STRONG GAUSSIAN APPROXIMATIONS IN THE LEFT TRUNCATED AND RIGHT CENSORED MODEL

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Abstract: In this paper, we consider the Tsai-Jewel-Wang estimator $F_n^0(x)$ of an unknown distribution function F^0 when the data are subject to random left truncation and right censorship. Strong Gaussian approximations of the product-limit process $\sqrt{n}[F_n^0(x) - F^0(x)]$ are constructed with rate $O((\log n)^{3/2}/n^{1/8})$. A functional law of the iterated logarithm for the maximal deviation of the estimator from the estimand is derived from the construction.

Key words and phrases: Cumulative hazard, left truncation, Gaussian approximations, right censorship, product-limit.

1. Introduction

Let X, T and S be positive independent random variables with continuous distribution functions (df) F^0, G^0 and L^0 respectively. Let $Y = \min(X, S)$ and $\delta = I(X \leq S)$. If $Y \geq T$, one observes (Y, T, δ). If $Y < T$, nothing is observed. We think of X as survival time, the observation of which is subjected to right censorship, S, and left truncation, T, mechanisms. δ indicates whether the observed Y is a censored item or not. This is the left truncation, right censorship (LTRC) model. Denote the df of Y by J. By the independent assumption, we have $1 - J = (1 - F^0)(1 - L^0)$. Let (X_i, T_i, S_i), $i = 1, \ldots, N$, be i.i.d. as (X, T, S), where the population size N is fixed but unknown. The basic problem is to estimate F^0 from the empirical data (Y_i, T_i, δ_i), $i = 1, \ldots, n$, where n is the number of observed triplets.

As a consequence of truncation, the number of observed pairs, n, is a $\text{Bin}(N, \alpha)$ random variable, with $\alpha := P(T \leq Y)$. By the Strong Law of Large Numbers, $n/N \to \alpha$ almost surely as $N \to \infty$. Conditional on the value of n, (Y_i, T_i, δ_i), $i = 1, \ldots, n$ are still i.i.d., but with the joint conditional distribution of (Y, T) given by $H(y, t) = P\{Y \leq y, T \leq t \mid T \leq Y\} = \alpha^{-1} \int_0^y G^0(t \wedge z) \, dJ(z)$ for y, $t > 0$. The marginal distribution functions are denoted by

$$F(y) := H(y, \infty) = \alpha^{-1} \int_0^y G^0(z) \, dJ(z),$$

$$G(t) := H(\infty, t) = \alpha^{-1} \int_0^\infty G^0(t \wedge z) \, dJ(z).$$

Here and in the following, $\int_a^b = \int_{(a,b]}$ for $0 \leq a < b \leq \infty$. Empirical counterparts of these distribution functions are denoted by $H_n(y,t)$, $F_n(y)$ and $G_n(t)$, respectively. For $0 \leq z < \infty$, let

$$C(z) = G(z) - F(z-) = \frac{1}{\alpha} P(T \leq z \leq S)\,[1 - F(z-)]. \qquad (1.1)$$

$C(z)$ is consistently estimated by $C_n(z) = G_n(z) - F_n(z-)$. The product-limit (PL) estimator of F^0 (Tsai, Jewel and Wang (1987)) is

$$1 - F_n^0(t) = \prod_{i:Y_i \leq t} \left[1 - \frac{1}{nC_n(Y_i)} \right]^{\delta_i}, \qquad (1.2)$$

assuming no ties in the data. This reduces to the Kaplan-Meier PL-estimator when $T = 0$, and to the Lynden-Bell (1971) estimator when there is no right censoring.

Gu and Lai (1990) and Lai and Ying (1991) obtained a functional law of the iterated logarithm for a slightly modified form of the TJW estimator using martingale theory. Gijbels and Wang (1993) and Zhou (1996) established almost sure representation of the TJW estiamtor in terms of sums of normed i.i.d. random processes. A stronger result is claimed by Zhou and Yip (1999). By invoking the approximation theorems of Komlós, Major and Tusnády (1975) for the univariate empirical process, they were able to obtain strong approximation of the product-limit process $Z_n(t) = \sqrt{n}\,[F_n^0(t) - F^0(t)]$ by a two-parameter Gaussian process at the almost sure rate of $O(\log^2 n/\sqrt{n})$. From the construction, they inferred the functional law of the iterated logarithm for the PL-process.

The KMT theorems are not directly applicable in our situation. The PL-estimator in (1.2) depends on C_n, which is the difference of the marginal empirical distribution functions of Y and T, given the event $\{T \leq Y\}$. Since F^0 and G^0 are arbitrary, these processes are not independent in general. Therefore, the claim by Zhou and Yip cannot be true without severe restrictions and, even in that case, a constructive proof would be non-trivial.

The same comments hold for the random truncation model. Both are irreducible two-dimensional models. For this latter model, Tse (2000) has established strong approximation of the PL-process by a two-parameter Kiefer type process at the almost sure rate of $O((\log n)^{3/2}/n^{1/8})$. We show the counterpart for the LTRC model in this paper. The results include the random censorship and trucation models as limiting cases. The approximation rate $O((\log n)^{3/2}/n^{1/8})$ is not as fast as that claimed by Zhou and Yip, but is still good enough to let us deduce almost sure statements like the Law of the Iterated Logarithm from that of the corresponding Gaussian processes.

For the random censorship model ($T = 0$), the conditional event $\{T \leq Y\}$ is trivially satisfied and the dependence problem does not arise. Note however, even

in this case, the data lead to the natural formation of two empirical processes, one for all the observed $Y's$ and the other for the observed but uncensored $Y's$ only. Burke, Csörgő and Horváth (1981, 1988) revealed to us the true one-dimensional nature of the model by embedding the latter as a sub-process of the former. The construction is non-trivial. It is only then that the KMT approximation theorems are applicable. Heuristically, the difference between the random censorship and truncation mechanisms is due to the fact that we have strictly less information in the latter model since the very existence of a truncated item is hidden from the observer.

In Section two, we present the main results. Auxiliary results and proofs are relegated to Section three. Examples of applications can be found in Gijbels and Wang (1993) and Zhou and Yip (1999).

2. Main Theorems

We have seen that the data in the LTRC model can be organized into the edf F_n and G_n. The information from the indicator variable has not been taken into account yet. For this, we use the observed uncensored $Y's$ to construct the edf $n\,F_{1n}(z) = \sum_{i=1}^{n} I(Y_i \le z, \delta = 1)$ which is a consistent estimator of $F_1(z) = P(Y \le z, \delta = 1 \mid T \le Y)$. Note that F_1 is a sub-distribution of F. The triplets F_n, G_n and F_{1n} contain all the relevant information from the data for the estimation of F^0. The corresponding empirical processes are defined as:

$$\alpha_n^1(z_1) := \sqrt{n}\,[\,F_n(z_1) - F(z_1)\,], \qquad \alpha_n^2(z_2) := \sqrt{n}\,[\,G_n(z_2) - G(z_2)\,],$$
$$\beta_{n1}(u) := \sqrt{n}\,[\,F_{1n}(u) - F_1(u)\,].$$

The notation is intended to remind us that β_{n1} is a sub-process of α_n^1, whereas α_n^1 and α_n^2 together form a two-component random process with covariance depending on (the arbitrary) F^0 and G^0.

For any df K, let $a_K = \inf\{z : K(z) > 0\}$ and $b_K = \sup\{z : K(z) < 1\}$. Following Tse (2000), we suppose that $a_{G^0} = a_J = 0$ throughout. The cumulative hazard function associated with F^0 is

$$\Lambda^0(t) := \int_0^t \frac{d\,F^0(z)}{1 - F^0(z-)} = \int_0^t \frac{d\,F_1(z)}{C(z)}, \qquad 0 \le t < \infty,$$

which is consistently estimated by

$$\Lambda_n^0(t) := \int_0^t \frac{d\,F_{1n}(z)}{C_n(z)}, \qquad 0 \le t < \infty.$$

The cumulative hazard process is $\hat{Z}_n(t) := \sqrt{n}\,[\,\Lambda_n^0(t) - \Lambda^0(t)\,]$. For the theorems below, we assume that F^0, G^0 and L^0 satisfy the condition

$$\int_0^\infty \frac{d\,F_1(z)}{C^3(z)} < \infty. \tag{2.1}$$

The condition, while not optimal, serves to maintain finite variances of the limiting Gaussian processes near the lower end point and simplifies the proofs.

For $0 < t < b < b_J$, let

$$l(t) := \int_0^t \frac{d\,F_1(u)}{C^2(u)}. \tag{2.2}$$

Theorem 2.1. *Suppose* (2.1) *is satisfied. On a rich enough probability space, one can define a sequence of independent and identically distributed mean zero Gaussian processes* $\{B_n(t), 0 < t < b\}$, *for* $b < b_J$, *with* $\mathrm{Cov}\,[\,B_n(s), B_n(t)\,] = l(\min(s,t))$, *for* $0 < s,\, t < b < b_J$ *such that, almost surely,*

$$\sup_{0 \le t \le b} |\,\hat{Z}_n(t) - B_n(t)\,| = O\left(\frac{\log n}{n^{1/6}}\right),$$

$$\sup_{0 \le t \le b} |\,Z_n(t) - [1 - F^0(t)]\,B_n(t)\,| = O\left(\frac{\log n}{n^{1/6}}\right).$$

Note that the statements above are conditional on n, the observed sample size, and that the approximations hold on fixed intervals $[0, b]$. Both serve to simplify the argument below. Extending to the more general formulation of Tse (2000) can be accomplished with more work.

While weak convergence results follow readily from Theorem 2.1, almost sure statements cannot be obtained from them since the covariances between different members in the sequences are not specifed. For that purpose, we need the next theorem.

Theorem 2.2. *Assume* (2.1) *is satisfied. On a rich enough probability space, one can construct a two-parameter mean zero Gaussian process* $B(t, u)$ *for* $t \ge 0$ *and* $u \ge 0$ *with* $\mathrm{Cov}\,[\,B(s,n), B(t,m)\,] = \sqrt{\frac{n}{m}}\, l(s)$, *for* $n \le m, s < t$ *such that, almost surely,*

$$\sup_{0 \le t \le b} |\,\hat{Z}_n(t) - B(t,n)\,| = O\left(\frac{(\log n)^{3/2}}{n^{1/8}}\right),$$

$$\sup_{0 \le t \le b} |\,Z_n(t) - [1 - F^0(t)]\,B(t,n)\,| = O\left(\frac{(\log n)^{3/2}}{n^{1/8}}\right).$$

As a consequence of Theorem 2.2, we obtain the next theorem for the uniform consistency rate of the PL-estimator for the LTRC model. The proof parallels that of Theorem 1 in Csörgő and Horváth (1988) for the random censorship model with our Theorem 2.2 replacing the role of BCH's strong construction.

Theorem 2.3. *If* (2.1) *is satisfied, then the sequence* $\{(2 \log \log n)^{-1/2} Z_n(\cdot)\}$ *is almost surely relatively compact in the supremum norm of functions over* $[0, b]$,

and its set of limit points is $\{l(b)^{1/2} [1 - F^0(\cdot)] g(l(\cdot)/l(b)) : g \in S\}$, *where S is Strassen's set of absolutely continuous functions*

$$S = \Big\{ g \mid g : [0,1] \to \mathbf{R}, \ g(0) = 0, \ \int_0^1 \Big(\frac{d\,g(x)}{d\,x}\Big)^2 dx \ \leq\ 1 \Big\}.$$

Consequently, with $v^2(t) = (1 - F^0(t))^2 l(t)$,

$$\limsup_{n\to\infty} \Big(\frac{n}{2\log\log n}\Big)^{1/2} \sup_{t\in[0,b]} \mid F_n^0(t) - F^0(t) \mid = \sup_{t\in[0,b]} v(t),$$

$$\liminf_{n\to\infty} (n\log\log n)^{1/2} \sup_{t\in[0,b]} \frac{\mid F_n^0(t) - F^0(t) \mid}{1 - F^0(t)} = \frac{\pi}{8^{1/2}} (l(b))^{1/2},$$

$$\frac{\pi}{8^{1/2}} v(b) \leq \liminf_{n\to\infty} (n\log\log n)^{1/2} \sup_{t\in[0,b]} \mid F_n^0(t) - F^0(t) \mid \leq \frac{\pi}{8^{1/2}} (l(b))^{1/2}.$$

Note that in the absence of censorship, Theorems 2.1, 2.2 and 2.3 give corresponding results for the random truncation model. In the absence of truncation, the model reduces to the one-dimensional censorship model for which BCH's result is optimal.

3. Auxiliary Results and Proofs

As in the random truncation model, we need the results of Borisov (1982) and M. Csörgő and Horváth (1988), which are stated as Theorems 2.A and 2.B in Tse (2000). Our goal is to construct strong Gaussian approximations for $\hat{Z}_n$ and Z_n. Since the triplets $(\beta_{n1}, \alpha_n^1, \alpha_n^2)$ form the basic ingredients for estimation of F^0 in the LTRC model, the key to the construction in Theorem 2.1 is to have simultaneous strong approximations of the triplets at the specified rate. That this can be done is the content of the next theorem.

Theorem 3.1. *On a rich enough probability space, one can define three sequences of Gaussian processes $W_{n1}(t)$, $W_n^1(t)$ and $W_n^2(t)$ such that, for $s = (s_0, s_1, s_2) \in R^{3+}$, $\alpha_n(s) = (\beta_{n1}(s_0), \alpha_n^1(s_1), \alpha_n^2(s_2))$, and $W_n(s) = (W_{n1}(s_0), W_n^1(s_1), W_n^2(s_2))$, we have, almost surely, $\sup_{s\in R^{3+}} \mid \alpha_n(s) - W_n(s) \mid = O(\log n/n^{1/6})$. Moreover, W_n is a 3-dimensional vector-valued mean zero Gaussian process having the same covariance structure as the vector α_n:*

$$E\,W_{n1}(u)\,W_{n1}(v) = \min\,(F_1(u), F_1(v)\,) - F_1(u)\,F_1(v),$$
$$E\,W_n^1(u)\,W_n^1(v) = \min\,(F(u), F(v)\,) - F(u)\,F(v),$$
$$E\,W_n^2(u)\,W_n^2(v) = \min\,(G(u), G(v)\,) - G(u)\,G(v),$$
$$E\,W_{n1}(u)\,W_n^1(v) = \min\,(F_1(u), F_1(v)\,) - F_1(u)\,F(v),$$
$$E\,W_{n1}(u)\,W_n^2(v) = H_1(u, v) - F_1(u)\,G(v),$$
$$E\,W_n^1(u)\,W_n^2(v) = H(u, v) - F(u)\,G(v),$$

where $H_1(u, v) = P(Y \le u, T \le v, \delta = 1 \mid T \le Y)$ and $u, v \in R$.

Proof. By Borisov's theorem, we can construct a probability space with two sequences of random processes (α_n^1, α_n^2) and (W_n^1, W_n^2) defined on it such that, almost surely, $\sup_{u \in R^+} \mid \alpha_n^i(u) - W_n^i(u) \mid = O(\log n/n^{1/6})$, jointly for $i = 1,\ 2$. Our theorem requires that β_{n1} and W_{n1} be included in the approximation. For that purpose, we make a further specification on the construction of α_n^1 and W_n^1.

Let $F_2(u) = F(u) - F_1(u)$. This is the distribution function for the observed, censored Y items. Set $F_i(\infty) = \lim_{u \to \infty} F_i(u)$, $i = 1,\ 2$. It follows that $F(\infty) = \lim_{u \to \infty} F(u) = 1$. Let $Y_1^1, \ldots, Y_{n_1}^1$ denote the n_1 observed uncensored Y items in order of appearance, and $Y_1^2, \ldots, Y_{n_2}^2$ denote the n_2 observed censored Y items, also in order of appearance. Of course, $n_1 + n_2 = n$. Then the random variables $F_1(Y_1^1), \ldots, F_1(Y_{n_1}^1)$ are uniformly distributed on the interval $(0, F_1(\infty))$. Similarly, the shifted random variables $F_2(Y_1^2) + F_1(\infty), \ldots, F_2(Y_{n_2}^2) + F_1(\infty)$ are uniformly distributed on the interval $(F_1(\infty), 1)$.

Let $\tilde{\alpha}$ denote the empirical process of n i.i.d. unif$(0, 1)$ random variables and $\tilde{W}_n^1$ be copies of its Brownian Bridge limit. Define $\beta_{n1}(u) = \tilde{\alpha}_n(F_1(u))$, $\beta_{n2}(u) = \tilde{\alpha}_n(F_2(u) + F_1(\infty)) - \tilde{\alpha}_n(F_1(\infty))$, and $\alpha_n^1(u) = \beta_{n1}(u) + \beta_{n2}(u)$, together with their corresponding Gaussian limits $W_{n1}(u) = \tilde{W}_n^1(F_1(u))$, $W_{n2}(u) = \tilde{W}_n^1(F_2(u) + F_1(\infty)) - \tilde{W}_n^1(F_1(\infty))$, and $W_n^1(u) = W_{n1}(u) + W_{n2}(u)$. Then, the construction above for (α_n^1, α_n^2) and W_n^1, W_n^2 automatically includes the desired result for α_{n1} and W_{n1}.

Lastly, a lengthy but straightforward calculation analogous to that of Wang, Jewel and Tsai (1986) for the random truncation model shows that the joint covariance of W_{n1}, W_n^1 and W_n^2 are as stated in the theorem. We skip those details.

We now turn to the proof of Theorem 2.1. The construction holds on the probability space of Theorem 3.1.

Proof of Theorem 2.1. For the sake of simplicity, we often denote $\sup_{0 \le t \le b} |f(t)|$ by $\|f(\cdot)\|$. We start with the usual decomposition of $\hat{Z}_n(t)$:

$$\hat{Z}_n(t) = \int_0^t \frac{d\beta_{n1}}{C} + \int_0^t \frac{\sqrt{n}\,(C - C_n)}{C^2}\, dF_1 + R_{n1}(t),$$

$$R_{n1}(t) = \int_0^t \frac{\sqrt{n}\,(C - C_n)}{C^2}\, d(F_{n1} - F_1) + \int_0^t \frac{\sqrt{n}\,(C - C_n)^2}{C_n\, C^2}\, dF_{n1}.$$

Note that $\sqrt{n}\,(C - C_n) = \alpha_n^1 - \alpha_n^2$. An integration by parts yields

$$\hat{Z}_n(t) = \frac{\beta_{n1}(t)}{C(t)} + \int_0^t \frac{\beta_{n1}}{C^2}\, dG - \int_0^t \frac{\beta_{n1}}{C^2}\, dF + \int_0^t \frac{\alpha_n^1}{C^2}\, dF_1 - \int_0^t \frac{\alpha_n^2}{C^2}\, dF_1 + R_{n1}(t).$$

Define, for $t \in [0, b]$, the sequence of Gaussian processes

$$B_n(t) := \frac{W_{n1}(t)}{C(t)} + \int_0^t \frac{W_{n1}}{C^2} dG - \int_0^t \frac{W_{n1}}{C^2} dF + \int_0^t \frac{W_n^1}{C^2} dF_1 - \int_0^t \frac{W_n^2}{C^2} dF_1.$$

Clearly, $E\, B_n(t) = 0$. Lengthy calculation also gives the covariance of $B_n(t)$: $\mathrm{Cov}\,[B_n(s), B_n(t)] = l(s \wedge t) = l(s) \wedge l(t)$ for $s, t > 0$, where $l(t)$ is defined in (2.2). Define $l^{-1}(t)$ as the generalized inverse function of $l(t)$ in the same way as do Burke, Csörgő and Horváth (1981) for the random censorship model. Then the covariance formula above implies that $W_n(\cdot) = B_n(l^{-1}(\cdot))$ is a standard Wiener process on $[0, \infty]$ for each n, and hence $B_n(t) = W_n(l(t))$, for $t \geq 0$.

Theorem 2.1 is about the order of $\|\hat{Z}_n - B_n\| = \|R_{n1} + R_{n2}\|$, where

$$R_{n2} = \int_0^t \frac{\beta_{n1} - W_{n1}}{C^2} dG - \int_0^t \frac{\beta_{n1} - W_{n1}}{C^2} dF + \int_0^t \frac{\alpha_n^1 + W_n^1}{C^2} dF_1 - \int_0^t \frac{\alpha_n^2 + W_n^2}{C^2} dF_1.$$

By Theorem 2.1 of Zhou and Yip, $\|R_{n1}\| = O(\log\log n / \sqrt{n})$ almost surely. Suppose $\hat{Z}_n$ and B_n are constructed on the probability space of Theorem 3.1, using $\{\beta_{n1}, \alpha_n^1, \alpha_n^2\}$ and $\{W_{n1}, W_n^1, W_n^2\}$ defined there. Then $\|R_{n2}\| = O(\log n / n^{1/6})$. We get the first statement of the theorem. Finally, combining this statement with Theorem 2.2 of Zhou and Yip, we get the second statement of the theorem.

Proof of Theorem 2.2. This is similar to the proofs of Theorem 2.1 with the role of Theorem 2.A replaced by Theorem 2.B in Tse (2000).

Proof of Theorem 2.3. Observe that the process $\{[1 - F^0(t)]\, B(t, u), 0 < t \leq b, u \geq 0\}$ is equal in distribution to the process

$$\left\{ l(b)^{1/2} \left[1 - F^0(t) \right] u^{1/2}\, W\left(\frac{l(t)}{l(b)}, u \right), 0 < t \leq b, u \geq 0 \right\},$$

where $W(t, u)$ is a standard two-parameter Wiener process. Hence the first statement of the theorem follows from the standard Functional Law of the Iterated Logarithm for a two-parameter Wiener process (Theorem 1.14.1 in Csörgő and Révész (1981)). This implies that

$$\sup_{g \in S} \sup_{0 < t \leq b} \left| l(b)^{1/2} \left[1 - F^0(t) \right] g\left(\frac{l(t)}{l(b)} \right) \right| \leq \sup_{0 < t \leq b} v(t),$$

where the inequality is obtained from Riesz' lemma (Lemma 1.3.1 in Csörgő and Révész), according to which $|g(t)| \leq t^{1/2}$ on $(0, 1)$ for any $g \in S$. The opposite inequality is trivial.

The first liminf statement also follows from Theorem 2.2 and the representation of $[1 - F^0(t)]\, B(t, u)$ via Chung's second loglog law for the two-parameter Wiener process, obtained from Chung's original law as applied for partial sums

of independent Wiener processes. The second liminf law is a trivial consequence of the first.

Acknowledgements

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