

Convex Mixtures Imputation and Applications

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Supplementary Material

S1 Simulation Case 3

$$\text{Case 3} \left\{ \begin{array}{l} m_3(x) = 2x, \\ \sigma_3^2(x) = \begin{cases} 0.16, & 0 < x < 1, \\ 1.0, & -1 < x \leq 0, \end{cases} \\ p_3(x) = \frac{e^x}{1+e^x}, \\ X \sim U(-1, 1), \\ E(Y) = 0, E(Y_{obs}) = 0.318, P(\delta = 1) = 0.500, \sigma_{KR}^2 = 2.823. \end{array} \right.$$

Tables S1 and S2 list calculated results of Case 3.

Table S1: Average estimates using KR, IPW, k -NN and CM imputation under Case 3

| Method | | | Sample Size n | | | | | | | | | | | |
|---------|------|----|-----------------|----------------------|----------------------|-------|--------|----------------------|----------------------|-------|-------|----------------------|----------------------|-------|
| | | | 100 | | | | 500 | | | | 1000 | | | |
| h | k | | Bias | $n \cdot \text{VAR}$ | $n \cdot \text{MSE}$ | CCI | Bias | $n \cdot \text{VAR}$ | $n \cdot \text{MSE}$ | CCI | Bias | $n \cdot \text{VAR}$ | $n \cdot \text{MSE}$ | CCI |
| KR | 0.05 | - | 0.071 | 3.605 | 4.113 | 0.934 | 0.002 | 3.126 | 3.127 | 0.951 | 0.002 | 2.829 | 2.835 | 0.943 |
| | 0.15 | - | 0.014 | 3.139 | 3.160 | 0.952 | 0.004 | 3.050 | 3.057 | 0.950 | 0.006 | 2.780 | 2.811 | 0.939 |
| | 0.20 | - | 0.013 | 3.060 | 3.078 | 0.952 | 0.007 | 3.043 | 3.066 | 0.949 | 0.009 | 2.775 | 2.847 | 0.941 |
| | 0.30 | - | 0.018 | 2.979 | 3.013 | 0.951 | 0.014 | 3.028 | 3.131 | 0.944 | 0.016 | 2.773 | 3.035 | 0.934 |
| IPW | 0.05 | - | 0.072 | 3.626 | 4.138 | 0.933 | 0.002 | 3.175 | 3.176 | 0.950 | 0.002 | 2.889 | 2.893 | 0.944 |
| | 0.15 | - | 0.012 | 3.202 | 3.217 | 0.950 | 0.001 | 3.054 | 3.054 | 0.953 | 0.002 | 2.789 | 2.794 | 0.941 |
| | 0.20 | - | 0.009 | 3.130 | 3.138 | 0.954 | 0.001 | 3.042 | 3.043 | 0.953 | 0.002 | 2.794 | 2.799 | 0.942 |
| | 0.30 | - | 0.008 | 3.026 | 3.032 | 0.951 | 0.002 | 3.044 | 3.046 | 0.949 | 0.003 | 2.781 | 2.791 | 0.942 |
| k -NN | - | 1 | 0.006 | 3.304 | 3.308 | 0.945 | -0.000 | 3.677 | 3.677 | 0.948 | 0.002 | 3.379 | 3.383 | 0.952 |
| | - | 2 | 0.010 | 3.164 | 3.173 | 0.947 | 0.000 | 3.264 | 3.264 | 0.944 | 0.002 | 3.085 | 3.088 | 0.944 |
| | - | 4 | 0.019 | 3.108 | 3.143 | 0.956 | 0.001 | 3.144 | 3.144 | 0.945 | 0.002 | 2.938 | 2.944 | 0.941 |
| | - | 8 | 0.038 | 3.131 | 3.277 | 0.955 | 0.002 | 3.143 | 3.145 | 0.949 | 0.003 | 2.841 | 2.847 | 0.943 |
| | - | 16 | 0.095 | 3.287 | 4.181 | 0.922 | 0.006 | 3.119 | 3.139 | 0.948 | 0.004 | 2.790 | 2.805 | 0.941 |
| CM | 0.15 | 1 | 0.003 | 3.039 | 3.047 | 0.944 | 0.001 | 3.293 | 3.294 | 0.948 | 0.003 | 3.039 | 3.047 | 0.944 |
| | 0.20 | 1 | 0.004 | 3.035 | 3.049 | 0.944 | 0.002 | 3.286 | 3.288 | 0.948 | 0.004 | 3.035 | 3.049 | 0.944 |
| | 0.30 | 1 | 0.006 | 3.031 | 3.067 | 0.942 | 0.004 | 3.277 | 3.285 | 0.949 | 0.006 | 3.031 | 3.067 | 0.942 |
| | 0.15 | 2 | 0.003 | 2.918 | 2.925 | 0.939 | 0.001 | 3.123 | 3.124 | 0.947 | 0.003 | 2.918 | 2.925 | 0.939 |
| | 0.20 | 2 | 0.003 | 2.915 | 2.927 | 0.940 | 0.002 | 3.116 | 3.118 | 0.946 | 0.003 | 2.915 | 2.927 | 0.940 |
| | 0.30 | 2 | 0.006 | 2.910 | 2.942 | 0.939 | 0.004 | 3.110 | 3.119 | 0.948 | 0.006 | 2.910 | 2.942 | 0.939 |
| | 0.15 | 4 | 0.003 | 2.859 | 2.869 | 0.941 | 0.002 | 3.085 | 3.086 | 0.946 | 0.003 | 2.859 | 2.869 | 0.941 |
| | 0.20 | 4 | 0.004 | 2.857 | 2.872 | 0.941 | 0.002 | 3.080 | 3.083 | 0.946 | 0.004 | 2.857 | 2.872 | 0.941 |
| | 0.30 | 4 | 0.006 | 2.854 | 2.892 | 0.943 | 0.004 | 3.075 | 3.084 | 0.949 | 0.006 | 2.854 | 2.892 | 0.943 |
| | 0.05 | 8 | 0.002 | 2.830 | 2.836 | 0.942 | 0.002 | 3.123 | 3.124 | 0.945 | 0.002 | 2.830 | 2.836 | 0.942 |
| | 0.15 | 8 | 0.003 | 2.811 | 2.822 | 0.938 | 0.003 | 3.101 | 3.105 | 0.950 | 0.003 | 2.811 | 2.822 | 0.938 |
| | 0.20 | 8 | 0.004 | 2.809 | 2.826 | 0.939 | 0.003 | 3.098 | 3.103 | 0.949 | 0.004 | 2.809 | 2.826 | 0.939 |
| | 0.05 | 16 | 0.004 | 2.788 | 2.801 | 0.941 | 0.005 | 3.102 | 3.115 | 0.948 | 0.004 | 2.788 | 2.801 | 0.941 |
| | 0.15 | 16 | 0.004 | 2.783 | 2.802 | 0.939 | 0.006 | 3.094 | 3.110 | 0.947 | 0.004 | 2.783 | 2.802 | 0.939 |
| | 0.20 | 16 | 0.005 | 2.783 | 2.809 | 0.939 | 0.006 | 3.093 | 3.114 | 0.948 | 0.005 | 2.783 | 2.809 | 0.939 |

S1. SIMULATION CASE 3

Table S2: Average estimates using CMIPW and CR imputation under Case 3

| Method | | Sample Size n | | | | | | | | | | | | |
|--------|------|-----------------|----------------------|----------------------|-------|-------|----------------------|----------------------|-------|-------|----------------------|----------------------|-------|-------|
| | | 100 | | | | 500 | | | | 1000 | | | | |
| h | k | Bias | $n \cdot \text{VAR}$ | $n \cdot \text{MSE}$ | CCI | Bias | $n \cdot \text{VAR}$ | $n \cdot \text{MSE}$ | CCI | Bias | $n \cdot \text{VAR}$ | $n \cdot \text{MSE}$ | CCI | |
| CMIPW | 0.05 | 1 | -0.004 | 3.013 | 3.045 | 0.943 | -0.001 | 3.275 | 3.276 | 0.943 | 0.002 | 3.097 | 3.102 | 0.949 |
| | 0.15 | 1 | -0.007 | 2.965 | 3.053 | 0.941 | -0.003 | 3.243 | 3.277 | 0.939 | 0.002 | 3.041 | 3.045 | 0.945 |
| | 0.20 | 1 | -0.009 | 2.966 | 3.123 | 0.938 | -0.006 | 3.235 | 3.330 | 0.941 | 0.002 | 3.039 | 3.044 | 0.943 |
| | 0.05 | 2 | -0.004 | 2.911 | 2.941 | 0.946 | -0.001 | 3.208 | 3.210 | 0.943 | 0.002 | 3.014 | 3.017 | 0.941 |
| | 0.15 | 2 | -0.007 | 2.871 | 2.957 | 0.943 | -0.003 | 3.180 | 3.216 | 0.939 | 0.002 | 2.971 | 2.975 | 0.941 |
| | 0.20 | 2 | -0.009 | 2.875 | 3.030 | 0.935 | -0.006 | 3.172 | 3.272 | 0.941 | 0.002 | 2.969 | 2.973 | 0.941 |
| | 0.05 | 4 | -0.004 | 2.888 | 2.923 | 0.947 | -0.001 | 3.144 | 3.146 | 0.948 | 0.002 | 2.947 | 2.951 | 0.939 |
| | 0.15 | 4 | -0.007 | 2.856 | 2.948 | 0.941 | -0.003 | 3.120 | 3.156 | 0.934 | 0.002 | 2.896 | 2.899 | 0.939 |
| | 0.20 | 4 | -0.009 | 2.862 | 3.025 | 0.937 | -0.006 | 3.113 | 3.210 | 0.936 | 0.002 | 2.895 | 2.899 | 0.940 |
| | 0.05 | 8 | -0.004 | 2.878 | 2.912 | 0.942 | -0.001 | 3.123 | 3.125 | 0.949 | 0.002 | 2.902 | 2.906 | 0.940 |
| | 0.15 | 8 | -0.007 | 2.849 | 2.942 | 0.937 | -0.004 | 3.099 | 3.137 | 0.941 | 0.002 | 2.835 | 2.839 | 0.938 |
| | 0.20 | 8 | -0.009 | 2.857 | 3.021 | 0.936 | -0.006 | 3.091 | 3.193 | 0.933 | 0.002 | 2.833 | 2.837 | 0.938 |
| | 0.05 | 16 | -0.004 | 2.854 | 2.881 | 0.944 | -0.001 | 3.087 | 3.089 | 0.943 | 0.002 | 2.873 | 2.877 | 0.946 |
| | 0.15 | 16 | -0.006 | 2.838 | 2.919 | 0.943 | -0.003 | 3.068 | 3.104 | 0.941 | 0.002 | 2.798 | 2.802 | 0.939 |
| | 0.20 | 16 | -0.009 | 2.847 | 2.995 | 0.940 | -0.006 | 3.061 | 3.159 | 0.941 | 0.002 | 2.798 | 2.803 | 0.939 |
| | CR | 0.15 | 1 | 0.004 | 3.272 | 3.274 | 0.947 | -0.003 | 3.321 | 3.325 | 0.947 | -0.001 | 3.071 | 3.072 |
| 0.20 | | 1 | 0.003 | 3.252 | 3.253 | 0.948 | -0.005 | 3.308 | 3.320 | 0.946 | -0.003 | 3.079 | 3.090 | 0.948 |
| 0.30 | | 1 | -0.003 | 3.188 | 3.189 | 0.950 | -0.011 | 3.323 | 3.380 | 0.945 | -0.009 | 3.067 | 3.148 | 0.951 |
| 0.05 | | 2 | 0.010 | 3.154 | 3.164 | 0.948 | 0.001 | 3.269 | 3.269 | 0.948 | 0.001 | 3.064 | 3.066 | 0.945 |
| 0.20 | | 2 | 0.006 | 3.193 | 3.196 | 0.950 | -0.005 | 3.133 | 3.143 | 0.952 | -0.004 | 2.960 | 2.973 | 0.944 |
| 0.30 | | 2 | -0.000 | 3.149 | 3.149 | 0.954 | -0.010 | 3.153 | 3.205 | 0.952 | -0.009 | 2.946 | 3.032 | 0.943 |
| 0.15 | | 4 | 0.015 | 3.123 | 3.144 | 0.955 | -0.002 | 3.105 | 3.107 | 0.948 | -0.001 | 2.889 | 2.890 | 0.941 |
| 0.20 | | 4 | 0.014 | 3.134 | 3.153 | 0.954 | -0.004 | 3.093 | 3.102 | 0.948 | -0.003 | 2.899 | 2.909 | 0.939 |
| 0.30 | | 4 | 0.007 | 3.097 | 3.102 | 0.953 | -0.010 | 3.116 | 3.167 | 0.947 | -0.009 | 2.886 | 2.964 | 0.943 |
| 0.15 | | 8 | 0.030 | 3.117 | 3.206 | 0.952 | -0.001 | 3.107 | 3.108 | 0.951 | -0.001 | 2.835 | 2.835 | 0.942 |
| 0.20 | | 8 | 0.029 | 3.127 | 3.209 | 0.952 | -0.003 | 3.100 | 3.106 | 0.951 | -0.003 | 2.847 | 2.856 | 0.939 |
| 0.30 | | 8 | 0.022 | 3.088 | 3.137 | 0.951 | -0.009 | 3.127 | 3.167 | 0.951 | -0.009 | 2.836 | 2.911 | 0.946 |
| 0.15 | | 16 | 0.072 | 3.205 | 3.725 | 0.942 | 0.002 | 3.084 | 3.086 | 0.952 | 0.000 | 2.796 | 2.796 | 0.942 |
| 0.20 | | 16 | 0.071 | 3.198 | 3.695 | 0.941 | -0.000 | 3.077 | 3.077 | 0.954 | -0.002 | 2.810 | 2.814 | 0.946 |
| 0.30 | | 16 | 0.063 | 3.144 | 3.547 | 0.947 | -0.006 | 3.104 | 3.122 | 0.951 | -0.008 | 2.802 | 2.860 | 0.942 |

S2 Simulation Case 4

$$\text{Case 4} \left\{ \begin{array}{l} m_4(x) = 2x + 1, \quad x \in (-3, 4) \\ \sigma_4^2(x) = 1 \\ p_4(x) = \frac{e^{2.5x}}{1+e^{2.5x}} \\ X \sim 0.3U(-3, 0) + 0.7U(0, 4), \\ E(Y) = 2.90, \quad E(Y_{obs}) = 5.009, \quad P(\delta = 1) = 0.679, \quad \sigma_{KR}^2 = 88.275. \\ \sigma_{CM}^2 = 88.275 + 0.242 \times \frac{1}{k} \left(1 + \frac{1}{k}\right) \\ \sigma_{CR}^2 = 88.275 + 0.267 \times \frac{1}{k} \end{array} \right.$$

Table S3 lists calculated results of Case 4 using sample sizes 1000 to 3000.

S2. SIMULATION CASE 4

Table S3: Average estimates using all imputation estimators under simulation Case 4

| Method | | | Sample Size n | | | | | | | | | | | |
|---------|-----|---|-----------------|----------------------|----------------------|-------|-------|----------------------|----------------------|-------|-------|----------------------|----------------------|-------|
| | | | 1000 | | | | 2000 | | | | 3000 | | | |
| h | k | | Bias | $n \cdot \text{VAR}$ | $n \cdot \text{MSE}$ | CCI | Bias | $n \cdot \text{VAR}$ | $n \cdot \text{MSE}$ | CCI | Bias | $n \cdot \text{VAR}$ | $n \cdot \text{MSE}$ | CCI |
| KR | 0.8 | - | 0.579 | 118.789 | 453.643 | 0.585 | 0.446 | 149.409 | 547.509 | 0.609 | 0.350 | 168.826 | 537.329 | 0.682 |
| | 1.0 | - | 0.536 | 99.137 | 386.341 | 0.591 | 0.420 | 110.974 | 464.544 | 0.589 | 0.346 | 109.398 | 468.950 | 0.613 |
| IPW | 1.0 | - | 0.440 | 109.245 | 302.776 | 0.718 | 0.315 | 128.036 | 326.155 | 0.749 | 0.234 | 130.136 | 294.553 | 0.790 |
| | 1.4 | - | 0.388 | 68.301 | 218.457 | 0.697 | 0.316 | 61.803 | 261.625 | 0.604 | 0.281 | 55.096 | 291.600 | 0.478 |
| k -NN | - | 1 | 0.186 | 57.295 | 92.070 | 0.878 | 0.139 | 68.908 | 107.469 | 0.872 | 0.103 | 77.745 | 109.745 | 0.909 |
| | - | 2 | 0.240 | 42.569 | 100.386 | 0.781 | 0.178 | 52.970 | 116.515 | 0.809 | 0.138 | 58.590 | 115.358 | 0.837 |
| CM | 0.2 | 1 | 0.188 | 57.189 | 92.394 | 0.874 | 0.140 | 68.687 | 107.909 | 0.868 | 0.104 | 77.491 | 110.164 | 0.907 |
| | 0.4 | 1 | 0.191 | 57.112 | 93.659 | 0.871 | 0.144 | 68.579 | 109.853 | 0.863 | 0.108 | 77.394 | 112.388 | 0.906 |
| | 0.2 | 2 | 0.241 | 42.565 | 100.730 | 0.781 | 0.179 | 52.915 | 117.160 | 0.807 | 0.139 | 58.559 | 116.144 | 0.831 |
| CMIPW | 0.4 | 2 | 0.245 | 42.504 | 102.365 | 0.778 | 0.183 | 52.827 | 119.675 | 0.803 | 0.142 | 58.488 | 119.124 | 0.826 |
| | 0.2 | 1 | 0.187 | 57.202 | 92.023 | 0.874 | 0.139 | 68.710 | 107.307 | 0.871 | 0.103 | 77.515 | 109.478 | 0.909 |
| | 0.8 | 1 | 0.190 | 57.038 | 92.966 | 0.871 | 0.142 | 68.457 | 108.767 | 0.864 | 0.106 | 77.252 | 111.145 | 0.906 |
| CR | 0.2 | 2 | 0.230 | 44.343 | 97.345 | 0.803 | 0.169 | 55.232 | 112.230 | 0.830 | 0.128 | 61.281 | 110.294 | 0.849 |
| | 0.8 | 2 | 0.218 | 48.868 | 96.328 | 0.837 | 0.160 | 59.520 | 110.990 | 0.847 | 0.121 | 65.454 | 109.098 | 0.865 |
| | 0.8 | 1 | 0.132 | 63.452 | 80.797 | 0.913 | 0.075 | 78.328 | 89.472 | 0.918 | 0.033 | 90.690 | 93.948 | 0.947 |
| CR | 1.0 | 1 | 0.112 | 66.217 | 78.664 | 0.922 | 0.053 | 83.397 | 89.012 | 0.933 | 0.010 | 95.601 | 95.905 | 0.955 |
| | 1.0 | 2 | 0.165 | 49.747 | 76.912 | 0.888 | 0.092 | 65.304 | 82.236 | 0.917 | 0.044 | 73.405 | 79.241 | 0.942 |
| | 1.4 | 2 | 0.144 | 53.990 | 74.619 | 0.901 | 0.072 | 70.533 | 80.865 | 0.930 | 0.024 | 79.968 | 81.766 | 0.951 |

S3 On Wine Quality Data

S3.1 Modified weights of weighted k -NN estimators

Definition: Weighted k -NN and CM estimator for tolerance interval $[0, T]$.

Let uniform k -NN weights, $\frac{1}{k}$, be replaced with a triangular set of weights:

$k = 2$, weights are 0.5 for each of two nearest neighbors (NN);

$k = 4$, center 2 NN have weight $\frac{3}{4}$; outer 2 NN use the remainig $\frac{1}{4}$.

$k = 8$, center 2 NN use weight $\frac{1}{2}$; next 2 NN use weight $\frac{1}{4}$, outer 4 NN use weight $\frac{1}{4}$.

$k = 16$, center 2 NN use weight $\frac{3}{8}$; next 2 NN use weight $\frac{2}{8}$; then, next 4 NN use weight $\frac{2}{8}$; the outer 8 NN use weight $\frac{1}{8}$.

$k = 32$, center 2 NN use weight $\frac{1}{4}$, next 2 NN use weight $\frac{3}{16}$; further 4 NN use weight $\frac{3}{16}$; then, next 8 NN use weight $\frac{3}{16}$; outer 16 NN also use weight $\frac{3}{16}$.

S3.2 Descriptions and computations of wine quality data

S3. ON WINE QUALITY DATA7

Table S4: Description of red and white wine data (Cortez et al., 2009, Table 1)

| Attribute | Red wine | | | White wine | | |
|----------------------|----------|-------|-------|------------|-------|-------|
| | Min | Max | Mean | Min | Max | Mean |
| Fixed acidity | 4.6 | 15.9 | 8.3 | 3.8 | 14.2 | 6.9 |
| Volatile acidity | 0.1 | 1.6 | 0.5 | 0.1 | 1.1 | 0.3 |
| Citric acid | 0.0 | 1.0 | 0.3 | 0.0 | 1.7 | 0.3 |
| Residual sugar | 0.9 | 15.5 | 2.5 | 0.6 | 65.8 | 6.4 |
| Chlorides | 0.01 | 0.61 | 0.08 | 0.01 | 0.35 | 0.05 |
| Free sulfur dioxide | 1 | 72 | 14 | 2 | 289 | 35 |
| Total sulfur dioxide | 6 | 289 | 46 | 9 | 440 | 138 |
| Density | 0.990 | 1.004 | 0.996 | 0.987 | 1.039 | 0.994 |
| pH | 2.7 | 4.0 | 3.3 | 2.7 | 3.8 | 3.1 |
| Sulphates | 0.3 | 2.0 | 0.7 | 0.2 | 1.1 | 0.5 |
| Alcohol | 8.4 | 14.9 | 10.4 | 8.0 | 14.2 | 10.4 |

Table S5: MAD(se) and PA(T)(se) of k -NN, KR prediction for red wine

| Method | k/h | MAD | PA($T = 0.25$) | PA($T = 0.5$) | PA($T = 1.0$) |
|---------------------|-----|--------------|------------------|-----------------|-----------------|
| Weighted k -NN | 1 | 0.555(0.025) | 0.534(0.018) | 0.534(0.018) | 0.921(0.010) |
| | 2 | 0.559(0.018) | 0.332(0.019) | 0.691(0.017) | 0.904(0.011) |
| | 4 | 0.551(0.014) | 0.370(0.019) | 0.596(0.017) | 0.876(0.011) |
| | 8 | 0.540(0.012) | 0.338(0.018) | 0.583(0.015) | 0.876(0.010) |
| | 16 | 0.538(0.011) | 0.294(0.019) | 0.568(0.016) | 0.875(0.009) |
| | 32 | 0.537(0.011) | 0.254(0.021) | 0.569(0.016) | 0.879(0.009) |
| KR | 0.1 | 0.627(0.011) | 0.105(0.009) | 0.458(0.012) | 0.833(0.009) |
| | 0.5 | 0.615(0.012) | 0.127(0.010) | 0.468(0.012) | 0.836(0.009) |
| | 1.0 | 0.594(0.013) | 0.216(0.014) | 0.498(0.014) | 0.849(0.009) |
| | 1.5 | 0.567(0.015) | 0.287(0.017) | 0.532(0.016) | 0.870(0.010) |
| | 1.8 | 0.559(0.014) | 0.258(0.018) | 0.543(0.016) | 0.877(0.010) |
| | 2.0 | 0.557(0.013) | 0.235(0.019) | 0.554(0.016) | 0.877(0.010) |
| | 2.2 | 0.554(0.012) | 0.216(0.018) | 0.560(0.016) | 0.878(0.010) |
| | 2.5 | 0.551(0.011) | 0.202(0.017) | 0.568(0.016) | 0.880(0.009) |

S3. ON WINE QUALITY DATA₉

Table S6: MAD(se) and PA(T)(se) of CM prediction for red wine

| | h\k | 1 | 2 | 4 | 8 | 16 | 32 |
|----------------------|-----|--------------|--------------|--------------|--------------|--------------|--------------|
| MAD | 0.1 | 0.555(0.025) | 0.546(0.018) | 0.536(0.015) | 0.522(0.012) | 0.518(0.012) | 0.516(0.011) |
| | 0.5 | 0.555(0.025) | 0.546(0.018) | 0.534(0.015) | 0.520(0.012) | 0.516(0.012) | 0.514(0.011) |
| | 1.0 | 0.559(0.025) | 0.549(0.018) | 0.538(0.015) | 0.523(0.012) | 0.518(0.012) | 0.515(0.011) |
| | 1.5 | 0.562(0.023) | 0.555(0.017) | 0.545(0.014) | 0.533(0.012) | 0.529(0.011) | 0.526(0.011) |
| | 1.8 | 0.560(0.021) | 0.555(0.016) | 0.545(0.014) | 0.536(0.012) | 0.533(0.011) | 0.532(0.011) |
| | 2.0 | 0.558(0.020) | 0.554(0.015) | 0.545(0.013) | 0.537(0.012) | 0.535(0.011) | 0.535(0.011) |
| | 2.2 | 0.557(0.019) | 0.553(0.015) | 0.545(0.013) | 0.538(0.011) | 0.537(0.011) | 0.537(0.011) |
| | 2.5 | 0.555(0.019) | 0.552(0.014) | 0.544(0.012) | 0.539(0.011) | 0.538(0.011) | 0.539(0.010) |
| PA ($T = 0.25$) | 0.1 | 0.534(0.018) | 0.365(0.019) | 0.390(0.018) | 0.364(0.018) | 0.324(0.018) | 0.290(0.020) |
| | 0.5 | 0.533(0.018) | 0.364(0.019) | 0.390(0.018) | 0.367(0.018) | 0.328(0.018) | 0.295(0.020) |
| | 1.0 | 0.525(0.018) | 0.347(0.019) | 0.372(0.018) | 0.354(0.018) | 0.324(0.019) | 0.297(0.020) |
| | 1.5 | 0.498(0.018) | 0.326(0.019) | 0.329(0.018) | 0.314(0.018) | 0.297(0.020) | 0.276(0.020) |
| | 1.8 | 0.482(0.018) | 0.325(0.019) | 0.312(0.018) | 0.293(0.019) | 0.278(0.020) | 0.257(0.021) |
| | 2.0 | 0.474(0.018) | 0.324(0.018) | 0.307(0.018) | 0.283(0.019) | 0.269(0.020) | 0.247(0.021) |
| | 2.2 | 0.469(0.018) | 0.326(0.018) | 0.303(0.018) | 0.277(0.019) | 0.261(0.021) | 0.238(0.022) |
| | 2.5 | 0.464(0.018) | 0.327(0.019) | 0.298(0.018) | 0.268(0.020) | 0.252(0.021) | 0.229(0.022) |
| PA ($T = 0.5$) | 0.1 | 0.534(0.018) | 0.696(0.017) | 0.613(0.017) | 0.603(0.016) | 0.591(0.016) | 0.595(0.016) |
| | 0.5 | 0.534(0.018) | 0.694(0.017) | 0.615(0.017) | 0.605(0.016) | 0.593(0.016) | 0.596(0.016) |
| | 1.0 | 0.533(0.018) | 0.655(0.017) | 0.599(0.017) | 0.595(0.016) | 0.588(0.016) | 0.593(0.016) |
| | 1.5 | 0.534(0.018) | 0.582(0.017) | 0.567(0.017) | 0.571(0.016) | 0.571(0.015) | 0.576(0.015) |
| | 1.8 | 0.537(0.017) | 0.570(0.017) | 0.561(0.017) | 0.565(0.015) | 0.566(0.016) | 0.571(0.015) |
| | 2.0 | 0.538(0.017) | 0.568(0.016) | 0.560(0.016) | 0.565(0.015) | 0.566(0.015) | 0.570(0.015) |
| | 2.2 | 0.540(0.017) | 0.568(0.016) | 0.561(0.016) | 0.565(0.015) | 0.566(0.015) | 0.569(0.015) |
| | 2.5 | 0.542(0.017) | 0.568(0.016) | 0.562(0.016) | 0.566(0.015) | 0.567(0.016) | 0.569(0.015) |
| PA ($T = 1.0$) | 0.1 | 0.921(0.010) | 0.904(0.010) | 0.878(0.011) | 0.880(0.010) | 0.880(0.009) | 0.885(0.009) |
| | 0.5 | 0.921(0.010) | 0.904(0.010) | 0.878(0.011) | 0.880(0.010) | 0.880(0.009) | 0.886(0.009) |
| | 1.0 | 0.912(0.011) | 0.896(0.011) | 0.875(0.011) | 0.878(0.010) | 0.879(0.010) | 0.885(0.009) |
| | 1.5 | 0.885(0.011) | 0.879(0.011) | 0.870(0.011) | 0.874(0.010) | 0.877(0.010) | 0.882(0.009) |
| | 1.8 | 0.878(0.011) | 0.874(0.011) | 0.869(0.011) | 0.875(0.010) | 0.878(0.010) | 0.883(0.009) |
| | 2.0 | 0.873(0.011) | 0.872(0.011) | 0.869(0.011) | 0.875(0.010) | 0.878(0.009) | 0.882(0.009) |
| | 2.2 | 0.871(0.011) | 0.871(0.011) | 0.869(0.011) | 0.875(0.010) | 0.878(0.009) | 0.881(0.009) |
| | 2.5 | 0.868(0.011) | 0.869(0.011) | 0.868(0.011) | 0.875(0.010) | 0.877(0.009) | 0.880(0.009) |

Table S7: MAD(se) and PA(T)(se) of k -NN, KR prediction for white wine

| Method | k/h | MAD | PA($T = 0.25$) | PA($T = 0.5$) | PA($T = 1.0$) |
|---------------------|-----|--------------|------------------|-----------------|-----------------|
| Weighted k -NN | 1 | 0.658(0.024) | 0.469(0.014) | 0.469(0.014) | 0.891(0.010) |
| | 2 | 0.642(0.015) | 0.285(0.012) | 0.637(0.013) | 0.869(0.009) |
| | 4 | 0.628(0.012) | 0.327(0.013) | 0.542(0.012) | 0.831(0.009) |
| | 8 | 0.610(0.010) | 0.305(0.013) | 0.531(0.011) | 0.829(0.007) |
| | 16 | 0.604(0.010) | 0.279(0.013) | 0.516(0.011) | 0.826(0.007) |
| | 32 | 0.600(0.009) | 0.261(0.014) | 0.515(0.011) | 0.830(0.006) |
| KR | 0.1 | 0.660(0.008) | 0.505(0.006) | 0.505(0.006) | 0.731(0.005) |
| | 0.5 | 0.651(0.008) | 0.511(0.006) | 0.512(0.006) | 0.736(0.005) |
| | 1.0 | 0.633(0.009) | 0.509(0.007) | 0.521(0.007) | 0.766(0.006) |
| | 1.5 | 0.636(0.011) | 0.406(0.010) | 0.497(0.010) | 0.813(0.007) |
| | 1.8 | 0.633(0.011) | 0.330(0.011) | 0.494(0.010) | 0.821(0.008) |
| | 2.0 | 0.627(0.011) | 0.292(0.012) | 0.493(0.011) | 0.828(0.007) |
| | 2.2 | 0.622(0.010) | 0.267(0.013) | 0.493(0.010) | 0.831(0.007) |
| | 2.5 | 0.617(0.009) | 0.245(0.013) | 0.490(0.009) | 0.832(0.006) |

S3. ON WINE QUALITY DATA₁₁

Table S8: MAD(se) and PA(T)(se) of CM prediction for white wine

| | h\k | 1 | 2 | 4 | 8 | 16 | 32 |
|----------------------|-----|--------------|--------------|--------------|--------------|--------------|--------------|
| MAD | 0.1 | 0.658(0.024) | 0.634(0.015) | 0.618(0.012) | 0.597(0.011) | 0.590(0.010) | 0.584(0.009) |
| | 0.5 | 0.659(0.024) | 0.634(0.015) | 0.617(0.012) | 0.596(0.011) | 0.589(0.010) | 0.583(0.009) |
| | 1.0 | 0.661(0.024) | 0.636(0.015) | 0.619(0.012) | 0.598(0.011) | 0.590(0.010) | 0.584(0.009) |
| | 1.5 | 0.664(0.023) | 0.641(0.015) | 0.625(0.012) | 0.605(0.010) | 0.598(0.010) | 0.592(0.009) |
| | 1.8 | 0.663(0.022) | 0.641(0.014) | 0.626(0.012) | 0.608(0.010) | 0.602(0.010) | 0.597(0.009) |
| | 2.0 | 0.662(0.022) | 0.641(0.014) | 0.626(0.012) | 0.609(0.010) | 0.604(0.010) | 0.600(0.009) |
| | 2.2 | 0.661(0.021) | 0.640(0.014) | 0.625(0.012) | 0.610(0.010) | 0.605(0.010) | 0.602(0.009) |
| | 2.5 | 0.660(0.021) | 0.640(0.013) | 0.625(0.011) | 0.610(0.010) | 0.606(0.009) | 0.605(0.009) |
| PA ($T = 0.25$) | 0.1 | 0.469(0.014) | 0.306(0.012) | 0.339(0.013) | 0.321(0.013) | 0.297(0.013) | 0.282(0.014) |
| | 0.5 | 0.469(0.014) | 0.304(0.012) | 0.338(0.013) | 0.321(0.013) | 0.298(0.013) | 0.283(0.014) |
| | 1.0 | 0.465(0.014) | 0.295(0.012) | 0.330(0.013) | 0.313(0.013) | 0.294(0.013) | 0.281(0.014) |
| | 1.5 | 0.450(0.014) | 0.281(0.012) | 0.300(0.012) | 0.290(0.013) | 0.280(0.013) | 0.271(0.014) |
| | 1.8 | 0.438(0.014) | 0.278(0.012) | 0.284(0.012) | 0.279(0.013) | 0.272(0.013) | 0.263(0.014) |
| | 2.0 | 0.429(0.014) | 0.274(0.012) | 0.275(0.013) | 0.271(0.013) | 0.264(0.013) | 0.255(0.014) |
| | 2.2 | 0.419(0.014) | 0.269(0.012) | 0.267(0.012) | 0.263(0.013) | 0.259(0.013) | 0.249(0.014) |
| | 2.5 | 0.403(0.014) | 0.259(0.012) | 0.257(0.012) | 0.254(0.013) | 0.251(0.013) | 0.243(0.014) |
| PA ($T = 0.5$) | 0.1 | 0.469(0.014) | 0.641(0.013) | 0.555(0.012) | 0.547(0.011) | 0.533(0.011) | 0.533(0.011) |
| | 0.5 | 0.469(0.014) | 0.638(0.013) | 0.555(0.012) | 0.547(0.012) | 0.534(0.011) | 0.534(0.011) |
| | 1.0 | 0.469(0.014) | 0.617(0.013) | 0.546(0.012) | 0.541(0.012) | 0.531(0.011) | 0.531(0.011) |
| | 1.5 | 0.468(0.014) | 0.550(0.012) | 0.519(0.012) | 0.521(0.011) | 0.519(0.011) | 0.522(0.011) |
| | 1.8 | 0.466(0.014) | 0.519(0.012) | 0.507(0.012) | 0.513(0.011) | 0.514(0.011) | 0.517(0.011) |
| | 2.0 | 0.466(0.014) | 0.509(0.012) | 0.504(0.012) | 0.510(0.011) | 0.512(0.011) | 0.515(0.011) |
| | 2.2 | 0.467(0.014) | 0.504(0.012) | 0.502(0.012) | 0.508(0.011) | 0.511(0.011) | 0.513(0.011) |
| | 2.5 | 0.467(0.014) | 0.501(0.012) | 0.501(0.012) | 0.508(0.011) | 0.510(0.011) | 0.510(0.011) |
| PA ($T = 1.0$) | 0.1 | 0.891(0.010) | 0.870(0.009) | 0.832(0.009) | 0.832(0.007) | 0.830(0.007) | 0.836(0.006) |
| | 0.5 | 0.890(0.010) | 0.869(0.009) | 0.832(0.009) | 0.832(0.007) | 0.831(0.007) | 0.837(0.006) |
| | 1.0 | 0.886(0.010) | 0.866(0.009) | 0.831(0.009) | 0.832(0.007) | 0.830(0.007) | 0.837(0.006) |
| | 1.5 | 0.860(0.010) | 0.845(0.010) | 0.823(0.009) | 0.827(0.007) | 0.828(0.007) | 0.834(0.006) |
| | 1.8 | 0.839(0.011) | 0.831(0.010) | 0.819(0.009) | 0.823(0.008) | 0.825(0.007) | 0.831(0.006) |
| | 2.0 | 0.832(0.010) | 0.827(0.010) | 0.817(0.009) | 0.821(0.007) | 0.824(0.007) | 0.830(0.006) |
| | 2.2 | 0.825(0.010) | 0.824(0.009) | 0.815(0.009) | 0.821(0.007) | 0.824(0.007) | 0.830(0.006) |
| | 2.5 | 0.819(0.009) | 0.819(0.008) | 0.814(0.008) | 0.821(0.007) | 0.825(0.007) | 0.830(0.006) |