Supplementary Document for Asymptotic Overshoot for Arithmetic IID Random Variables

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5 appendix

5.1 4 Lemmas

Lemma 1 and Lemma 2 are used to prove Theorem 1, Lemma 3 and Lemma 4 are used to prove Theorem 2 and Lemma 2 is used to prove Theorem 3.

Lemma 1 Let x_1, x_2, $\cdots$ be iid random variables with $\mu = E(x_1) > 0$. Then

$$\frac{1 - E(\exp(-\alpha S_{\tau_+}))}{E(S_{\tau_+})} = \frac{1}{\mu} \exp\left(-\sum_{k=1}^{\infty} \frac{1}{k} E(e^{-\alpha S_k^+}) \right),$$

where where $S_k^+ = \max\{0, S_k\}$.

Proof of Lemma 1: Spitzer (1960) provided equations (13) and (14). Let x_1, x_2,... be iid random variables, for $\alpha \geq 0$ and $\mu = E(x_i) \geq 0$, then

$$E(e^{-\alpha S_{\tau_+}}) = 1 - \exp\left(-\sum_{k=1}^{\infty} \frac{1}{k} E[e^{-\alpha S_k}; S_k > 0] \right), \tag{13}$$

For the same condition except that $\mu > 0$, then

$$E(S_{\tau_+}) = \mu \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} P(S_k \leq 0) \right). \tag{14}$$

By combining equations (13) and (14), Lemma 1 is proven.

Lemma 2 For $\alpha > 0$, $\beta > 0$ and $h > 0$, then

(a): $$\int_{-\pi/h}^{\pi/h} \frac{\exp(-\alpha h - iht)}{1 - \exp(-\alpha h - iht)} dt = 0$$

(b): $$\frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} \frac{1}{1 - \exp(-\beta h + iht)} dt = 1$$

(c): $$\frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} \frac{\exp(-\alpha h - iht) \log(1 - \exp(iht))}{1 - \exp(-\alpha h - iht)} dt = \log(1 - e^{-\alpha h})$$

(d): $$\int_{-\pi/h}^{\pi/h} \frac{\log(1 - \exp(iht))}{1 - \exp(-\beta h + iht)} dt = 0$$

Proof of Lemma 2

Let $z = e^{-iht}$, then

$$\int_{-\pi/h}^{\pi/h} \frac{\exp(-\alpha h - iht)}{1 - \exp(-\alpha h - iht)} dt = \frac{1}{-ih} \oint_{|z|=1} \frac{e^{-\alpha h} z}{(1 - e^{-\alpha h} z) z} dz = 0,$$

because the residue is at $z = e^{\alpha h}$ outside the circle $r = 1$. This proves equation (a).

To prove equation (b), let $z = e^{iht}$. Then

$$\frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} \frac{1}{1 - \exp(-\beta h + iht)} dt = \frac{h}{2\pi i h} \oint_{|z|=1} \frac{dz}{(1 - e^{-\alpha h} z) z} = \frac{2\pi i}{2\pi i} = 1,$$

because there exist one residue $z = 0$ inside the circle $r = 1$.

Lemma 3 is from Siegmund (1985, chap 8).

Lemma 4 For $\beta > 0$ and $h > 0$, then

$$\text{(a):} \qquad \int_0^{2\pi/h} \frac{\exp(-\beta h + iht)}{(1-\exp(-\beta h + iht))^2} dt = 0$$

$$\text{(b):} \qquad \int_0^{2\pi/h} \frac{\exp(-\beta h + iht)(1-\exp(iht))}{(1-\exp(-\beta h + iht))^2} dt = 0$$

$$\text{(c):} \qquad \int_0^{2\pi/h} \frac{\exp(-\beta h + iht)\log(1-\exp(iht))}{(1-\exp(-\beta h + iht))^2} dt = 0$$

Proof: Let $z = \exp(-\beta h + iht)$, then equations (a) and (b) can be shown by doing the complex integrals straight forward. Equation (a) becomes

$$\frac{1}{ih} \oint_{|z|=e^{-\beta h}} \frac{dz}{(1-z)^2} = 0,$$

and equation (b) becomes

$$\frac{1}{ih} \oint_{|z|=e^{-\beta h}} \frac{1-e^{\beta h}z}{(1-z)^2} dz = 0,$$

because the residues in both cases are not inside the circle $|z| = e^{-\beta h}$.

To prove equation (c), $f(r)$ is defined as:

$$f(r) = \int_0^{2\pi/h} \frac{\exp(-\beta h + iht)\log(1-\exp(-r + iht))}{(1-\exp(-\beta h + iht))^2} dt.$$

Then $f(\infty) = 0$ and

$$f'(r) = \int_0^{2\pi/h} \frac{\exp(-\beta h + iht)}{(1-\exp(-\beta h + iht))^2} \frac{\exp(-r + iht)}{(1-\exp(-r + iht))} dt.$$

Let $z = e^{iht}$, then

$$f'(r) = \frac{1}{ih} \oint_{|z|=1} \frac{(e^{-\beta h - r}z)dz}{(1-e^{-r}z)(1-e^{-\beta h}z)^2} = 0, \text{ iff } r > 0.$$

So $f(r) = 0$ for $r > 0$. As long as $\beta > 0$, $f(r)$ is continuous at $r = 0$ from right. So, $f(0) = f(\infty) = 0$, and the proof is done.

5.2 Proof of Theorem 1

By Lemma 1, the goal is to calculate the term $\sum_{k=1}^{\infty} \frac{1}{k} E(e^{-\alpha S_k^+})$ to get the proof.

$$\begin{aligned}
& \sum_{k=1}^{\infty} \frac{1}{k} E(e^{-\alpha S_k^+}) \\
= & \sum_{k=1}^{\infty} \frac{1}{k} \left[\sum_{j=1}^{\infty} e^{-\alpha jh} P(S_k = jh) + \lim_{\beta \downarrow 0} \sum_{j=0}^{\infty} e^{-\beta jh} P(S_k = -jh) \right] && (15) \\
= & \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} dt \frac{\xi(t) e^{-\alpha h - iht}}{1 - e^{-\alpha h - iht}} + \lim_{\beta \downarrow 0} \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} dt \frac{\xi(t)}{1 - e^{-\beta h + iht}} && (16) \\
= & \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} dt \left[\frac{\left(\xi(t) + \log \frac{\mu}{h} + \log(1 - e^{iht})\right) e^{-\alpha h - iht}}{1 - e^{-\alpha h - iht}} \right] - \log(1 - e^{-\alpha h}) \\
& + \lim_{\beta \downarrow 0} \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} dt \left[\frac{(\xi(t) + \log \frac{\mu}{h} + \log(1 - e^{iht}))}{1 - e^{-\beta h + iht}} \right] - \log \frac{\mu}{h} && (17) \\
= & \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} dt [\left(\xi(t) + \log(\mu(1 - e^{ith})/h)\right) (\frac{e^{-\alpha h - iht}}{1 - e^{-\alpha h - iht}} + \frac{1}{1 - e^{iht}})] + \log \frac{h/\mu}{1 - e^{-\alpha h}} \\
& && (18)
\end{aligned}$$

From equation (15) to equation (16), the following two equations are used.

$$P(S_k = jh) = \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} dt \phi^k(t) e^{-ijht},$$

because S_k is an iid sum and the definition; and

$$\xi(t) = \sum_{k=1}^{\infty} \frac{\phi^k(t)}{k} = -\log(1 - \phi(t)).$$

From equation (16) to equation (17), Lemma 2 is used. From equation (17) to equation (18), the Dominant Convergent Theorem is applied to exchange the orders of the limit and the integral. The integral region of the second integral of equation (17) is split into two parts: $|t| < \epsilon$ and $\epsilon \leq |t| \leq \pi/h$, where ϵ is positive and much smaller than π/h ($0 < \epsilon << \pi/h$, $<<$ means 'much smaller than') such that Taylor expansion over t can be done in the region $|t| < \epsilon$. The integrated function in the region $\epsilon \leq |t| \leq \pi/h$ is bounded by an integrable function such that the Dominant Convergent Theorem can be applied. For the region $|t| < \epsilon$, Taylor expansion is used for each function: $\xi(t) = -\log(-i\mu t) - i\frac{Ex_1^2}{2\mu} t + O(t^2)$, $\log(1 - e^{iht}) = \log(-iht) + iht/2 + O(t^2)$ such that the numerator is $-i\frac{(Ex_1^2/\mu - h)}{2} t + O(t^2)$. $|\frac{1}{1 - e^{-\beta h + iht}}| = |\frac{1}{(1 - e^{-\beta h}) - ihe^{-\beta h} t + O(t^2)}| \leq \frac{1}{ht + O(t^2)}|$. Therefore $[\frac{(\xi(t) + \log \frac{\mu}{h} + \log(1 - e^{iht}))}{1 - e^{-\beta h + iht}}] = O(t)$ and the Dominant Convergent Theorem can be applied to get equation (18).

Proof of Corollary 1 The integral region is split into four parts:$I_1 = \{t : -\pi/h \le t < -\epsilon/h\}$, $I_2 = \{t : -\epsilon/h \le t < 0\}$, $I_3 = \{t : 0 \le t < \epsilon/h\}$, and $I_4 = \{t : \epsilon/h \le \pi/h < 0\}$, where $0 < \epsilon << 1$. Because $\limsup |\phi(t)| < 1$, so $\limsup |\xi(t)|$ is bounded. Furthermore $\log(\mu(1-e^{ith})/h)$, $\frac{e^{-\alpha h - iht}}{1-e^{-\alpha h - iht}}$ and $\frac{1}{1-e^{iht}}$ are all bounded in the regions I_1 and I_4. In Theorem 1, it is shown that

$$G(t) = -\frac{h}{2\pi}[(\xi(t) + \log(\mu(1-e^{ith})/h))\,(\frac{e^{-\alpha h - iht}}{1-e^{-\alpha h - iht}} + \frac{1}{1-e^{iht}})] + \log\frac{h/\mu}{1-e^{-\alpha h}}$$

is an integrable function on $[-\pi/h, \pi/h]$. So

$$\lim_{h\downarrow 0}\int_{I_1+I_4} dt\ G(t) = \int_{-\infty}^{-\infty} dt\ G(t) + \int_{\infty}^{\infty} dt\ G(t) = 0.$$

For regions I_2 and I_3, and $\alpha h << 1$, given any $\alpha > 0$, Taylor expansion is allpied on $\log(\mu(1-e^{ith})/h)$, $\frac{e^{-\alpha h - iht}}{1-e^{-\alpha h - iht}}$ and $\frac{1}{1-e^{iht}}$. By the definition of $\xi(t) = -\log(1-\phi(t))$, it is known that $\Re(\xi(-t)) = \Re(\xi(t))$, and $\Im(\xi(-t) = -\Im(\xi(t))$. This lets the last equality of the following equations hold.

$$\begin{aligned}
& \lim_{\epsilon\downarrow 0}\lim_{h\downarrow 0}\int_{I_2+I_3} dt\ G(t) \\
= & -\lim_{\epsilon\downarrow 0}\lim_{h\downarrow 0}\int_{-\epsilon/h}^{\epsilon/h} dt\frac{1}{2\pi}[\xi(t) + \log(-i\mu t) + O(ht)](\frac{1}{\alpha+it} + \frac{i}{t})(1+O(ht)) \\
= & -\lim_{\epsilon\downarrow 0}\int_{-\infty}^{\infty} dt\frac{1}{2\pi}\left[(\Re(\xi(t)) + \log(\mu|t|))\frac{\alpha}{\alpha^2+t^2} - (\Im(\xi(t)) - sign(t)\pi/2)(\frac{-t}{\alpha^2+t^2} + \frac{1}{t})\right](1+O(\epsilon)) \\
= & \frac{1}{\pi}\int_0^{\infty}[\frac{\alpha^2}{\alpha^2+t^2}\frac{\Im(\xi(t)) - \pi/2}{t} - \frac{\alpha}{\alpha^2+t^2}(\Re(\xi(t)) + \log(\mu t))]dt.
\end{aligned}$$

5.3 Proof of Theorem 2

Proof: Lemma 3 provided a formula to compute.

$$
\begin{aligned}
& \sum_{n=1}^{\infty} \frac{E(S_n^-)}{n} \\
= & \lim_{\beta \downarrow 0} \frac{\partial}{\partial \beta} \sum_{n=1}^{\infty} \frac{E(e^{\beta S_n^-})}{n} \\
= & \lim_{\beta \downarrow 0} \frac{\partial}{\partial \beta} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} \frac{1}{n} e^{-\beta jh} P(S_n = -jh) \\
= & \lim_{\beta \downarrow 0} \frac{\partial}{\partial \beta} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} dt \frac{\phi^n(t)}{n} e^{itjh - \beta jh} \\
= & -\lim_{\beta \downarrow 0} \frac{h^2}{2\pi} \int_{-\pi/h}^{\pi/h} dt \frac{\xi(t) e^{-\beta h + iht}}{(1 - e^{-\beta h + iht})^2} \qquad (19) \\
= & -\lim_{\beta \downarrow 0} \frac{h^2}{2\pi} \int_{-\pi/h}^{\pi/h} dt \frac{\left(\xi(t) + \log(\frac{\mu}{h}) + \log(1 - e^{iht}) - \frac{E(x_1^2) - h\mu}{2\mu h}(1 - e^{iht})\right) e^{-\beta h + iht}}{(1 - e^{-\beta h + iht})^2} \qquad (20) \\
= & -\frac{h^2}{2\pi} \int_{-\pi/h}^{\pi/h} dt \frac{\Re(\xi(t)) + \log\left(\frac{\mu}{h}\right) + \log\left(2|\sin\frac{ht}{2}|\right) - \frac{E(x_1^2) - h\mu}{2\mu h}(1 - \cos(ht))}{2(\cos(ht) - 1)} \qquad (21) \\
= & -\frac{E(x_1^2)}{4E(x_1)} + \frac{h}{4} - \frac{h^2}{2\pi} \int_{-\pi/h}^{\pi/h} dt \frac{\Re(\xi(t)) + \log\left(\frac{\mu}{h}\right) + \log\left(2|\sin\frac{ht}{2}|\right)}{2(\cos(ht) - 1)}. \qquad (22)
\end{aligned}
$$

From (19) to (20), Lemma 4 is used. From (20) to (21), the Dominant Convergent Theorem is applied and the the equation $\frac{e^{iht}}{(1-e^{iht})^2} = \frac{1}{2(\cos(ht)-1)}$ which is a real number, so only the real part of the rest of the function is considered. From (21) to (22), simple algebra is used. To apply the Dominant Convergent Theorem, the integral region of (20) is split into two parts like the proof for Theorem 1: $\int_{|t|<\epsilon}$ and $\int_{\epsilon \le |t| < \pi/h}$, where $0 < \epsilon << \pi/h$. For the region $\epsilon \le |t| < \pi/h$, the function is bounded by an integrable function. For the region $|t| << \epsilon$, Taylor expansion is used to get the order of the function:

$$\xi(t) + \left(\log\left(\frac{\mu}{h}\right) + \log\left(1 - e^{iht}\right) - \frac{E(x_1^2) - h\mu}{2\mu h}\left(1 - e^{iht}\right)\right) = O(t^2),$$

and $(1 - e^{-\beta h + iht})^2$ is of the order $O(t^2)$ when $\beta = 0$. So the Dominant Convergent Theorem can be applied.

From Lemma 3, it can be written

$$
\begin{aligned}
\frac{E(S_{\tau_+}^2)}{2E(S_{\tau_+})} &= \frac{E(x_1^2)}{2E(x_1)} + \sum_{n=1}^{\infty} \frac{E(S_n^-)}{n} \\
&= \frac{E(x_1^2)}{4E(x_1)} + \frac{h}{4} - \frac{h^2}{4\pi} \int_{-\pi/h}^{\pi/h} dt \frac{Re[\xi(t) + \log(\frac{\mu}{h}) + \log(2|sin(\frac{ht}{2})|)]}{cos(ht) - 1}
\end{aligned}
$$

and the proof is done.

Proof of Corollary 2 The proof is similar to the proof for Corollary 1, so a succinct version is provided here. The integral region is split into four parts: $I_1 = \{t : -\pi/h \leq t < -\epsilon/h\}$, $I_2 = \{t : -\epsilon/h \leq t < 0\}$, $I_3 = \{t : 0 \leq t < \epsilon/h\}$, and $I_4 = \{t : \epsilon/h \leq \pi/h < 0\}$, where $0 < \epsilon << 1$. The integrals on regions I_1 and I_4 can be shown to be zero because they are integrable function integrated on $(-\infty, -\infty)$ and (∞, ∞) after taking the limit $h \downarrow 0$. For region I_2 and I_3, Taylor expansion can be applied on $\log(\mu/h) + \log(2|\sin(ht/2)|) = \log(\mu t) + O(ht)$ and $\frac{1}{\cos(ht)-1} = -\frac{2}{h^2t^2}(1 + O(ht))$, and then let $\epsilon \downarrow 0$. Because the integrated function is an even function, so the integral on I_2 and I_3 are the same and the proof is done.

5.4 Proof of Theorem 3

Spitzer (1960) provides equation (14):

$$E(S_{\tau_+}) = E(x_1)\exp[\sum_{n=1}^{\infty}\frac{P(S_n \leq 0)}{n}].$$

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{P(S_n \leq 0)}{n} \\
= &\lim_{\beta\downarrow 0}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{e^{\beta S_n}P(S_n = -kh)}{n} \\
= &\lim_{\beta\downarrow 0}\frac{h}{2\pi}\int_{-\pi/h}^{\pi/h}dt\frac{\xi(t)}{1-e^{-\beta h+iht}} &(23)\\
= &\lim_{\beta\downarrow 0}\frac{h}{2\pi}\int_{-\pi/h}^{\pi/h}dt\left(\frac{\xi(t)+\log(\mu/h)+\log(1-e^{iht})}{1-e^{-\beta h+iht}}\right) - \log(\mu/h) &(24)\\
= &\frac{h}{2\pi}\int_{-\pi/h}^{\pi/h}dt\frac{\Re(\xi(t))+\log(\frac{\mu}{h})+\log(2\sin(\frac{ht}{2}))}{2} - \frac{\Im(\xi(t))-\frac{\pi}{2}+\frac{ht}{2}}{2\tan(ht/2)} - \log(\frac{\mu}{h}) &(25)
\end{aligned}$$

From (23) to (24), equations (b) and (d) of Lemma 2 are applied. From (24) to (25), the Dominant Convergent Theorem is applied and the real part of the function is considered. To apply the Dominant Convergent Theorem, the integral region is split to $\int_{|t|<\epsilon}$ and $\int_{\epsilon\leq|t|<\pi/h}$, where $0 < \epsilon << \pi/h$. For the region $\epsilon \leq |t| < \pi/h$, the function is bounded by an integrable function. For the region $|t| < \epsilon$, Taylor expansion is applied to show that it is bounded by an integrable function. The following equations are applied to take

the real part of the function,

$$
\begin{aligned}
&(1-e^{iht})(1-e^{-iht}) = (1-\cos(ht))^2 + \sin^2(ht) = 2 - 2\cos(ht) = 4\sin^2(ht/2) \\
&\Re(\log(1-e^{iht})) = .5\log\left((1-\cos(ht))^2 + \sin^2(ht)\right) = \log(2\sin(ht/2)) \\
&\Im(\log(1-e^{iht})) = \tan^{-1}\frac{-\sin(ht)}{1-\cos(ht)} = -\tan^{-1}(\cot(ht/2)) = -\pi/2 + ht/2 \\
&\frac{1-\cos(ht)}{(1-e^{iht})(1-e^{-iht})} = \frac{1-\cos(ht)}{2-2\cos(ht)} = .5 \\
&\frac{\sin(ht)}{(1-e^{iht})(1-e^{-iht})} = \frac{2\sin(ht/2)\cos(ht/2)}{4\sin^2(ht/2)} = \frac{1}{2\tan(ht/2)}.
\end{aligned}
$$

By equation (14), the proof is done.