the probability functions of run lengths of one-sided cumulative sum (CUSUM) schemes when observations are exponentially distributed. Average run lengths (ARLs), standard deviations of run lengths (SDRLs) and percentiles of run length distributions can then be obtained by recursively evaluating the probability functions. The run length distributions of CUSUM schemes are found to be highly skewed, and consequently conclusions based on ARL alone can be misleading. Knowledge of run length distributions would provide a comprehensive understanding of CUSUM schemes. A comparison of the performance of CUSUM schemes is presented, and general considerations in the design of CUSUM schemes are discussed.

Key words and phrases: Quality control, cumulative sum (CUSUM), average run length (ARL), percentiles, exponential distribution, Poisson distribution.

1. Introduction

The main objective of this paper is to provide a comprehensive understanding of the properties of run length distributions of one-sided cumulative sum (CUSUM) schemes when observations are exponentially distributed. Such CUSUM schemes are used, for example, to monitor the intensity of a homogeneous Poisson process as associated with the occurrence of rare events.

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed continuous random variables with a common probability density function $f(x)$. The lower-sided and upper-sided CUSUM schemes are defined as

$$T_0 = v, \quad T_t = \min\{0, T_{t-1} + (X_t - k)\}, \quad t = 1, 2, \ldots,$$

and

$$S_0 = u, \quad S_t = \max\{0, S_{t-1} + (X_t - k)\}, \quad t = 1, 2, \ldots,$$

respectively where k is a suitably chosen positive constant. Note that $-h < v \leq 0$ and $0 \leq u < h$ where h is the control chart limit. The lower-sided CUSUM scheme is intended to detect a downward shift in the mean and it signals an out-of-control alarm at the first t for which $T_t \leq -h$. Similarly, the upper-sided CUSUM scheme is intended to detect an upward shift in the mean and it signals an out-of-control alarm at the first t for which $S_t \geq h$.

If CUSUM schemes are used to monitor the rate of occurrences of defects for a homogeneous Poisson process, and if X_i corresponds to an exponential interarrival time, then an upward shift in the mean indicates an improvement in quality while a downward shift in the mean indicates a deterioration in the quality. A theoretical treatment of detection of failure rate increases is given by Lorden and Eisenberger (1973). Vardeman and Ray (1985) described an application of the exponential CUSUM scheme in controlling the intensity of a Poisson process. By repeatedly solving simple first-order differential equations, Vardeman and Ray obtained exact expressions for the ARLs of CUSUM schemes when the observations are exponentially distributed. Various methods have been used to approximate probability distributions for run lengths of CUSUM schemes. These include procedures developed by Brook and Evans (1972), Reynolds (1975), Khan (1978), Zacks (1981), Abel and Avenhaus (1984, comments on Zacks' paper), Woodall (1983, 1984) and Waldmann (1986). The cited papers did not approximate the run length distribution for exponential observations.

Exact expressions for the probability functions of run lengths have not been obtained before because of the complexity of the derivation. The simple form of the exponential probability distribution allows exact expressions for the run length distribution to be derived, and these are presented in Section 2. The recursive approach given in this paper is related to the differential equation approach used by Vardeman and Ray (1985). Integral equations for the probability functions of run lengths are solved instead of differential equations. A comparison of the performance of CUSUM schemes based on ARLs and percentiles is given in Section 3. A CUSUM scheme is found to be more effective in detecting small shifts in the mean when k is close to the target mean. As the absolute difference between k and the target mean increases, a CUSUM scheme becomes less effective in detecting small shifts in the mean but becomes more effective in detecting large shifts in the mean. General considerations in the design of CUSUM schemes are considered in Section 4. Design procedures developed by Page (1961), Bowker and Lieberman (1972), and Woodall (1985) for normal observations can be adapted here for the design of CUSUM schemes for exponential observations. The main advantage of knowing the probability function of run length is to enable a quality control engineer to use a CUSUM scheme with confidence.

2. Probability Functions of Run Lengths of CUSUM Schemes

We shall restrict attention in this article to the case in which X_1 is exponentially distributed with mean β, that is, where

$$f(x) = \begin{cases} \beta^{-1}\exp(-x/\beta), & x \geq 0, \\ 0, & x < 0. \end{cases} \tag{2.1}$$

If Y is the number of events occurring in a given unit of time and $1/\beta$ is the rate of occurrence, then the time between occurrences of two events is an exponential random variable with mean β. Define N as the index at which an out-of-control alarm is first given by a CUSUM scheme and $\Pr(n,u)$ as the probability that $N = n$ given that the initial value is u. The probability function $\Pr(n,u)$ of a CUSUM scheme with chart parameters h and k when X_1 is exponentially distributed with mean β is the same as the probability function $\Pr(n,u/\beta)$ of a CUSUM scheme with parameters h/β and k/β when X_1 is exponentially distributed with mean 1. Therefore, we shall, without loss of generality, consider the case in which $\beta = 1$. Recursive exact expressions for $\Pr(n,u)$ are derived in this section.

For $n \geq 2$, the basic recurrent relationships for run length probabilities of a lower-sided CUSUM scheme are

$$\Pr(n,u) = \Pr(n-1,0)\Pr(X_1 \geq k-u) + \int_{-h}^{0} \Pr(n-1,x)f(x+k-u)dx, \tag{2.2}$$

for $-h < u \leq -h+k$, and

$$\Pr(n,u) = \Pr(n-1,0)\Pr(X_1 \geq k-u) + \int_{u-k}^{0} \Pr(n-1,x)f(x+k-u)dx, \tag{2.3}$$

for $-h+k < u \leq 0$. Similarly, recurrent relationships for run length probabilities of an upper-sided CUSUM scheme are

$$\Pr(n,u) = \Pr(n-1,0)\Pr(X_1 \leq k-u) + \int_{0}^{h} \Pr(n-1,x)f(x+k-u)dx, \tag{2.4}$$

for $0 \leq u < k$, and

$$\Pr(n,u) = \int_{u-k}^{h} \Pr(n-1,x)f(x+k-u)dx, \tag{2.5}$$

for $k \leq u < h$. Recursive equations similar to equations (2.2)–(2.5) for a general process were first derived by Ewan and Kemp (1960).

We shall now derive exact expressions for $\Pr(n,u)$, $n = 1,2,\ldots$ for lower-sided CUSUM schemes. Let L denote the smallest integer such that $h \leq Lk$. The derivation of exact expressions for run length distributions will be illustrated here for cases $L = 1$ and $L = 2$. Constants c_{nmj} will be used to facilitate the derivation of run length distribution. Note that n is the run length, integer m is selected such that $-h + (m-1)k < u \leq -h + mk$ and integer j provides a label for the terms in the expression $\Pr(n, u)$.

First, consider the case $L = 2$. For $-h < u \leq -h + k$, the CUSUM scheme will signal an out-of-control alarm at the first observation, X_1, if $u - k < u + X_1 - k \leq -h$. Hence

$$\begin{aligned}\Pr(1, u) &= \Pr(u - k < u + X_1 - k \leq -h)\\ &= 1 - \exp(-k + h + u)\\ &= 1 + c_{111}\exp(u).\end{aligned} \tag{2.6}$$

For $-h + k < u \leq 0$, T_1 will be greater than $-h$, which implies that

$$\Pr(1, u) = 0. \tag{2.7}$$

For $-h < u \leq -h + k$, equations (2.2), (2.6) and (2.7) imply that

$$\begin{aligned}\Pr(2, u) &= 0 \cdot \Pr(X_1 \geq k - u)\\ &\quad + \int_{-h}^{-h+k} [1 + c_{111}\exp(x)]\exp(-x - k + u)dx\\ &\quad + \int_{-h+k}^{0} 0 \cdot \exp(-x - k + u)dx\\ &= \exp(-k + u)[\exp(h) - \exp(h - k)] + c_{111}\exp(-k + u)k\\ &= c_{211}\exp(u).\end{aligned} \tag{2.8}$$

For $-h + k < u \leq 0$, equations (2.3), (2.6) and (2.7) imply that

$$\begin{aligned}&\Pr(2, u)\\ &= 0 \cdot \Pr(X_1 \geq k - u)\\ &\quad + \int_{u-k}^{-h+k} [1 + c_{111}\exp(x)]\exp(-x - k + u)dx\\ &\quad + \int_{-h+k}^{0} 0 \cdot \exp(-x - k + u)dx\\ &= 1 - \exp(-k + u)\exp(h - k) + c_{111}\exp(-k + u)[-h + k - (u - k)]\\ &= 1 + c_{221}\exp(u) + c_{222}\exp(u)(u - k).\end{aligned} \tag{2.9}$$

Exact expressions for $\Pr(n, u)$, $n = 3, 4, \ldots$, are obtained by evaluating the integrals (2.2) or (2.3) repeatedly, depending on whether $-h < u \leq -h + k$ or $-h + k < u \leq 0$ and identifying the general pattern of c_{nmj} where $1 \leq j \leq m \leq 2$. A complete summary of these expressions for $\Pr(n, u)$ can be found in the appendix. For $L = 1$ and $-h < u \le 0$,

$$\Pr(1,u) = 1 - \exp(-k + h + u), \tag{2.10}$$

and

$$\Pr(n,u) = \left\{ \frac{\exp(h)}{1+h}[1 - \exp(-k) - \exp(-k)h] \right\} \cdot \left\{ \exp(-k)(1+h) \right\}^{n-1} \exp(u), \tag{2.11}$$

for $n = 2, 3, \ldots$. These are similar to the probability function of a geometric random variable.

The same approach was used to derive exact expressions for run length probability functions of upper-sided CUSUM schemes and a complete summary of these expressions can be found in Gan (1989c). With $L = 1$ and $0 \le u < h$,

$$\Pr(n,u) = c_{n11} + c_{n12}\exp(u), \tag{2.12}$$

$n = 1, 2, \ldots$ where $c_{111} = 0$, $c_{112} = \exp(-k)\exp(-h)$, $c_{211} = c_{112}$, $c_{212} = c_{112}(h-1)\exp(-k)$, and $c_{n11} = c_{n-1,11} + c_{n-1,12}$, $c_{n12} = \exp(-k)\big[hc_{n-1,12} - c_{n-1,12} - c_{n-1,11}\exp(-h)\big]$.

Recursive evaluation of the exact formulas of $\Pr(n,u)$ can be used to evaluate moments and percentiles of the run length distribution. Let N_u denote the run length of a CUSUM scheme given that the initial value is u. The ARL is denoted by $E(N_u)$ and the standard deviation of run length (SDRL) is

$$\text{SDRL} = \sqrt{E(N_u^2) - [E(N_u)]^2}. \tag{2.13}$$

Recursive expressions for $\Pr(n,u)$ listed in the appendix and Gan (1989c) require little modification for translation into computer programming statements. FORTRAN programs implementing the procedures described in this paper were developed by Gan (1989a,b). All the computations are performed in double precision to prevent excessive rounding errors. The ARLs obtained using the recursive expressions of $\Pr(n,u)$ are identical to those obtained by Vardeman and Ray (1985).

3. Relative Performance of CUSUM Schemes

The ARLs, SDRLs and percentiles of 5 lower-sided and 5 upper-sided CUSUM schemes with the same in-control ARL of 500 are displayed in Tables 1 and 2 respectively. The starting values of these CUSUM schemes are set at zero. For a faster detection of an initial out-of-control situation, Lucas and Crosier (1982) suggested using nonzero starting values.

All CUSUM schemes have the same in-control ARL but the percentiles reveal substantial differences in the run length distributions. Any out-of-control alarm issued when a process is in control is considered a false alarm. The main difference between these CUSUM schemes involves probabilities of early false alarms. The lower-sided CUSUM scheme with $\beta = 1$, $h = 6.506$ and $k = 0.800$ signals within the first 46 samples with probability 0.05. However, a CUSUM scheme with a smaller k (and h such that the in-control ARL is 500) would signal within a smaller number of samples with the same probability.

Besides having a smaller chance of early false alarms, a CUSUM scheme with k close to 1 (target mean) is more effective in detecting a small shift in the mean. A lower-sided CUSUM scheme with $k = 0.7$, for example, will remain inactive as long as the observation obtained is greater than 0.7. Thus, the effectiveness of a lower-sided CUSUM scheme in detecting a small shift in the mean increases as k increases to 1. This phenomena is reflected in both ARLs and percentiles. Similar argument holds for upper-sided CUSUM schemes.

When the shift in the mean increases, a CUSUM scheme with k such that $|k-1|$ is large, is more effective. The reason is clear by inspecting the chart parameters of CUSUM schemes. An upper-sided CUSUM scheme with $h = 6.617$ and $k = 1.500$ requires an observation of at least 8.117 to signal an out-of-control alarm. However, an upper-sided CUSUM scheme with $h = 19.594$ and $k = 1.010$ requires an observation of at least 20.604. A lower-sided CUSUM scheme with k close to 1 is not effective in detecting a large downward shift in the mean since the smallest mean possible is zero and the most extreme observation that can be obtained is zero. It was pointed out earlier that a lower-sided CUSUM scheme with $h = 6.506$ and $k = 0.800$ is very effective in detecting a small downward shift in the mean but very ineffective in detecting a large downward shift in the mean. Note that this CUSUM scheme requires 9 zeros for it to accumulate beyond the chart limit $-h$ because of the relative sizes of k and h.

A Shewhart scheme is obtained by plotting sample observation against sample number. A Shewhart scheme signals an out-of-control alarm at the first sample for which the observation is more extreme than the chart limit. The lower-sided CUSUM scheme with $k = 0.002$ and $h = 0.000$ and the upper-sided CUSUM scheme with $k = 6.215$ and $h = 0.000$ are both Shewhart schemes. A Shewhart scheme which put all its weight on the present observation is thus very effective in detecting very large shifts in the mean. The upper-sided Shewhart scheme performs slightly better than the upper-sided CUSUM schemes when the mean is larger than 6. The largest downward shift in the mean is at most 1 which is not large enough for the lower-sided Shewhart scheme to perform better than the lower-sided CUSUM schemes.

4. Design of CUSUM Schemes

Tables 1 and 2 show that the run length distributions of CUSUM schemes are highly skewed. The ARL and the median run length (MRL) are substantially different and conclusions based on ARL alone can be misleading. For example, the lower-sided CUSUM scheme with $k = 0.500$ and $h = 1.905$ has an in-control ARL of 500, but it would be difficult for a less statistically skilled quality control engineer to accept that he will get runs less than 32 rather frequently, about one in 20 times. Also, half of all the run lengths are less than 349.

The main advantage of knowing the run length distribution of CUSUM schemes is to be able to provide quality control engineers a comprehensive understanding of the run length properties of CUSUM schemes. The confidence of a quality control engineer in control charts can be quickly eroded when he or she encounters a few short run lengths but no assignable causes can be found. Information like MRL and probabilities of early false alarms, for example, are useful to quality control engineers for a good understanding of the working of a CUSUM scheme.

Reported design and analyses of CUSUM schemes have generally been based on ARL considerations but these have been widely criticized (see Barnard (1959) and Bissell (1969)). The two main reasons for this trend are due to (1) computations of the run length distributions are extremely difficult in most cases, and (2) in-control run length distributions are 'roughly' geometric and hence they are 'roughly' characterized by ARL. Reason (2) somewhat justifies the use of ARL in the design of a CUSUM scheme. The results in Section 3 show that a CUSUM scheme with k close to 1 is sensitive in detecting small shifts in the mean and the probability function of run length of such a CUSUM scheme is substantially different from that of a geometric random variable. I therefore propose that once a CUSUM scheme has been selected from CUSUM schemes with the same in-control ARL, then selected percentiles of run length distribution of the CUSUM scheme are computed. These percentiles or probabilities can then be used to explain the run length properties of a CUSUM scheme to quality control engineers.

The design procedures based on ARL developed by Page (1961), Bowker and Lieberman (1972), and Woodall (1985) for normal observations can be adapted here in an obvious way for monitoring exponential observations. Tables of ARLs, SDRLs and percentiles of CUSUM schemes with in-control ARL = 100, 200, ..., 1000 are available from the author or can be computed easily using the computer program developed by Gan (1989a,b).

Alternatively, we can adapt design procedures recommended by Page, Bowker and Lieberman, and Woodall to be based on MRL instead of ARL. The MRL has the advantage of ease in interpretation in relation to a highly skewed distribution. If probability of early false alarm of a CUSUM scheme is of main concern, then the $5th$ percentile of the run length, for example, may be used in place of the ARL or computed to provide additional guidance in the selection of a CUSUM scheme. Whichever design procedure is used, it is recommended here that once a CUSUM scheme is selected, selected percentiles or probabilities of run length distribution are computed to provide a better understanding of the working of a CUSUM scheme.

5. Conclusions

When observations are exponentially distributed, exact values of run length probabilities can be evaluated recursively from the formulas provided in the appendix and Gan (1989c). These can be used to evaluate the ARLs, SDRLs and percentiles of run length distributions. When the observations are normally distributed, CUSUM schemes are known to be more effective than the Shewhart scheme in detecting small shifts in the mean but slightly less effective in detecting large shifts in the mean. Upper-sided CUSUM schemes have similar properties when the observations are exponentially distributed. The largest downward shift in the mean is at most 1 which is not large enough for the lower-sided Shewhart scheme to perform better than the lower-sided CUSUM schemes. When k is close to the target mean, a CUSUM scheme is more effective in detecting small shifts in the mean. As the absolute difference between k and target mean increases, a CUSUM scheme becomes more effective in detecting larger shifts in the mean. A two-sided CUSUM scheme is obtained by running a lower-sided and an upper-sided CUSUM schemes simultaneously. Thus, a comprehensive understanding of the lower-sided and upper-sided CUSUM schemes would provide a better understanding of the two-sided CUSUM scheme.

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Appendix: Probability Function of Run Length of a Lower-Sided CUSUM Scheme

Consider a lower-sided CUSUM scheme with control chart parameters h and k and let L be the smallest integer such that $h \leq Lk$, $L \geq 2$. Note that 0^0 is defined to be 1. The general form for $\Pr(n, u)$ and explicit expressions for $\{c_{nmj}\}$ were first worked out for cases $L = 2, 3, 4$ and $L \geq 5$. They were then combined together to give the results in this appendix.

For $L \geq 2$,

$$\begin{aligned}
&\Pr(1,u) \\
&= 1 + c_{111}\exp(u), && -h < u \leq -h+k, && \text{(A1)} \\
&= 0, && -h+k < u \leq 0. && \text{(A2)}
\end{aligned}$$

For $L \geq 3$ and $2 \leq n \leq L-1$,

$$\begin{aligned}
&\Pr(n,u) \\
&= \sum_{j=1}^{m} c_{nmj}\exp(u)(u-(j-1)k)^{j-1}, && -h+(m-1)k < u \leq -h+mk, \\
& && m = 1,2,\ldots,n-1, && \text{(A3)} \\
&= 1 + \sum_{j=1}^{n} c_{nnj}\exp(u)(u-(j-1)k)^{j-1}, && -h+(n-1)k < u \leq -h+nk, && \text{(A4)} \\
&= 0, && -h+nk < u \leq 0. && \text{(A5)}
\end{aligned}$$

For $L \geq 2$ and $n = L$,

$$\begin{aligned}
&\Pr(n,u) \\
&= \sum_{j=1}^{m} c_{nmj}\exp(u)(u-(j-1)k)^{j-1}, && -h+(m-1)k < u \leq -h+mk, \\
& && m = 1,2,\ldots,L-1, && \text{(A6)} \\
&= 1 + \sum_{j=1}^{L} c_{nLj}\exp(u)(u-(j-1)k)^{j-1}, && -h+(L-1)k < u \leq 0. && \text{(A7)}
\end{aligned}$$

For $L \geq 2$ and $n \geq L+1$,

$$\begin{aligned}
&\Pr(n,u) \\
&= \sum_{j=1}^{m} c_{nmj}\exp(u)(u-(j-1)k)^{j-1}, && -h+(m-1)k < u \leq -h+mk, \\
& && m = 1,2,\ldots,L-1, && \text{(A8)} \\
&= \sum_{j=1}^{L} c_{nLj}\exp(u)(u-(j-1)k)^{j-1}, && -h+(L-1)k < u \leq 0. && \text{(A9)}
\end{aligned}$$

The constants $\{c_{nmj}\}$ are defined as follows:

For $L \geq 2$ and $n = 1, 2$,

$$c_{111} = -\exp(-k)\exp(h), \tag{A10}$$

$$c_{1ab} = 0, \quad a = 2,\ldots,L, \quad b = 1,2,\ldots,a, \tag{A11}$$

$$c_{221} = \exp(-k)\{-\exp(h-k) + c_{111}(-h+k)\}, \tag{A12}$$

$$c_{211} = \exp(-k)\{\exp(h) + c_{111}h\} + c_{221}, \tag{A13}$$

$$c_{222} = -c_{111}\exp(-k), \tag{A14}$$

$$c_{2ab} = 0, \quad a = 3,\ldots,L, \quad b = 1,2,\ldots,a, \quad \text{(for } L \geq 3 \text{ only)}. \tag{A15}$$

For $L \geq 3$ and $n = 3$,

$$c_{331} = \exp(-k)\left\{ -\exp(h-2k) + \sum_{j=1}^{2} c_{22j} \frac{(-h+(3-j)k)^j}{j} \right\}, \tag{A16}$$

$$c_{321} = \exp(-k)\left\{ c_{211}(-h+k) - \sum_{j=1}^{2} c_{22j} \frac{(-h+(2-j)k)^j}{j} + \exp(h-k) \right\} + c_{331}, \tag{A17}$$

$$c_{311} = \exp(-k)hc_{211} + c_{321}, \tag{A18}$$

$$c_{3ab} = -c_{2,a-1,b-1}\exp(-k)/(b-1), \quad a = 2,3, \quad b = 2,\dots,a, \tag{A19}$$

$$c_{3ab} \equiv 0, \quad a = 4,\dots,L, \quad b = 1,2,\dots,a, \quad (\text{ for } L \geq 4 \text{ only}). \tag{A20}$$

For $L \geq 5$ and $4 \leq n \leq L-1$,

$$c_{nn1} = \exp(-k)\left\{ -\exp(h-(n-1)k) + \sum_{j=1}^{n-1} c_{n-1,n-1,j} \frac{(-h+(n-j)k)^j}{j} \right\}, \tag{A21}$$

$$c_{n,n-1,1} = \exp(-k)\left\{ \sum_{j=1}^{n-2} c_{n-1,n-2,j} \frac{(-h+(n-j-1)k)^j}{j} - \sum_{j=1}^{n-1} c_{n-1,n-1,j} \frac{(-h+(n-j-1)k)^j}{j} + \exp(h-(n-2)k) \right\} + c_{nn1}, \tag{A22}$$

$$c_{na1} = \exp(-k)\left\{ \sum_{j=1}^{a-1} c_{n-1,a-1,j} \frac{(-h+(a-j)k)^j}{j} - \sum_{j=1}^{a} c_{n-1,aj} \frac{(-h+(a-j)k)^j}{j} \right\} + c_{n,a+1,1}, \quad a = n-2,\dots,2, \tag{A23}$$

$$c_{n11} = \exp(-k)hc_{n-1,11} + c_{n21}, \tag{A24}$$

$$c_{nab} = -c_{n-1,a-1,b-1}\exp(-k)/(b-1), \quad a = 2,\dots,n, \; b = 2,\dots,a, \tag{A25}$$

$$c_{nab} = 0, \quad a = n+1,\dots,L, \quad b = 1,2,\dots,a. \tag{A26}$$

For $L \geq 4$ and $n = L$,

$$c_{nn1} = \exp(-k)\left\{ -\exp(h-(n-1)k) + \sum_{j=1}^{n-1} c_{n-1,n-1,j} \frac{(-h+(n-j)k)^j}{j} \right\}, \tag{A27}$$

$$c_{n,L-1,1} = \exp(-k)\left\{ \sum_{j=1}^{L-2} c_{n-1,L-2,j} \frac{(-h+(L-1-j)k)^j}{j} \right.$$

$$-\sum_{j=1}^{L-1} c_{n-1,L-1,j}\frac{(-h+(L-1-j)k)^j}{j}+\exp(h-(L-2)k)\Bigg\}+c_{nL1}, \quad \text{(A28)}$$

$$c_{na1}=\exp(-k)\Bigg\{\sum_{j=1}^{a-1} c_{n-1,a-1,j}\frac{(-h+(a-j)k)^j}{j}$$

$$-\sum_{j=1}^{a} c_{n-1,aj}\frac{(-h+(a-j)k)^j}{j}\Bigg\}+c_{n,a+1,1}, \quad a=2,\dots,L-2, \quad \text{(A29)}$$

$$c_{n11}=\exp(-k)hc_{n-1,11}+c_{n21}, \quad \text{(A30)}$$

$$c_{nab}=-c_{n-1,a-1,b-1}\exp(-k)/(b-1), \quad a=2,\dots,L,\ b=2,\dots,a. \quad \text{(A31)}$$

For $L \ge 2$ and $n = L+1$,

$$c_{nL1}=\exp(-k)\Bigg\{c_{n-1,L1}+\sum_{j=2}^{L} c_{n-1,Lj}(-(j-1)k)^{j-1}$$

$$+\sum_{j=1}^{L-1} c_{n-1,L-1,j}\frac{(-h+(L-j)k)^j}{j}+\exp(h-(L-1)k)$$

$$+\sum_{j=1}^{L} c_{n-1,Lj}\left[\frac{(-(j-1)k)^j}{j}-\frac{(-h+(L-j)k)^j}{j}\right]\Bigg\}, \quad \text{(A32)}$$

$$c_{na1}=\exp(-k)\Bigg\{\sum_{j=1}^{a-1} c_{n-1,a-1,j}\frac{(-h+(a-j)k)^j}{j}$$

$$-\sum_{j=1}^{a} c_{n-1,aj}\frac{(-h+(a-j)k)^j}{j}\Bigg\}+c_{n,a+1,1},$$

$$a=2,\dots,L-1 \text{ (for } L\ge 3 \text{ only)}, \quad \text{(A33)}$$

$$c_{n11}=\exp(-k)hc_{n-1,11}+c_{n21}, \quad \text{(A34)}$$

$$c_{nab}=-c_{n-1,a-1,b-1}\exp(-k)/(b-1), \quad a=2,\dots,L,\ b=2,\dots,a. \quad \text{(A35)}$$

For $L \ge 2$ and $n \ge L+2$,

$$c_{nL1}=\exp(-k)\Bigg\{c_{n-1,L1}+\sum_{j=2}^{L} c_{n-1,Lj}(-(j-1)k)^{j-1}$$

$$+\sum_{j=1}^{L-1} c_{n-1,L-1,j}\frac{(-h+(L-j)k)^j}{j}$$

$$+\sum_{j=1}^{L} c_{n-1,Lj}\left[\frac{(-(j-1)k)^j}{j}-\frac{(-h+(L-j)k)^j}{j}\right]\Bigg\}, \quad \text{(A36)}$$

$$c_{na1} = \exp(-k)\left\{ \sum_{j=1}^{a-1} c_{n-1,a-1,j} \frac{(-h+(a-j)k)^j}{j} - \sum_{j=1}^{a} c_{n-1,aj} \frac{(-h+(a-j)k)^j}{j} \right\} + c_{n,a+1,1},$$

$$a = 2, \ldots, L-1 \text{ (for } L \geq 3 \text{ only)}, \tag{A37}$$

$$c_{n11} = \exp(-k) h c_{n-1,11} + c_{n21}, \tag{A38}$$

$$c_{nab} = -c_{n-1,a-1,b-1} \exp(-k)/(b-1), \quad a = 2, \ldots, L, \; b = 2, \ldots, a. \tag{A39}$$

Table 1. Average run lengths, standard deviations of run lengths and percentiles of run length distributions of lower-sided CUSUM schemes.

					Percentiles														
β	h	k	ARL	SDRL	0.001	0.01	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.950	0.990	0.999
1.00	0.000	0.002	500.0	499.5	1	6	26	53	112	179	256	347	458	602	804	1151	1497	2301	3451
	0.737	0.300	500.0	496.5	4	8	29	56	114	181	257	348	458	601	803	1147	1491	2290	3433
	1.905	0.500	500.0	493.5	6	11	32	59	117	183	259	349	459	601	801	1143	1485	2279	3415
	4.267	0.700	500.0	486.8	11	18	38	65	122	187	262	351	459	599	797	1134	1471	2255	3375
	6.506	0.800	500.0	478.9	15	24	46	72	128	192	266	353	460	598	792	1124	1456	2226	3329
0.75	0.000	0.002	375.1	374.6	1	4	20	40	84	134	192	260	344	452	603	863	1123	1726	2588
	0.737	0.300	183.3	179.7	3	5	13	23	44	68	95	128	168	220	293	417	542	831	1245
	1.905	0.500	118.4	111.6	5	7	13	19	32	47	64	84	109	141	186	264	341	521	778
	4.267	0.700	75.2	61.7	9	11	16	20	28	36	45	57	70	88	113	155	198	297	439
	6.506	0.800	63.6	43.6	12	15	20	23	30	36	43	51	61	73	90	120	150	219	318
0.50	0.000	0.002	250.3	249.8	1	3	13	27	56	90	128	174	229	301	402	576	749	1151	1726
	0.737	0.300	49.0	45.2	3	4	6	9	14	20	27	35	45	58	77	108	139	212	316
	1.905	0.500	25.5	18.8	5	6	7	9	11	14	16	20	24	29	37	50	63	93	136
	4.267	0.700	20.2	9.0	8	9	10	11	13	15	16	18	20	23	26	32	38	51	70
	6.506	0.800	21.7	7.0	10	12	13	14	16	17	19	20	22	24	27	31	35	45	58
0.25	0.000	0.002	125.4	124.9	1	2	7	14	28	45	64	87	115	151	201	288	375	576	863
	0.737	0.300	9.7	6.0	3	3	4	4	5	6	7	8	9	11	13	17	22	31	45
	1.905	0.500	8.1	2.5	5	5	5	6	6	7	7	8	8	9	10	11	13	17	22
	4.267	0.700	10.1	1.7	7	7	8	8	9	9	9	10	10	11	11	12	13	15	18
	6.506	0.800	12.4	1.6	9	10	10	11	11	11	12	12	13	13	14	14	15	17	19
0.10	0.000	0.002	50.5	50.0	1	1	3	6	12	18	26	35	46	61	81	116	150	231	346
	0.737	0.300	4.2	1.0	3	3	3	3	3	4	4	4	4	5	5	6	6	7	9
	1.905	0.500	5.3	0.6	4	4	5	5	5	5	5	5	5	5	6	6	6	7	8
	4.267	0.700	7.6	0.6	7	7	7	7	7	7	7	8	8	8	8	8	8	9	10
	6.506	0.800	9.8	0.6	9	9	9	9	9	10	10	10	10	10	10	10	11	11	12
0.06	0.000	0.002	30.5	30.0	1	1	2	4	7	11	16	21	28	37	49	70	90	139	208
	0.737	0.300	3.5	0.6	3	3	3	3	3	3	3	3	4	4	4	4	4	5	6
	1.905	0.500	5.0	0.3	4	4	4	5	5	5	5	5	5	5	5	5	5	6	6
	4.267	0.700	7.1	0.3	7	7	7	7	7	7	7	7	7	7	7	7	8	8	8
	6.506	0.800	9.2	0.4	9	9	9	9	9	9	9	9	9	9	9	10	10	10	10

Table 2. Average run lengths, standard deviations of run lengths and percentiles of run length distributions of upper-sided CUSUM schemes.

					Percentiles														
β	h	k	ARL	SDRL	0.001	0.01	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.950	0.990	0.999
1.00	0.000	6.215	500.0	499.5	1	6	26	53	112	179	256	347	458	602	804	1151	1497	2301	3451
	6.617	1.500	500.0	496.2	3	9	29	56	115	181	257	348	458	601	802	1146	1490	2289	3431
	9.814	1.200	500.0	487.0	6	15	38	64	122	187	262	351	459	599	797	1134	1472	2256	3377
	15.635	1.050	500.0	457.5	16	32	62	90	146	207	278	361	463	594	779	1096	1412	2147	3198
	19.594	1.010	500.0	430.3	25	46	81	112	168	226	293	371	467	590	763	1060	1357	2045	3031
1.05	0.000	6.215	371.9	371.4	1	4	20	40	83	133	190	258	341	448	598	856	1113	1711	2566
	6.617	1.500	324.5	320.5	2	7	20	38	76	118	168	226	298	390	520	742	964	1480	2218
	9.814	1.200	289.5	276.3	5	12	27	42	75	112	155	205	267	346	458	649	841	1285	1921
	15.635	1.050	264.2	223.7	13	25	45	62	91	122	157	198	247	311	401	555	709	1067	1578
	19.594	1.010	263.4	200.5	21	36	59	77	108	138	170	207	251	308	388	524	660	976	1429
1.10	0.000	6.215	284.2	283.7	1	3	15	30	64	102	145	197	260	342	457	654	850	1307	1960
	6.617	1.500	221.1	217.1	2	5	15	27	53	82	115	155	203	265	353	504	654	1004	1503
	9.814	1.200	183.4	170.3	4	10	21	31	52	74	101	132	169	218	287	405	523	796	1188
	15.635	1.050	165.1	127.9	11	21	35	47	66	85	105	129	157	193	244	331	418	620	909
	19.594	1.010	169.0	114.2	17	29	46	59	79	98	118	139	165	197	242	317	393	568	818
1.50	0.000	6.215	63.0	62.5	1	1	4	7	14	23	32	44	58	76	101	144	188	288	432
	6.617	1.500	33.9	30.0	1	2	5	7	11	15	20	25	32	40	52	73	94	142	210
	9.814	1.200	29.9	20.9	2	4	7	9	13	17	21	25	29	35	43	57	71	103	148
	15.635	1.050	35.4	18.3	5	8	13	16	20	24	28	32	36	41	48	59	70	94	127
	19.594	1.010	40.9	18.5	7	11	17	20	25	30	34	38	42	48	55	65	76	98	129
2.00	0.000	6.215	22.4	21.9	1	1	2	3	5	8	12	16	21	27	36	51	66	101	151
	6.617	1.500	13.0	10.0	1	1	2	3	5	7	8	10	13	15	19	26	33	48	70
	9.814	1.200	13.6	8.2	1	2	4	5	7	9	10	12	14	16	19	24	29	41	56
	15.635	1.050	18.0	8.4	2	4	7	9	11	13	15	17	19	21	24	29	34	43	57
	19.594	1.010	21.3	8.9	4	6	9	11	14	16	18	20	22	25	28	33	38	48	61
3.00	0.000	6.215	7.9	7.4	1	1	1	1	2	3	4	6	7	9	12	18	23	35	52
	6.617	1.500	5.9	4.0	1	1	1	2	3	3	4	5	6	7	9	11	14	19	28
	9.814	1.200	6.9	3.9	1	1	2	3	4	4	5	6	7	8	10	12	14	19	26
	15.635	1.050	9.4	4.4	1	2	3	4	6	7	8	9	10	11	13	15	18	22	29
	19.594	1.010	11.2	4.8	1	3	5	6	7	8	9	11	12	13	15	18	20	25	31

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