Statistica Sinica **8**(1998), 1265-1270

INTELLIGENT SEARCH FOR 2^{13-6} AND 2^{14-7} MINIMUM ABERRATION DESIGNS

Jiahua Chen

University of Waterloo

Abstract: Among all 2^{n-k} regular fractional factorial designs, minimum aberration designs are often preferred. When 2^{n-k} (the run-size) is no more than 64, most (possibly all) minimum aberration designs have been found. When $2^{n-k} = 128$, the search for minimum aberration designs becomes very hard. The results in Chen and Wu (1991) and Chen (1992) include all 128-run minimum aberration designs with $k \leq 5$. When n is close to 2^{n-k}, Tang and Wu (1996) also obtain some minimum aberration designs. In this paper, we search for 128-run minimum aberration designs with $k = 6$ and $k = 7$. By combining theoretical understanding with intelligent computer search, both cases are solved.

Key words and phrases: Defining contrast subgroup, fractional factorial design, wordlength pattern.

1. Introduction

We consider the problem of finding the minimum aberration 2^{13-6} and 2^{14-7} fractional factorial designs. In general, a 2^{n-k} design enables us to study n 2-level factors with 2^{n-k} runs. Since experimental runs are not conducted at all possible level combinations, some effects are aliased. A good design should avoid the aliasing of important effects. Usually, lower order interactions are considered to be more important than higher order interactions. The minimum aberration criterion is based on this belief (Fries and Hunter (1980)). There have been extensive discussions on minimum aberration designs in the literature. We refer to Chen and Wu (1991) for more details.

Searching for minimum aberration designs is not a trivial task. Fortunately, for many applications, both the number of factors n and the fractions k are not large. In these cases, designs with minimum aberration are easy to obtain and have been well documented. See Box, Hunter and Hunter (1978) and Franklin (1984).

When either n or k becomes large, as is common in applications, the corresponding minimum aberration designs are not so easy to obtain. Combinatorial techniques become demanding. Chen and Wu (1991) and Chen (1992) have obtained minimum aberration designs for $k \leq 5$ and all n. Combinatorial theory and the computer have been used to produce a complete catalog of designs with run-size up to 32 in Chen, Sun and Wu (1993). In addition, all resolution IV or higher 64-run designs are given in the same paper. There are other results in the literature. For example, by studying complementary designs, minimum aberration designs when n is close to 2^{n-k} were obtained by Tang and Wu (1996). Chen and Hedayat (1996) defined the notion of weak minimum aberration and categorized many designs by using projective geometry techniques. However, the task of finding many 128-run minimum aberration designs still awaits. In this paper, we specifically work on finding minimum aberration 2^{13-6} and 2^{14-7} designs.

2. Some Preparations and Useful Existing Results

To fix ideas, view an 8-run 2^{5-2} fractional factorial design as a submatrix of the regular 8×8 Hadamard matrix with 0 and 1 entries. The 8 columns can be named I, A, B, C, AB, AC, BC and ABC such that, for example, column AB is the sum of columns A and B module 2, and column I is a column of 0's. We can use any 5 columns, except column I, to form a 2^{5-2} fractional factorial design. If the first 5 non-zero columns are selected and their corresponding factors are named 1, 2, 3, 4, 5, then

$$I = 124 = 135 = 2345. \tag{1}$$

This means, for example, that the columns for factors 1, 2, and 4 have sum 0 (mod 2). The statistical implication is the aliasing of the main effect of factor 1 and the two-factor interaction between factors 2 and 4. We say 1 and 24 are aliased. Similarly, 2 and 14, 23 and 45, etc. are aliased.

We call (1) the defining contrast subgroup of the 2^{5-2} design. The set {I, 124, 135, 2345} forms an Abelian group with multiplication being the corresponding column summation mod 2. Each element in this group is called a word, the symbols for factors 1, 2 and so on are called letters, and the wordlength of a word equals the number of letters in the word.

It is not hard to see that a defining contrast subgroup uniquely determines a fractional factorial design. Thus, searching for an optimal design is equivalent to searching for an optimal defining contrast subgroup.

Let x_n be the number of length-n words in a defining contrast subgroup. We call $X = (x_1, x_2, \ldots)$ the wordlength pattern. The corresponding design has resolution R if x_R is the first non-zero component of X. Assume that the two 2^{n-k} designs D_1 and D_2 have the wordlength patterns X and Y, and $m = \min\{k : x_k \neq y_k\}$. Then D_1 has less aberration if $x_m < y_m$. A design with the smallest aberration, among all 2^{n-k} designs with a fixed n and k, is a minimum aberration design. Informally, a minimum aberration design has fewer pairs of low order interactions aliased.

A straightforward way to find an MA 2^{5-2} design is to exhaust all the possible choices of 5 out of the 7 columns. For this particular example, we find all of them have the same wordlength pattern. In fact, they are all equivalent. To find an MA 2^{13-6} design, the same method does not work as there are '127 choose 13' possible choices. At the same time, most of these choices create poor designs and many are equivalent. Thus, even if we search through only a relatively small fraction of them, we may have obtained an MA design already. The problem is: how do we know when to stop the search?

A simple fact is that not all vectors can be wordlength patterns. If X is the wordlength pattern of the current least aberration design, and we can show that any vector Y with less aberration cannot possibly be a wordlength pattern, then we have found an MA design.

For this purpose, namely to rule out the possibility of wordlength patterns with less aberration, knowing many necessary conditions is desirable. For convenience, we list some of them here. For a 2^{n-k} design with wordlength pattern X, we have $\sum_{m=1}^{n} x_m = 2^k - 1$; $\sum_{m=1}^{n} m x_m = n2^{k-1}$. By the way, we assume that the words in the defining contrast subgroup contain all the letters from 1 to n. Otherwise, the design cannot be a minimum aberration design (Chen and Wu (1991)).

Let w_1 and w_2 be two words in a defining contrast subgroup. Then $w_1 * w_2$ will be another word with all the letters either in w_1 or w_2, but not both. For example, $1234 * 3456 = 1256$. Because of this, each letter appears in exactly 2^{k-1} words in the defining contrast subgroup of a 2^{n-k} design if it appears at all. The words, which do not contain a specific letter, form the defining contrast subgroup of a $2^{(n-1)-(k-1)}$ design.

Another resource is the resolution bound given by Verhoeff (1987) in coding theory. It is known that a defining contrast subgroup is also a linear code. The resolution of a design is the minimum distance of its corresponding linear code. Hence, a bound on the minimum distance of certain linear codes is also a bound on the resolution of corresponding fractional factorial designs. By referring to Verhoeff (1987), we can rule out many possibilities.

The wordlength pattern of the MA 2^{12-5} design will be useful (Chen (1992)). It is given by $(0, 0, 0, 1, 8, 12, 8, 1, 0, 0, 0, 1)$. If a hypothetical wordlength pattern implies a 2^{12-5} design with less aberration than this design, it cannot be a wordlength pattern of any fractional factorial design.

3. Minimum Aberration 2^{13-6} and 2^{14-7} Designs

3.1. An MA 2^{13-6} design

After searching through some plausible designs, we found a 2^{13-6} design with the wordlength pattern $(0, 0, 0, 2, 16, 18, 10, 9, 4, 2, 0, 0)$. Its defining contrast subgroup is generated by

$$8 = 12345;\ \ 9 = 1236;\ \ t_{10} = 124567;\ \ t_{11} = 134567;\ \ t_{12} = 2347;\ \ t_{13} = 567, \quad (2)$$

where t_{10}, t_{11} are factors 10, 11 and so on.

From Verhoeff (1987), the maximum resolution of a 2^{13-6} design is 4. Thus, we must have $x_4 \geq 1$. Our conclusions are as follows.

Conclusion 1. The wordlength pattern of a 2^{13-6} fractional factorial design must have $x_4 > 1$ if $x_1 = x_2 = x_3 = 0$.

Proof. The previous discussion indicates that the smallest possible value of x_4 is 1. If $x_4 = 1$, we can find a subgroup of size $2^5 - 1$ which does not contain this length 4 word. Such a subgroup would define a 2^{12-5} design with resolution V which is impossible in view of the resolution IV MA 2^{12-5} design given earlier.

Consequently a 2^{13-6} design with less aberration, if it exists, must have a smaller x_5, or equal x_5 but smaller x_6, etc. We show no such possibilities exist.

Conclusion 2. The 2^{13-6} fractional factorial design given by (2) has minimum aberration.

Proof. We have seen that an MA 2^{13-6} design must have $x_4 = 2$ and $x_1 = x_2 = x_3 = 0$. If the two length-4 words share a letter, say 1, we can pick all the words which do not contain letter 1 to form a group. This group then defines a 2^{12-5} design with resolution V. This is impossible.

When two length-4 words do not share letters, we can write them as 1238, 4569. That is, we have selected 7 independent and two additional columns from the regular 128×128 Hadamard matrix to form a design. Four more columns should be selected from the remaining $127 - 9$ non-identity columns. It is easy to see that to keep $x_3 = 0$ and $x_4 = 2$, we cannot select columns corresponding to 2 or 3-factor interactions between factors 1, 2, ..., 7. Further, we cannot select columns corresponding to 4-factor interactions containing 123 or 456. With many similar considerations, it leaves only 42 choices for the remaining 4 columns. Hence, the task becomes well suited for the computer. A complete search confirms that the design given by (2) has minimum aberration.

3.2. An MA 2^{14-7} design

After searching through some plausible designs, we found a 2^{14-7} design with the wordlength pattern $(0, 0, 0, 3, 24, 36, 16, 11, 24, 12, 0, 1, 0, 0)$. The defining contrasts are:

$$8 = 123;\ 9 = 456;\ t_{10} = 1245;\ t_{11} = 1346;\ t_{12} = 12467;\ t_{13} = 13567;\ t_{14} = 23457. \tag{3}$$

We proceed to show that this is in fact an MA 2^{14-7} design. As before, we divide the proof into two steps.

Conclusion 3. An MA 2^{14-7} design must have $x_4 = 3$.

Proof. It is impossible to have $x_4 \leq 2$, or we can easily derive a 2^{13-6} design with less aberration than the MA 2^{13-6} design we have just obtained.

Conclusion 4. The 2^{14-7} design given by (3) is an MA design.

Assume that an MA 2^{14-7} design has $x_4 = 3$. Then any two of the 3 length-4 words cannot share a letter. Otherwise, we can pick a subgroup consisting of words which do not contain this letter. This will again result in a 2^{13-6} design with less aberration than the MA 2^{13-6} design we have just obtained.

Since the length-4 words do not share letters, two of them can be written as $8 = 123,\ 9 = 456$ without loss of generality. That is, from the regular 128×128 Hadamard matrix, we have selected 7 independent and two additional columns to form the design. We need only to look for another 5 columns. With the restrictions of $x_3 = 0$ and $x_4 = 3$, a large number of the 127 non-zero columns are not feasible. For example, any column of two-factor interactions or three-factor interactions containing 12, 13, 23, 45, 46, 56 is ruled out. We end up choosing 5 out of only 42 columns. This makes a complete computer search possible. A program written for this purpose verified that what we have now is the MA.

Remark. In fact, we first obtained a design with a slightly larger aberration than the current one. In the process of eliminating the possibility of finding better designs, we found the design given in (3). In addition, we were able to simplify the proof of (2) based on what we have learned.

Remark. It is known that the wordlength pattern does not uniquely determine a fractional factorial design (Chen and Lin (1991)). Thus, there could exist other non-equivalent 2^{13-6} and 2^{14-7} minimum aberration designs. The problem of uniqueness can be solved by investigating all minimum aberration designs obtained during the computer search. This, however, is beyond the scope of this paper. At the same time, our proofs indicate that other 2^{13-6} or 2^{14-7} minimum aberration designs, if any, have to share the structure of length-four words. This implies that the number of clearly estimable two-factor interactions cannot be improved for either design.

Acknowledgement

The author wishes to thank Professor C. F. Jeff Wu for his encouragement and the Consulting Editor for helpful suggestions. This research is supported by the Natural Sciences and Engineering Research Council of Canada.

References

Box, G. E. P., Hunter, W. G. and Hunter, J. S. (1978). *Statistics for Experimenters.* John Wiley, New York.

Chen, H. and Hedayat, A. S. (1996). 2^{n-l} designs with weak minimum aberration. *Ann. Statist.* **24**, 2536-2548.

Chen, J. (1992). Some results on 2^{n-k} fractional factorial designs and search for minimum aberration designs. *Ann. Statist.* **20**, 2124-2141.

Chen, J. and Lin, D. K. J. (1991). On the identity relationships of 2^{k-p} designs. *J. Statist. Plann. Inference* **28**, 95-98.

Chen, J., Sun, D. X. and Wu, C. F. J. (1993). A catalogue of two-level and three-level fractional factorial designs with small runs. *Int. Statist. Rev.* **61**, 131-145.

Chen, J. and Wu, C. F. J. (1991). Some results on s^{n-k} fractional factorial designs with minimum aberration or optimal moments. *Ann. Statist.* **19**, 1028-1041.

Franklin, M. F. (1984). Constructing tables of minimum aberration p^{n-m} designs. *Technometrics* **26**, 225-232.

Fries, A. and Hunter, W. G. (1980). Minimum aberration 2^{k-p} designs. *Technometrics* **22**, 601-608.

Tang, B. and Wu, C. F. J. (1996). Characterization of minimum aberration 2^{n-k} designs in terms of their complementary designs. *Ann. Statist.* **24**, 2549-2559.

Verhoeff, T. (1987). An updated table of minimum-distance bounds for binary linear codes. *IEEE Trans. Inform. Theory* **33**, 665-680.

Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, Ont N2L 3G1, Canada.

E-mail: jhchen@math.uwaterloo.ca

(Received April 1997; accepted January 1998)