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A TEST FOR STATIONARITY OF SPATIO-TEMPORAL RANDOM FIELDS ON PLANAR AND SPHERICAL DOMAINS

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Abstract: A formal test for weak stationarity of spatial and spatio-temporal random fields is proposed. We consider the cases where the spatial domain is planar or spherical, and we do not require distributional assumptions for the random fields. The method can be applied to univariate or to multivariate random fields. Our test is based on the asymptotic normality of certain statistics that are functions of estimators of covariances at certain spatial and temporal lags under weak stationarity. Simulation results for spatial as well as spatio-temporal cases on the two types of spatial domains are reported. We describe the results of testing the stationarity of Pacific wind data, and of testing the axial symmetry of climate model errors for surface temperature using the NOAA GFDL model outputs and the observations from the Climate Research Unit in East Anglia and the Hadley Centre.

Key words and phrases: Asymptotic normality, axial symmetry, climate model output, increasing domain asymptotics, inference, pacific wind data, stationarity.

1. Introduction

When dealing with spatial or spatio-temporal data from environmental applications, one often makes simplifying assumptions on the covariance structure, such as stationarity. For a spatial random field, $Z(\mathbf{s})$, defined in $D \subset \mathbb{R}^d$, we say that it is (weakly) stationary if the mean is constant across the spatial domain, D, and the covariance only depends on the lag between two spatial locations. In our notation, $\mathrm{E}\{Z(\mathbf{s})\} = \mu \in \mathbb{R}$ for all $\mathbf{s} \in D$, and $\mathrm{Cov}\{Z(\mathbf{s}_1), Z(\mathbf{s}_2)\} = C(\mathbf{s}_1 - \mathbf{s}_2)$ for an autocovariance function C and for all $\mathbf{s}_1, \mathbf{s}_2 \in D$. Therefore, if a spatial random field is nonstationary, the nonstationarity may be present in the mean and/or the covariance structure. In this paper, we focus on the nonstationarity of the random field in the covariance structure, assuming the mean of the random field is zero. If the random field does not have zero mean, we assume that we can remove the mean structure by subtracting a proper estimate of it and get a zero mean random field.

For a spatial random field defined in $\mathbb{R}^d$ or a spatio-temporal random field defined in $\mathbb{R}^d \times \mathbb{Z}$ $(d \geq 1)$, several authors have developed stationary or isotropic covariance models; see Gneiting, Genton, and Guttorp (2007) for a recent review. There also have been a number of papers that discuss hypothesis tests on separability of spatio-temporal covariance structure (Mitchell, Genton, and Gumpertz (2005, 2006); Fuentes (2006); Bevilacqua et al. (2010)). However, there are only few papers that discuss hypothesis tests of stationarity. Fuentes (2005) presented a test of stationarity for spatial random fields through a direct extension of the test of stationarity for time series of Priestley and Subba Rao (1969). The idea is based on spatial spectral analysis, but this approach requires that the spatial random field is on a regular grid. Dwivedi and Subba Rao (2010) developed a test of stationarity for time series based on the Discrete Fourier Transform but the asymptotic distribution of the test statistic is derived under the assumption that the process is either MA(∞) or is strongly stationary under some mixing conditions.

If the data cover a large portion of the Earth, as is common for satellite data or numerical model outputs, we need to consider a random field on the surface of the sphere. In this case the stationarity of the random field requires a different treatment. Here, even if the random field is isotropic, it may not be that it is stationary in latitude, unless the longitudinal lag is zero.

Global data from environmental applications often exhibit strong dependence of covariance on latitude (e.g. Stein (2007); Jun and Stein (2008)). Further, the covariance structures in the Northern and Southern Hemispheres need not be symmetric around the Equator. Thus, the left panel of Figure 1 shows the standard deviation of global surface temperature data (specifically, the difference between the temperature observation and climate model output for the corresponding quantity, see Section 5.2 for more details) with respect to latitude. Here dependence of the standard deviation of the data on latitude is not symmetric around the Equator. It is sometimes assumed that the random field is not isotropic, but stationary with respect to longitude (and time) and nonstationary with respect to latitude. For a global random field $Z(L, l)$ (L: latitude, l: longitude) on a sphere, we say that Z is *axially symmetric* if the covariance only depends on longitude through the difference between the two longitude values (Jones (1963)): $\mathrm{Cov}\{Z(L_1, l_1), Z(L_2, l_2)\} = C(L_1, L_2, l_1 - l_2)$ for all L_1, L_2, l_1, l_2 and an appropriate covariance function C. The covariance models used in Stein (2007), and Jun and Stein (2007, 2008) are not isotropic but assume axial symmetry.

Our interest lies in the axial symmetry. Figure 1 makes the dependence of the standard deviation of the data on latitude apparent; dependence on longitude is much weaker. Stein (2007) and Jun and Stein (2008) showed similar features for the total column ozone concentration data and concluded that axial symmetry was reasonable. Similarly, Cressie and Huang (1999) concluded that there was

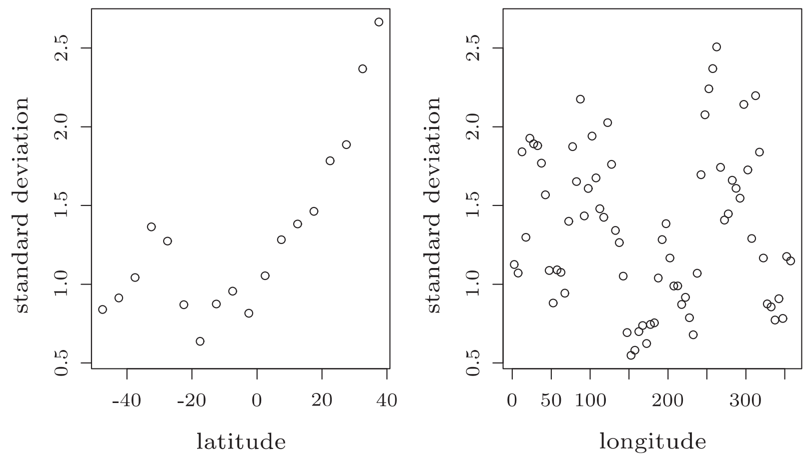

Figure 1. Standard deviations for the global surface temperature data (monthly average) in December, 1999, with respect to latitude (left panel) and longitude (right panel).

no clear nonstationarity in the Pacific wind data (see Section 5.1). In these examples, however, the judgment on whether or not the random field is stationary (or axially symmetric for random fields on a sphere) is subjective and is mostly based on empirical plots of covariances or variograms.

We present a formal hypothesis test for stationarity or axial symmetry of spatial and spatio-temporal random fields on planar and spherical domains. Recently, there has been a series of papers that have developed tests for various properties of the covariance structure for spatial and spatio-temporal random fields using empirical covariance estimators as test statistics. Guan, Sherman, and Calvin (2004) developed a test for isotropy of a spatial random field in $\mathbb{R}^2$ using the asymptotic joint normality of empirical covariance estimators at several spatial lags. Li, Genton, and Sherman (2007, 2008a,b) extended this idea to test separability and full symmetry for spatio-temporal random fields in both univariate and multivariate cases. Li et al. (2009) tested Taylor's hypothesis for the space-time covariance structure of rainfall data. However, the results in Guan, Sherman, and Calvin (2004), Li, Genton, and Sherman (2007, 2008a,b), and Li et al. (2009) are based on the assumption that the underlying spatial or spatio-temporal random fields are strongly stationary, whereas we test weak stationarity. Moreover, we consider random fields not only on planar domains but also on spherical domains. One of the main applications that we have in mind is that of testing the covariance structure of global data. Although the present work builds upon the results in Guan, Sherman, and Calvin (2004), Li, Genton, and Sherman (2008a,b), there are some significant differences.

The rest of the paper is organized as follows. Section 2 develops the asymptotic joint normality of our test statistics with some moment and mixing conditions. We consider spatial and spatio-temporal random fields as well as planar and spherical domain cases. The formal testing procedure is presented in Section 3. We report simulation results in Section 4. The application results to the Pacific Ocean wind data and global temperature data are given in Section 5. We conclude the paper with some discussion; the proof of the theorem is given in the Appendix.

2. Asymptotic Theory

We test weak stationarity of spatial and spatio-temporal random fields on either planar or spherical domains. The basic idea is to divide the spatial domain into two (or more) disjoint domains and to use the test statistic that is based on the differences between empirical estimators of covariances at given lags from the sub-domains. We establish the asymptotic normality of the empirical estimators and use an asymptotic chi-squared test based on it.

We first separate the spatial and spatio-temporal random fields. For each, we consider planar spatial domains and spherical spatial domains separately. For planar spatial domains, the asymptotic results are based on increasing domain asymptotics. For the spatio-temporal case, the spatial domain is fixed and the asymptotics come from the increasing time domain. We then discuss how the results can be extended to multivariate random fields.

We denote the random field as $Z(\mathbf{x})$. For a spatial random field, $\mathbf{x} = \mathbf{s} \in D$, where D is the spatial domain; for a spatio-temporal random field, $\mathbf{x} = (\mathbf{s}, t) \in D \times \mathbb{Z}$. In particular, if we consider a spatial random field on a sphere, we take $\mathbf{x} = (L, l) \in D \subset \mathcal{S}^2$, where L denotes the latitude, l the longitude, and $\mathcal{S}^2$ the surface of a sphere with radius R in $\mathbb{R}^3$. The lag, $\mathbf{k}$, may be spatial ($\mathbf{k} = \mathbf{h}$) or spatio-temporal ($\mathbf{k} = (\mathbf{h}, u)$).

2.1. Spatial random fields

We first suppose that the observations are taken over $D \subset \mathbb{R}^d$ for some $d \geq 1$. Guan, Sherman, and Calvin (2004) dealt with the case $d = 2$ and, while we also present results for $d = 2$, it is easy to show that our results hold for $d \geq 1$. We consider a regularly spaced domain; similar results may hold for irregularly spaced domains along the lines of Guan, Sherman, and Calvin (2004). Our results come from increasing domain asymptotics.

Consider a spatial random field $\{Z(\mathbf{s}) : \mathbf{s} \in D\}$. Suppose the observations are taken over $D \subset \mathbb{Z}^2$, a 2-dimensional space of integer lattice points. We assume Z has mean zero and denote the covariance function of Z as $C(\mathbf{s}, \mathbf{s} + \mathbf{h}) = \mathrm{Cov}\{Z(\mathbf{s}), Z(\mathbf{s} + \mathbf{h})\}$, where $\mathbf{s}, \mathbf{s} + \mathbf{h} \in D$. In the case that the random field Z

is weakly stationary, $C(\mathbf{s}, \mathbf{s}+\mathbf{h}) = C(\mathbf{0}, \mathbf{h}) = C_0(\mathbf{h})$ for all $\mathbf{s} \in D$, $\mathbf{h}$, and a stationary covariance function C_0. Now let $S(\mathbf{h}; D) = \{\mathbf{s} : \mathbf{s} \in D, \mathbf{s}+\mathbf{h} \in D\}$ and let $|S(\mathbf{h}; D)|$ be the cardinality of $S(\mathbf{h}; D)$. We consider the statistic,

$$\widehat{A}(\mathbf{h}; D) = \frac{1}{|S(\mathbf{h}; D)|} \sum_{\mathbf{s} \in S(\mathbf{h}; D)} Z(\mathbf{s}) Z(\mathbf{s}+\mathbf{h}). \tag{2.1}$$

If Z is weakly stationary with the covariance function C_0, then $\widehat{A}(\mathbf{h}; \cdot)$ is an estimator of the covariance $C_0(\mathbf{h}) = C(\mathbf{0}, \mathbf{h})$ for a spatial lag $\mathbf{h}$.

Our test is based on the difference of estimators from two disjoint spatial domains whose union is the entire domain D. Here, we let D_n be a domain that increases with n and consider $D_{n,1}$ and $D_{n,2}$ such that $D_{n,1} \sqcup D_{n,2} = D_n$ ($\sqcup$ denotes a union of disjoint sets). We require $|D_{n,1}|/|D_{n,2}| = O(1)$. An example of a spatial domain when $d = 2$ is given in Guan, Sherman, and Calvin (2004): let $B \subset (0, 2] \times (0, 1]$ be the interior of a simple closed curve such that $B \cap (0, 1] \times (0, 1]$ and $B \cap (1, 2] \times (0, 1]$ have nonempty interiors. Multiply B by n (call it B_n) and take $D_n = \{\mathbf{s} : \mathbf{s} \in B_n \cap \mathbb{Z}^2\}$, $D_{n,1} = D_n \cap (0, n] \times (0, n]$, and $D_{n,2} = D_n \cap (n, 2n] \times (0, n]$. The shape of D_n may be more general than this, but in general we require

$$|D_n| = O(n^2) \text{ and } |\partial D_n| = O(n), \tag{2.2}$$

where the boundary of a set D is $\partial D \equiv \{\mathbf{s} \in D : \exists\ \mathbf{s}' \notin D \text{ s.t. } d(\mathbf{s}, \mathbf{s}') = 1\}$, where $\mathbf{s} = (s_x, s_y)^T$, $d[(s_x, s_y), (s'_x, s'_y)] \equiv \max(|s_x - s'_x|, |s_y - s'_y|)$, and $|\partial D|$ is the number of points in ∂D.

For a set of spatial lags Λ, let $\widehat{\mathbf{G}}_{n,i} = \{\widehat{A}(\mathbf{h}; D_{n,i}) : \mathbf{h} \in \Lambda\}$ for $i = 1, 2$. If Z is strongly stationary, the asymptotic normality of $\widehat{\mathbf{G}}_{n,i}$ is a result in Guan, Sherman, and Calvin (2004). However, several assumptions required there may not be relevant here. In the following, we discuss the assumptions that we require for the test, and then we establish the asymptotic joint normality of $\widehat{\mathbf{G}}_n = (\widehat{\mathbf{G}}_{n,1}, \widehat{\mathbf{G}}_{n,2})^T$.

Start with

$$\alpha(p, k) \equiv \sup\{|P(A_1 \cap A_2) - P(A_1)P(A_2)| : A_i \in \mathcal{F}(E_i), |E_i| \leq p, \\ i = 1, 2, d(E_1, E_2) \geq k\},$$

where $|E|$ is the cardinality of the index set E, $\mathcal{F}(E)$ is the σ-algebra generated by the random variables $\{Z(\mathbf{s}) : \mathbf{s} \in E\}$, and $d(E_1, E_2)$ is the minimal "city block" distance between E_1 and E_2. We assume

$$\sup_p \frac{\alpha(p, k)}{p} = O(k^{-\epsilon}) \text{ for some } \epsilon > 2. \tag{2.3}$$

Thus, as distance k increases, dependence decreases at a polynomial rate in k. Guan, Sherman, and Calvin (2004) and Li, Genton, and Sherman (2008a) assume the same. Any m-dependent random field (observations separated by a distance larger than m are independent) satisfies this condition.

Consider the moment condition for a general random field Z that may be nonstationary:

$$\sup_n \mathrm{E}\{|\sqrt{|D_{n,i}|} \times |\widehat{A}(\mathbf{h}; D_{n,i})|^{2+\delta}\} \leq C_\delta, \tag{2.4}$$

for some $\delta > 0$, $C_\delta < \infty$, and $i = 1, 2$. Guan, Sherman, and Calvin (2004) require a similar moment condition for a strongly stationary random field. This condition is only slightly stronger than the existence of the (standardized) asymptotic variance of $\widehat{A}(\mathbf{h}; D_{n,i})$. Another condition is that $\lim_{n\to\infty} \sqrt{|D_{n,i}||D_{n,j}|} \times \mathrm{Cov}\{\widehat{A}(\mathbf{h}_1; D_{n,i}), \widehat{A}(\mathbf{h}_2; D_{n,j})\}$ exists and is finite for $i, j = 1, 2$ and $\mathbf{h}_1, \mathbf{h}_2 \in \Lambda$. This holds if we have

$$\sum_{\mathbf{s}_1 \in S(\mathbf{h}_1; D_{n,i})} \sum_{\mathbf{s}_2 \in S(\mathbf{h}_2; D_{n,j})} \mathrm{Cov}\{Z(\mathbf{s}_1)Z(\mathbf{s}_1 + \mathbf{h}_1), Z(\mathbf{s}_2)Z(\mathbf{s}_2 + \mathbf{h}_2)\} = O(n^2) \tag{2.5}$$

for $i, j = 1, 2$. This holds for any m-dependent process with finite fourth moment, or if $\mathrm{E}\{|Z(\mathbf{s})|^{4+\delta}\} < D_\delta$ for some $\delta > 0$ and $D_\delta < \infty$.

Theorem 1. *Let $\{Z(\mathbf{s}) : \mathbf{s} \in \mathbb{Z}^d\}$ be a random field that is observed in D_n satisfying* (2.2)*. If* (2.5) *holds, then the block matrix $\boldsymbol{\Sigma} = (\boldsymbol{\Sigma}_{ij})$ with $\boldsymbol{\Sigma}_{ij} \equiv \lim_{n\to\infty}(1/4)|D_n| Cov(\widehat{\mathbf{G}}_{n,i}, \widehat{\mathbf{G}}_{n,j})$, $i, j = 1, 2$, exists and is finite. If $\boldsymbol{\Sigma}$, $\boldsymbol{\Sigma}_{11}$, $\boldsymbol{\Sigma}_{22}$ are positive definite,* (2.3) *and* (2.4) *hold, and Z is weakly stationary with autocovariance function C_0, then the limiting distribution of $(\sqrt{|D_{n,1}|}(\widehat{\mathbf{G}}_{n,1} - \mathbf{G}), \sqrt{|D_{n,2}|}(\widehat{\mathbf{G}}_{n,2} - \mathbf{G}))^T$ is multivariate normal with mean zero and covariance matrix $\boldsymbol{\Sigma}$, where $\mathbf{G} = \{C_0(\mathbf{h}) : \mathbf{h} \in \Lambda\}$.*

Proof. See the Appendix.

For the spatial case with a spherical domain, a natural asymptotic framework is the so called fixed domain or infill asymptotics (Stein (1995); Lahiri (1996); Zhang and Zimmerman (2005)). To the best of our knowledge, there have been no results on statistics as in (2.1) under an infill asymptotics framework on the sphere.

2.2. Spatio-temporal random fields

The argument is similar to the one in Section 2.1, but here the asymptotics come from an increasing time domain. We take the space to be fixed and bounded, and regularly and irregularly spaced spatial domains cases are covered.

For $\mathbf{x} = (\mathbf{s}, t)^T$, $\mathbf{s} \in D \subset \mathbb{R}^d$, $t \in \mathbb{N}$, we assume that $Z(\mathbf{x})$ is a random field with mean zero. We take the spatial domain for observations in D to be fixed and the observations taken regularly over time, $T_n = \{1, \ldots, n\}$. We divide D into two disjoint, nonempty subsets, D_1 and D_2 such that $D_1 \sqcup D_2 = D$. As in Section 2.1, we require assumptions on the random field, although we do not require (2.2) since the spatial locations are considered to be fixed. For the mixing condition, let

$$\alpha(u) = \sup_{v \in \mathbb{N}} \Big[\sup_{A,B} \{ |P(A \cap B) - P(A)P(B)|, A \in \mathcal{F}_{-\infty}^{v}, B \in \mathcal{F}_{v+u}^{\infty} \} \Big], \tag{2.6}$$

where $\mathcal{F}_{-\infty}^{v}$ is the σ-algebra generated by the random field until $t = v$ and $\mathcal{F}_{v+u}^{\infty}$ is the σ-algebra generated by the random field from time $t = v + u$. We use a slightly different version of (2.6) than in Li, Genton, and Sherman (2008a,b), since we do not assume strong stationarity in time. We require the strong mixing condition

$$\alpha(u) = O(u^{-\epsilon}) \text{ for some } \epsilon > 0. \tag{2.7}$$

Bradley (2005) discussed several mixing conditions, including strong mixing, for processes that are not necessarily stationary, provided examples of processes that satisfy certain mixing conditions, and described relationships between mixing conditions.

We consider the statistic (for $\mathbf{k} = (\mathbf{h}, u)$),

$$\widehat{A}(\mathbf{k}; D) = \frac{1}{|S(\mathbf{h}; D)|(|T_n| - u)} \sum_{\mathbf{s} \in S(\mathbf{h}; D)} \sum_{t=1}^{n-u} Z(\mathbf{x}) Z(\mathbf{x} + \mathbf{k}).$$

If Z is weakly stationary, this is an estimator of the covariance $C_0(\mathbf{k}) = \mathrm{Cov}\{Z(\mathbf{x}), Z(\mathbf{x} + \mathbf{k})\}$. Let Λ be a set of space-time lags. We establish the asymptotic joint normality of $\widehat{\mathbf{G}}_n = (\widehat{\mathbf{G}}_{n,1}, \widehat{\mathbf{G}}_{n,2})^T$ for $\widehat{\mathbf{G}}_{n,i} = \{\widehat{A}(\mathbf{k}; D_i) : \mathbf{k} \in \Lambda\}$. To do this we consider additional assumptions. The moment condition is

$$\sup_n \mathrm{E}\{|\sqrt{|T_n|}\ |\widehat{A}(\mathbf{k}; D)|^{2+\delta}\} \leq C_\delta \text{ for some } \delta > 0, C_\delta < \infty. \tag{2.8}$$

To guarantee the existence and finiteness of the covariance matrix of $\widehat{\mathbf{G}}_n$, consider that, for $\mathbf{x}_i = (\mathbf{s}_i, t_i)^T \in D \times T_n$, $i = 1, 2$,

$$\sum_{t_1=1}^{n-u_1} \sum_{t_2=1}^{n-u_2} \sum_{\mathbf{s}_1 \in S(\mathbf{h}_1; D_1)} \sum_{\mathbf{s}_2 \in S(\mathbf{h}_2; D_2)} \mathrm{Cov}\{Z(\mathbf{x}_1) Z(\mathbf{x}_1 + \mathbf{k}_1), Z(\mathbf{x}_2) Z(\mathbf{x}_2 + \mathbf{k}_2)\} = O(n^2) \tag{2.9}$$

for all $\mathbf{k}_i = (\mathbf{h}_i, u_i)^T$ and $\mathbf{h}_i, u_i$ finite $(i = 1, 2)$.

Theorem 2. *Let $\{Z(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^d \times \mathbb{N}\}$ be a spatio-temporal random field observed in $D \times T_n$, where $D \subset \mathbb{R}^d$ and $T_n = \{1, \ldots, n\}$. Let D_1 and D_2 be non-empty subsets of D with $D = D_1 \sqcup D_2$. If* (2.9) *holds, then $\mathbf{\Sigma} = \lim_{n\to\infty} |T_n|\, Var(\widehat{\mathbf{G}}_n)$ exists and is finite. If $\mathbf{\Sigma}$ is positive definite,* (2.7) *and* (2.8) *hold, and Z is weakly stationary with autocovariance function C_0, and if we let $\mathbf{G} = \{C_0(\mathbf{k}), \mathbf{k} \in \Lambda\}$, then $\sqrt{|T_n|}(\widehat{\mathbf{G}}_n - (\mathbf{G}, \mathbf{G})^T)$ is asymptotically normal with mean zero and covariance matrix $\mathbf{\Sigma}$.*

This result is similar to Proposition 1 in Li, Genton, and Sherman (2008b), except that they required strong stationarity of the spatio-temporal random fields. The proof of Theorem 2 is similar to the proof of Theorem 1, given in the Appendix.

When the spatial domain is spherical, unlike the case in Section 2.1, the asymptotic results simply follow from Theorem 2.

2.3. Extension to multivariate random fields

There are several recent papers that develope cross-covariance models for multivariate processes (e.g. Gneiting, Kleiber, and Schlather (2010); Apanasovich and Genton (2010); Apanasovich, Genton, and Sun (2012)). The class of models developed in Jun (2011) gives nonstationary (axially symmetric) cross-covariances for multivariate processes on a sphere and the approach can easily be extended to spatio-temporal processes.

For a bivariate random field, $\mathbf{Z} = (Z_1, Z_2)^T$, our goal is to test whether the cross-covariance $\text{Cov}\{Z_1(\mathbf{x}_1), Z_2(\mathbf{x}_2)\}$ only depends on $\mathbf{x}_1$ and $\mathbf{x}_2$ through $\mathbf{x}_1 - \mathbf{x}_2$. Marginal weak stationarity of Z_1 and Z_2 does not necessarily imply weak stationarity of the cross-covariance structure of the two random fields. The basic idea of the asymptotic theory for the cross-covariance case is the same as in the univariate case and we simply need to modify the statistic $\widehat{A}$, with the spatio-temporal case as an example with $\mathbf{x} = (\mathbf{s}, t), \mathbf{k} = (\mathbf{h}, u)$, to write

$$\widehat{A}(\mathbf{k}; \cdot) = \frac{1}{|S(\mathbf{h}; \cdot)|(|T_n| - u)} \sum_{\mathbf{s} \in S(\mathbf{h}; \cdot)} \sum_{t=1}^{n-u} Z_1(\mathbf{x}) Z_2(\mathbf{x} + \mathbf{k}).$$

Here, we assume that the random fields Z_1 and Z_2 are observed over the same spatial and temporal domains.

The asymptotic joint normality of $\widehat{\mathbf{G}}_n$ holds for the spatial case under increasing domain asymptotics, as in Section 2.1, and for the spatio-temporal case under increasing time domain asymptotics as in Sections 2.2. The proof is similar to that of the previous theorems. Similar results are presented in Li, Genton, and Sherman (2008b), although their results are for strongly stationary multivariate processes.

3. Testing Procedure

3.1. Hypothesis and contrasts

We test the null hypothesis of weak stationarity of the random field. We write this as $\widehat{A}(\mathbf{k}; D_1) = \widehat{A}(\mathbf{k}; D_2)$, where $\widehat{A}(\cdot; D)$ is calculated from Z defined on D or $D \times T_n$, and $\mathbf{k} \in \Lambda$. Therefore we have $\mathbf{X}\widehat{\mathbf{G}}_n = \mathbf{0}$ where $\mathbf{X}$ is a full row rank matrix. In particular, for a spatial random field on $\mathbb{R}^2$, if we consider $\Lambda = \{(1,0),(0,1)\}$, then we may let $\mathbf{X} = (\mathbf{I}_2, -\mathbf{I}_2)$, where $\mathbf{I}_n$ denotes the identity matrix of dimension $n \times n$. Here, $\mathbf{X}$ is of dimension 2×4. Unlike the situations in Guan, Sherman, and Calvin (2004), Li, Genton, and Sherman (2007, 2008a,b), and Li et al. (2009), we do not have ambiguity in choosing the contrast matrix $\mathbf{X}$. Once we choose the lags to use through Λ, the contrast matrix $\mathbf{X}$ is automatically decided since $\mathbf{X}$ should simply form the differences of $\widehat{A}$ values in two different spatial regions.

We give an example of how to construct a test statistic when we have spatio-temporal random fields; the test statistic for the spatial case works in the same way. Suppose we consider a spatio-temporal random field and $|\Lambda| = m$. Let $\boldsymbol{\mathcal{G}} = \sqrt{|T_n|}(\widehat{\mathbf{G}}_n - (\mathbf{G}, \mathbf{G})^T)$, using the same notation as in Section 2.2. Define an $m \times 2m$ matrix $\mathbf{X}$ such that $\mathbf{X} = (\mathbf{I}_m, -\mathbf{I}_m)$. Then our test statistic is

$$\mathcal{T}_m = (\mathbf{X}\boldsymbol{\mathcal{G}})^T(\mathbf{X}\boldsymbol{\Sigma}\mathbf{X}^T)^{-1}\mathbf{X}\boldsymbol{\mathcal{G}} \tag{3.1}$$

and, due to Theorem 2, $\mathcal{T}_m$ is asymptotically chi-square distributed with degrees of freedom m.

We estimate the covariance matrix, $\boldsymbol{\Sigma}$, using subsampling (Guan, Sherman, and Calvin (2004); Li, Genton, and Sherman (2007, 2008a,b); Li et al. (2009)). The L_2-consistency of the subsampling estimator of the covariance matrix holds in our case due to (2.3), (2.4), and Theorem 1 in Ekström (2008). The results in Ekström (2008) do not require stationarity of the random field and assume a mixing condition weaker than (2.3). Therefore, if we replace $\boldsymbol{\Sigma}$ in (3.1) with $\hat{\boldsymbol{\Sigma}}$, the same asymptotic result holds by the multivariate Slutsky Theorem, without additional conditions. As noted in Guan, Sherman, and Calvin (2004), for the spatial case (in Section 2.1) in which the convergence of the asymptotic chi-square test statistic is slow, we explored both the asymptotic chi-square test and the calculation of p-values using subsampling. For the spatio-temporal case, we found that the convergence of the asymptotic chi-square test statistic was relatively fast.

3.2. Splitting the spatial domain

Our test statistic is based on the difference between estimators in two disjoint subsets of the spatial domain. In splitting the spatial domain, however, there is no obvious way of doing it. Furthermore, different collections of the random splitting sets may produce different p-values and, in some situations, the results of hypothesis testing may be different. Therefore, instead of splitting the domain into neighboring disjoint domains, it could be beneficial to randomly split the domain with respect to one of the spatial coordinates and repeat this procedure several times for both spatial coordinates. For example, Figure 2 gives an illustration of how to randomly split domains with respect to longitude or latitude for spherical data. This random splitting prevents us from getting spurious small p-values if it happens to be the case that there is a specific local spot where the values are either high or low (although the underlying random field is weakly stationary and the random field has mean zero).

In Section 2.1, we imposed a condition on the boundary of the spatial domain as in (2.2) for spatial processes. If we perform random splitting many times, there may be a case where the condition on the boundary of the spatial domain is not met. Thus, for this case, when we perform random splitting, we have to keep in mind that we may need to avoid certain splits of the spatial domain, although those that violate the condition in (2.2) are quite rare for a reasonably large n. When we do not have many spatial points to begin with, we have to ensure that there are enough pairs of spatial points in each splitted domain at the specified spatial lag. If the spatial domain is not regularly spaced, we may overlay a regular grid over the spatial domain and split the domain based on that grid.

It is not clear what the distribution of the collection of p-values from random splitting should be since the test statistics from different random splitting are not independent. Figure 3 displays the histograms and boxplots of the p-values from 130 random splits with respect to longitude and latitude for 1,000 simulations from the weakly stationary random field in Section 4.2. Although the histograms show that there are peaks near the boundaries in the distribution of p-values, the distribution is close to be uniform away from the end points. We cannot make a direct comparison between the boxplots in Figure 3 and those of p-values in Section 5, since the underlying covariance structure may be different. Nonetheless, we have a general idea of how the distribution of p-values should look under weak stationarity, from Figure 3.

We may also consider splitting into more than two spatial domains but risk having too little sample size in each sub-domain to get reliable $\widehat{A}$ values. Here, we have fixed the number of subdomains at two.

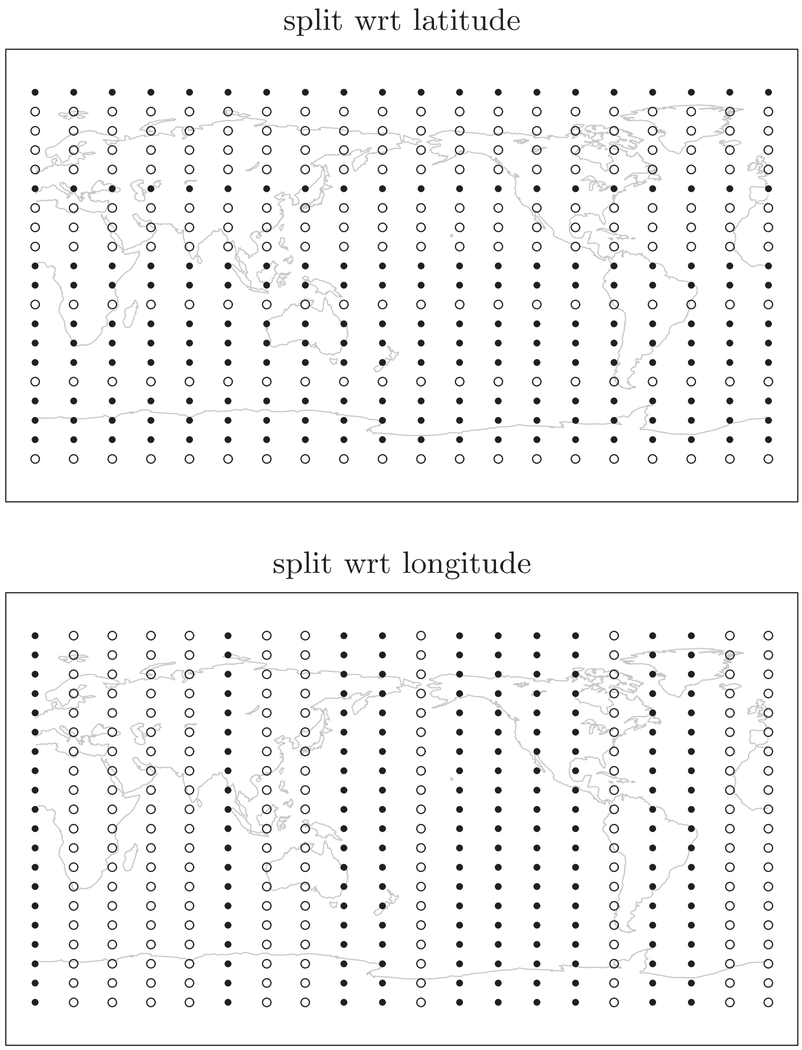

Figure 2. Two ways to split the spherical domain.

3.3. Choice of lags

Under our test setting, we may choose particular sets of spatial or temporal lags depending on what we want to test. For example, if we consider a spatio-temporal random field on a sphere, and if we want to test whether the random field is axially symmetric and stationary in time, we may choose a set of lags that only involve lags with respect to longitude and time. Similarly if we want to test if the random field is nonstationary with respect to latitude or not, we may choose a set of lags that only involves latitude. Although the distance on the globe depends on the two latitude points, if the longitudinal lag is zero, then

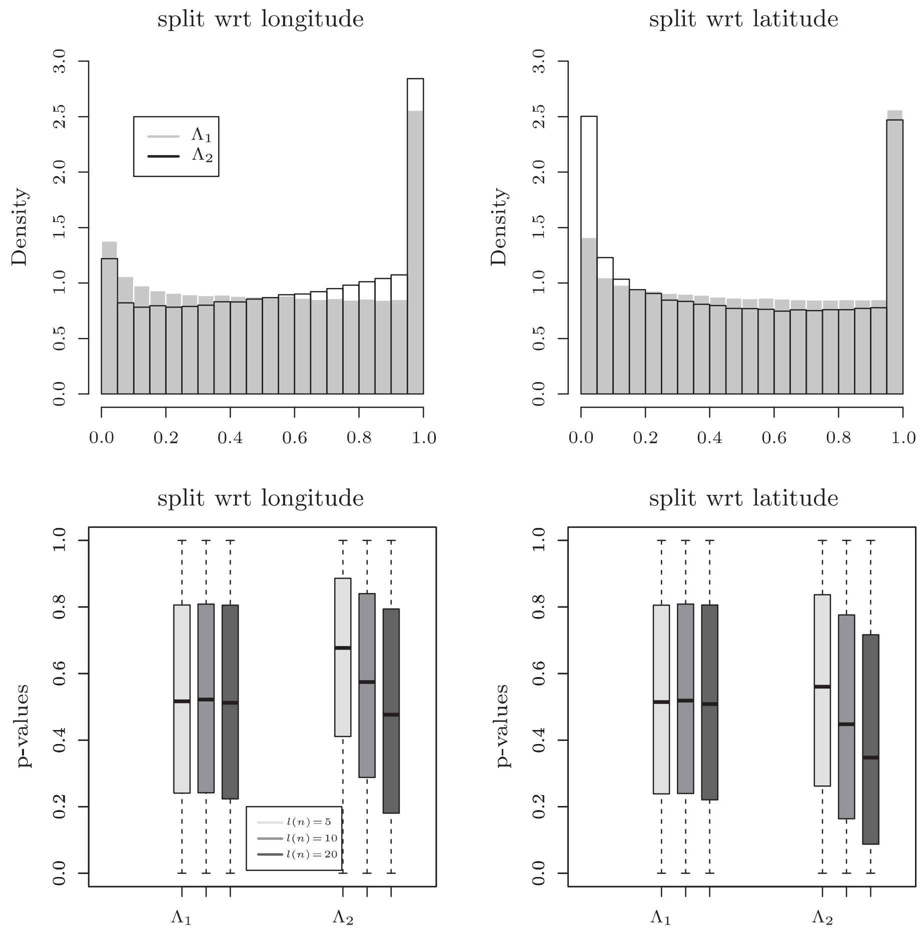

Figure 3. P-values from the test with the simulated data in Section 4.2. Histograms are based on 1,000 simulations and 130 random splits for all the subblock sizes together; boxplots are different for different subblock sizes. For Λ_1, Λ_2, and $l(n)$, see Section 4.2.

the distance only depends on the latitudinal lag. Therefore, we can test if the random field is actually nonstationary in latitude or not by using only latitudinal lags.

We acknowledge, however, that similarly to the tests proposed in Guan, Sherman, and Calvin (2004), Li, Genton, and Sherman (2008a,b), the results of our test of stationarity only hold for the particular lags that we consider.

4. Simulation Studies

4.1. Spatial random field on a regularly spaced planar domain

Consider a mean zero weakly stationary (isotropic) Gaussian random field

Table 1. Empirical sizes (%) of the test of stationarity using subsampling for the spatial case in Section 4.1.

		20×20			40×40		
m	c	Two lags	Four lags	Eight lags	Two lags	Four lags	Eight lags
2	0.5	3.3	5.2	7.8	2.8	3.3	4.4
	1.0	2.4	5.2	1.4	1.0	3.0	6.0
	1.5	3.3	1.8	3.3	1.0	2.3	10.2
5	0.5	10.8	7.0	7.8	13.0	4.7	3.7
	1.0	6.5	4.7	9.1	8.4	2.7	3.0
	1.5	10.8	12.2	23.3	4.0	2.4	5.6
8	0.5	24.7	9.3	10.6	24.7	8.3	5.7
	1.0	19.3	8.6	9.2	18.6	2.9	3.2
	1.5	18.8	15.3	17.4	13.1	5.4	5.4

Note: Nominal level is 5%. Empirical sizes are calculated based on 1,000 simulations and the largest standard error for sizes is 1.6%. Here m is the range of spatial dependence in (4.1) and the c values of 0.5, 1, and 1.5 correspond to the subblock size 2×2, 4×4, and 7×7 for 20×20 grids, and 3×3, 6×6, and 9×9 for 40×40 grids.

$Z(\mathbf{s}),\ \mathbf{s} \in D \subset \mathbb{Z}^2$. We assess the size of our test, and calculate the power of the test based on some nonstationary covariance models. The covariance/correlation structure that we used to assess the size of the test was

$$C_0(\mathbf{h}; m) = \begin{cases} 1 - \frac{3\|\mathbf{h}\|}{m} + \frac{\|\mathbf{h}\|^3}{2m^3} & \text{if } 0 \leq \|\mathbf{h}\| \leq m, \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

where m defines the range and strength of dependence. We tested the cases $m = 2, 5, 8$. We generated 1,000 realizations for each m on 40×20 or 80×40 rectangular grids. For the test statistic, we split the domain into two 20×20 or 40×40 neighboring squares (thus we did not perform the random splitting discussed in Section 3.2) and calculated $\widehat{A}$ on them. We used

$$\begin{aligned} \Lambda_1 &= \{\mathbf{h} : (1,0), (0,1)\}, \\ \Lambda_2 &= \{\mathbf{h} : (1,0), (0,1), (1,1), (-1,1)\}, \\ \Lambda_3 &= \{\mathbf{h} : (1,0), (0,1), (1,1), (-1,1), (2,1), (1,2), (-2,1), (-1,2)\}. \end{aligned}$$

We explored both the asymptotic chi-squared test and the subsampling approach to compute the p-values, using subsampling to estimate $\boldsymbol{\Sigma}$. We also applied the finite-sample adjustments as in Guan, Sherman, and Calvin (2004). Table 1 gives the empirical sizes based on p-values using subsampling. We do not report the outcomes from the asymptotic chi-squared test since the convergence of the test statistic is rather slow due to the small sample size n, but the results are qualitatively similar.

For the empirical power, we used the nonstationary model in Paciorek and Schervish (2006). If D is the regular grid of size $2n \times n$ ($n = 20$ or 40) and if $\mathbf{s}_1, \mathbf{s}_2 \in D$, we considered the nonstationary covariance function

$$C(\mathbf{s}_1, \mathbf{s}_2) = |\mathbf{\Sigma}_1|^{1/4} |\mathbf{\Sigma}_2|^{1/4} \left| \frac{\mathbf{\Sigma}_1 + \mathbf{\Sigma}_2}{2} \right|^{-1/2} \exp\left(\sqrt{Q_{12}}\right), \qquad (4.2)$$

with $Q_{12} = 2(\mathbf{s}_1 - \mathbf{s}_2)^T (\mathbf{\Sigma}_1 + \mathbf{\Sigma}_2)^{-1} (\mathbf{s}_1 - \mathbf{s}_2)$. We decomposed $\mathbf{\Sigma}_i = \mathbf{\Gamma}_i \mathbf{\Lambda}_i \mathbf{\Gamma}_i^T$ for $i = 1, 2$, where

$$\mathbf{\Gamma}_i = \frac{1}{d_i} \begin{pmatrix} \gamma_1(\mathbf{s}_i) & -\gamma_2(\mathbf{s}_i) \\ \gamma_2(\mathbf{s}_i) & \gamma_1(\mathbf{s}_i) \end{pmatrix}, \ \mathbf{\Lambda}_i = \begin{pmatrix} d_i^2 & 0 \\ 0 & \frac{d_i^2}{2} \end{pmatrix},$$

$d_i = \sqrt{\gamma_1^2(\mathbf{s}_i) + \gamma_2^2(\mathbf{s}_i)}$, $\gamma_1(\mathbf{s}) = \log\{s_x/(2n)\}$, $\gamma_2(\mathbf{s}) = \{s_x/(2n)\}^2 + \{s_y/n\}^2$, and $\mathbf{s} = (s_x, s_y)^T$. The forms of γ_1 and γ_2 were chosen arbitrarily to create a smooth nonstationary covariance structure through (4.2) (see Figure 4). Porcu, Mateu, and Christakos (2009) gave a more general framework for the nonstationary covariance models that includes the model in (4.2). In (4.2), we do not use the covariance function in (4.1) since the scale of $\sqrt{Q_{12}}$ is not comparable to the original distances between $\mathbf{s}_1$ and $\mathbf{s}_2$. Although the exponential covariance function does not have a compact support, we find that many of the covariance values calculated from (4.2) are close to zero in our domain, and the effective length of the support is small.

Table 1 gives the empirical size values of the test from subsampling when the nominal value is 5%. There is no clear association between the size values and the number of lags chosen. Overall the empirical sizes of the test are close to the nominal value, although the empirical sizes are somewhat large for small numbers of lags or when the dependence of the simulated process is strong ($m = 8$). The empirical sizes are closer to the nominal value when the sample size n is larger. The empirical powers using the nonstationary model in (4.2) from subsampling are given in Table 2. The empirical power values are significantly larger for $n = 40$, as expected. There is no clear association between the empirical power and the number of lags chosen. For $n = 40$, it seems that the empirical powers decreased as we used larger subblock lengths for each number of lags, although it was not the case when $n = 20$.

4.2. Spatio-temporal random field on a spherical domain

Consider a regularly spaced grid on the surface of a sphere and regularly spaced time points. This set-up is common for global spatio-temporal data (see Section 5.2). We simulated a random field from a first-order vector autoregressive model, VAR(1), $\mathbf{Z}_t = \mathbf{R}\ \mathbf{Z}_{t-1} + \boldsymbol{\epsilon}_t$, where $\mathbf{Z}_t = \{Z(\mathbf{s}_1, t), \ldots, Z(\mathbf{s}_k, t)\}^T$, $\mathbf{s}_i \in D$,

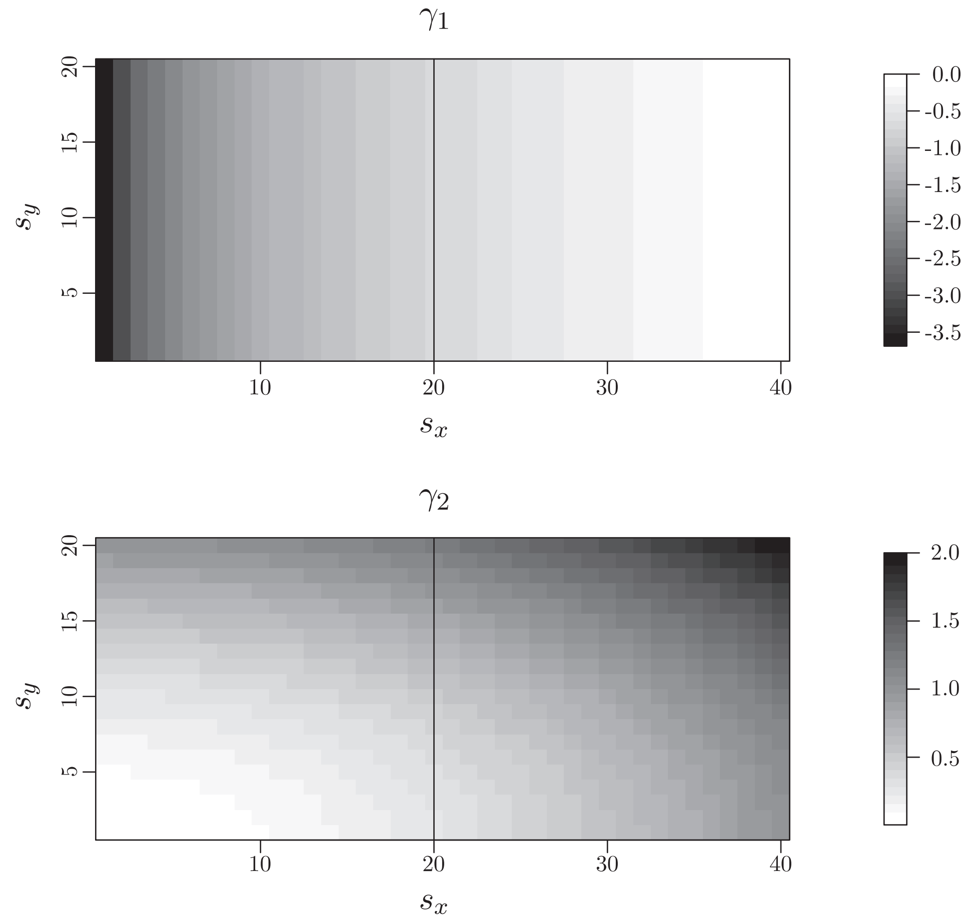

Figure 4. Image plots of the functions γ_1 and γ_2 over adjacent 20×20 spatial grids used to generate $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ in (4.2) of Section 4.1.

Table 2. Empirical powers (%) of the test of stationarity using subsampling for the spatial case in Section 4.1.

	20×20			40×40		
c	Two lags	Four lags	Eight lags	Two lags	Four lags	Eight lags
0.5	41.8	39.6	36.0	98.5	97.6	98.6
1.0	33.8	36.1	44.8	95.9	95.9	95.2
1.5	29.5	43.8	55.8	91.2	88.3	92.4

Note: Nominal level is 5%. Empirical powers are calculated based on 1,000 simulations and the largest standard error for powers is 1.6%. The value of c is defined in Table 1.

$i = 1, \ldots, k$, $t = 1, \ldots, T$. Here k is the number of observed points in the spatial domain D and T is the number of time points. We assumed that $\boldsymbol{\epsilon}_t$ is a mean zero Gaussian multivariate random field on a sphere with a spatially isotropic

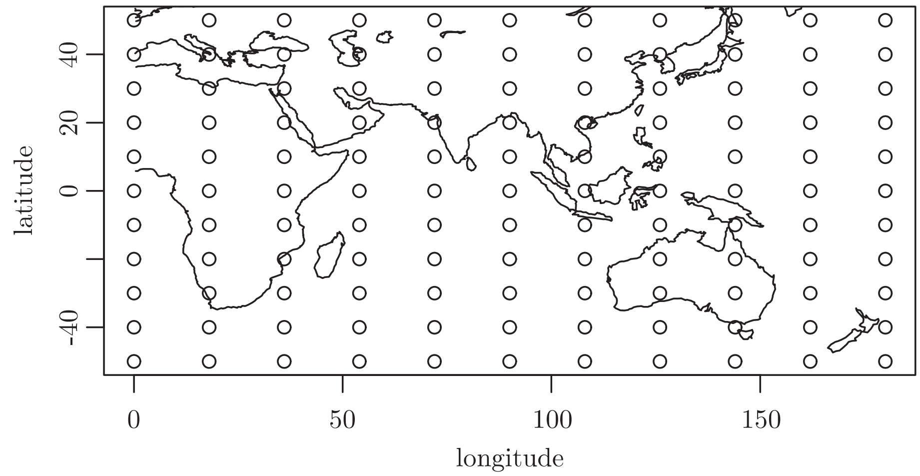

Figure 5. Locations on the sphere for the simulation in Section 4.2.

exponential correlation function and $\boldsymbol{\epsilon}_t$ is independent across time. The matrix $\mathbf{R}$ gives the autoregressive coefficients.

Figure 5 shows the spatial locations of points at which we simulated the data. There were 11 regularly spaced latitude points from 50° S to 40° N, and 11 regularly spaced longitude points from 0° to 180°. For the spatial covariance function, we used an isotropic function, $C_0(\mathbf{h}) = \exp(-||\mathbf{h}||/1,000)$ and for the temporal correlation, we used $\mathbf{R} = 0.4 \times \mathbf{I}_k$. The spatial distance used was chordal distance on the surface of a sphere, $\mathbf{I}_k$ is the identity matrix. Here $k = 11^2 = 121$ and $T = 400$. We estimated $\boldsymbol{\Sigma}$ using a subsampling technique. As described in Section 3.2, we tried random splitting of the domain with respect to latitude as well as with respect to longitude. We created 130 random splits for each splitting with respect to latitude and longitude. There are at most $\binom{11}{5} = 462$ ways of splitting the domain randomly into two disjoint domains and quite a few of them only contain pairs of spatial points with spatial lag greater than 1.

The assessment of the power of the test was done using a random field nonstationary with respect to latitude but stationary in longitude and time. We simulated the random field (with time structure still AR(1)) using the covariance function in Jun and Stein (2008). In particular, for a latitude point L and a longitude point l, we took

$$Z(L, l, 0) = \left\{ A(L)\frac{\partial}{\partial L} + B(L)\frac{\partial}{\partial l} \right\} W(L, l) \tag{4.3}$$

with $A(L) = 1+0.5P_1(\sin L)+P_2(\sin L)-0.5P_3(\sin L)$ and $B(L) = 1-2P_1(\sin L)-P_2(\sin L)+0.05P_3(\sin L)$, where P_n is the Legendre polynomial of degree n. The coefficients in the linear combinations for A and B were chosen arbitrarily. The

Table 3. Empirical sizes (%) and powers (%) of the test of axial symmetry and stationarity in time using subsampling for the spatial case in Section 4.2.

		SIZE		POWER	
split	$l(n)$	Λ_1	Λ_2	Λ_1	Λ_2
longitude	5	6.2	2.0	6.3	1.9
	10	6.5	5.2	6.5	5.2
	20	7.9	11.2	8.0	11.0
latitude	5	6.4	6.6	90.5	66.3
	10	6.6	11.7	90.5	69.6
	20	8.1	19.2	90.5	72.7

Note: Nominal level is 5%. Empirical sizes and powers are calculated based on 1,000 simulations and 130 random splits are performed. The largest standard error for sizes is 1.2% and that for powers is 1.5%. The size of subblock is $l(n)$.

random field W is a mean zero Gaussian random field with the covariance structure given by a Matérn covariance function,

$$\mathrm{Cov}\{W(L_1,l_1),W(L_2,l_2)\} = \Big(\frac{d}{1,000}\Big)^{1.5}\mathcal{K}_{1.5}\Big(\frac{d}{1,000}\Big),$$

where d is the chordal distance between the two points, (L_1,l_1) and (L_2,l_2), and $\mathcal{K}_\nu$ denotes the modified Bessel function of order ν. As discussed in Jun and Stein (2008), the model in (4.3) is axially symmetric but nonstationary with respect to latitude.

We used the following sets of lags:

$$\Lambda_1 = \{(L,l,u) : (1,0,0),(2,0,0)\},$$
$$\Lambda_2 = \{(L,l,u) : (0,1,1),(0,2,1),(0,1,2),(0,2,2)\}.$$

Here, the unit of latitude is 10° and that of longitude is 18°. The first set of lags, Λ_1, is to check the stationarity with respect to latitude, and Λ_2 is to check stationarity in longitude and time. The size of subblocks used was 5, 10, and 20.

Table 3 shows the empirical size and power for each split. It is interesting to note that when we split the domain with respect to longitude, we could not detect nonstationarity with respect to latitude. This is what we expect since if the random field is nonstationary with respect to latitude but stationary with respect to longitude, and if we split with respect to longitude, the expected difference of the estimators between the two splitted spatial domains should be zero. Therefore we should not see any difference in the empirical size and powers (except random variation) and both quantities should be close to the nominal level. However, when we split the domain with respect to latitude, we did detect nonstationarity with respect to latitude, as expected.

5. Applications

5.1. Pacific Ocean wind data

Consider the Pacific Ocean wind data that was studied by Wikle and Cressie (1999), Cressie and Huang (1999), Guan, Sherman, and Calvin (2004), and Li, Genton, and Sherman (2007). The data consist of the East-West component of the wind velocity vector from a small region over the tropical western Pacific Ocean for the period from November 1992 to February 1993. The time resolution is 6 hours and there are a total 480 time points. The spatial domain is a regularly spaced 17×17 grid with $1.875°$ (lon) by $1.9047°$ (lat) resolution. Wikle and Cressie (1999) considered spatial isotropic covariance models and Cressie and Huang (1999) fitted space-time stationary covariance models to these data. Guan, Sherman, and Calvin (2004) found some evidence against isotropy for this data set. In particular, they found that the variogram for the North-South direction is not bounded when the lag distance gets larger, which may suggest nonstationarity with respect to latitude.

We tested the stationarity assumption in the space and time domains using our procedure. The test is based on an asymptotic chi-squared distribution, and the time length $T = 480$ is large enough. Before we performed the hypothesis test, we needed to remove the fixed mean part of the data so that the residuals would be close to be mean zero. Wikle and Cressie (1999) and Li, Genton, and Sherman (2007) removed the temporal average of the data (average over time at each grid pixel) while Guan, Sherman, and Calvin (2004) did not explain if any procedure was done to make the process close to be mean zero before their analysis; the results in Cressie and Huang (1999) indicate that they used the raw data without removing the mean part. In our preliminary data analysis, however, there seems to be a persistent spatial pattern in the data over time and certainly removing temporal average should help to get rid of some of these patterns (see Figure 6(a)). We also found that even after removing temporal average from the data, there is still a non-negligible trend in the data (see Figure 6(b)). As with the tests in Guan, Sherman, and Calvin (2004) and Li, Genton, and Sherman (2007), our test assumes that the spatio-temporal field is mean zero. To see the effect of the fixed mean part, we performed the test on the raw data (after removing the grand mean), on the data with only temporal average removed, on the data with only spatial average removed, and on the data with both spatial and temporal averages removed.

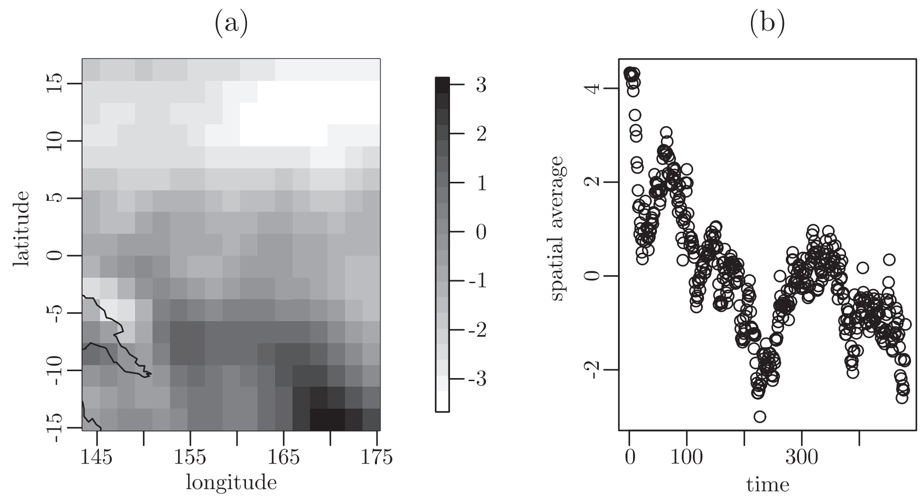

Figure 6. (a) Temporal average of the raw wind speed data. (b) Spatial average of the wind speed after removing temporal average from the raw data.

We considered several combinations of space-time lags for the test,

$$\begin{aligned}
\Lambda_1 =&\{(L,l,u) : (1,0,0),(2,0,0)\},\\
\Lambda_2 =&\{(L,l,u) : (0,1,0),(0,2,0)\},\\
\Lambda_3 =&\{(L,l,u) : (1,0,0),(0,1,0),(1,1,0)\},\\
\Lambda_4 =&\{(L,l,u) : (1,0,1),(0,1,1),(1,1,1)\}.
\end{aligned}$$

Here, the unit for the lag with respect to longitude is 1.875° and the one with respect to latitude is 1.9047°. The set Λ_1 is to test stationarity with respect to latitude, Λ_2 for longitude, Λ_3 for space, and Λ_4 for space and time together. We tried various sizes of subblocks and random splittings of the domain, some with respect to latitude and some with respect to longitude. For both cases of splitting with respect to latitude and longitude, we performed 1,000 random splittings as described in Section 3.2.

Figure 7 shows the p-values from the test. First notice that the test results on raw data and the data with spatial and temporal averages removed give large p-values for Λ_1 when we split the domain with respect to longitude, while p-values are small for the same lags when the split is with respect to latitude. This is a clear sign that the nonstationarity with respect to latitude is significant. We saw a similar phenomenon in the simulated data (Table 3). On the other hand, the p-values for Λ_2, Λ_3, and Λ_4 for the split with respect to longitude are quite different for the two data sets, probably because the temporal average (shown in Figure 6(a)) has a significant effect on the test statistics. We conclude that it

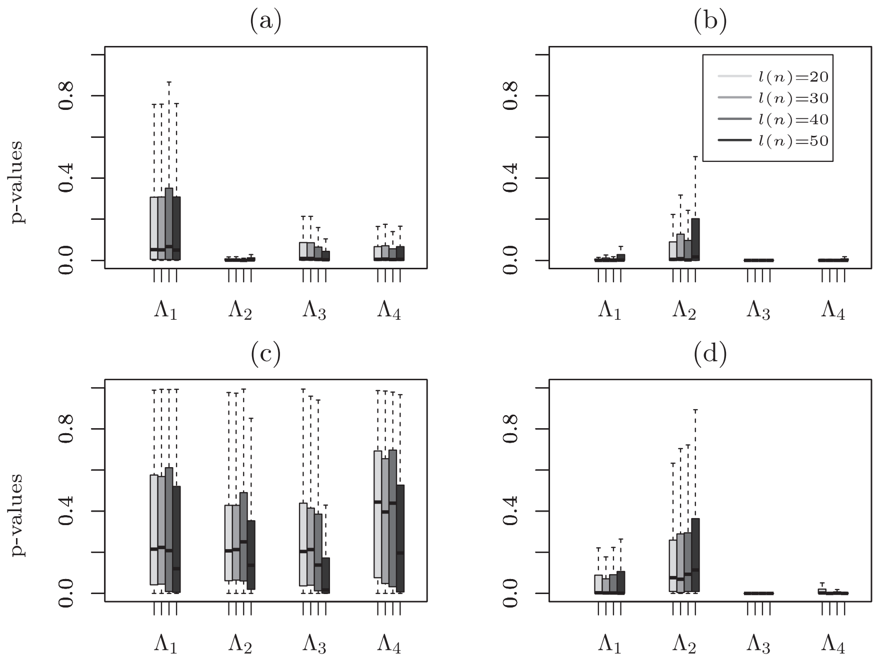

Figure 7. Boxplots of p-values for the tests with the wind data in Section 5.1, based on 1,000 random splitting of the domains. (a) Results on the raw data after removing the grand mean, with the split with respect to longitude. (b) Same data as (a), with the split with respect to latitude. (c) Results on the data after removing both spatial and temporal average, with the split with respect to longitude. (d) Same data as (c), with the split with respect to latitude. The subblock size is $l(n)$.

is necessary to remove both spatial and temporal averages from these data. We also checked that the p-values for the data with only temporal average removed and those for the data with spatial and temporal averages removed are quite similar (not shown). The p-values for Λ_2 for the data with spatial and temporal averages removed are quite large for both splitting schemes. This supports the argument that the random field is indeed axially symmetric.

5.2. Global surface temperature data

We analyzed monthly global surface temperature data over 50 years. In particular, we used observations of the combined data set of the land and the ocean from the Climate Research Unit, East Anglia and the Hadley Centre. The climate model outputs used in this paper are from the NOAA GFDL model (GFDL-CM2.0). Jun, Knutti, and Nychka (2008) modeled the difference between observations and climate model outputs as climate model errors (we call this

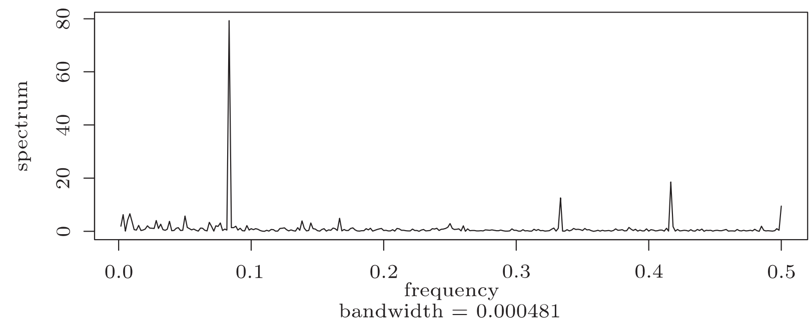

Figure 8. Raw periodogram of "diff" time series at spatial location 47.5° S and 47.5° E.

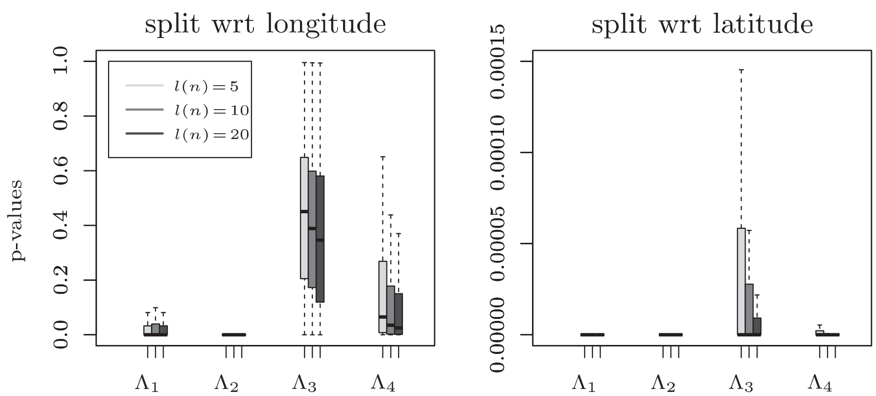

Figure 9. Boxplots of p-values for the tests with the "diff" process in Section 5.2. The distributions of p-values are generated from 1,000 random splitting of the longitude and latitude and $l(n)$ indicates different sizes of subblocks used in subsampling. Different scales of the y-axis are used in the two figures.

"diff" from now on) and used a simple nonstationary covariance model. They only considered a spatial component and their model is isotropic in space except that the covariance is larger over the land than over the ocean. Here we test the axial symmetry (and stationarity in time) as well as nonstationarity with respect to latitude.

Even though we removed the fixed mean part using the observations as an effective mean, we may still have some non-negligible mean part due to the complex error structure of the climate models. In particular, it turns out that the "diff" random field has non-negligible seasonality in many grid pixels. Figure 8 shows the periodogram of the "diff" time series at the grid pixel with latitude 47.5° S and longitude 47.5° E. The figure was generated using an R function, *spec.pgram*. Notice the peaks at several frequencies which correspond to an annual cycle and cycles of a few years length. Therefore, we used least squares with sine and cosine curves for the frequency of one year up to six years to remove seasonality. After this step, we did not see any noticeable mean structure.

The lags used for the test were

$$\begin{aligned}\Lambda_1 =&\{(L,l,u):(1,0,0)\},\\ \Lambda_2 =&\{(L,l,u):(1,0,0),(2,0,0)\},\\ \Lambda_3 =&\{(L,l,u):(0,1,1),(0,1,2)\},\\ \Lambda_4 =&\{(L,l,u):(0,1,1),(0,1,2),(0,2,1),(0,2,2)\}.\end{aligned}$$

Thus, Λ_1 and Λ_2 are to test for stationarity with respect to latitude, and Λ_3 and Λ_4 are for stationarity with respect to longitude and time. We performed random splitting 1,000 times with respect to longitude as well as an additional 1,000 times with respect to latitude. Figure 9 gives the p-values for each split. First notice that p-values for Λ_3 and Λ_4 are quite large when the split is with respect to longitude while they are quite small when the split is with respect to latitude (notice the scale difference of the y-axis for the two figures). In fact, the p-values for Λ_3 when the split is done with respect to longitude seem to have a similar distribution as the one in Figure 3. This is a clear sign that the random field is axially symmetric. The p-values for Λ_1 and Λ_2 for both ways to split are close to zero, which is an indication that the random field is strongly nonstationary with respect to latitude. We conclude that even if we model the difference between the observations and the climate model outputs, nonstationarity is significant with respect to latitude in the difference random field. Hence axially symmetric models that are nonstationary with respect to latitude found in Jun and Stein (2007, 2008) seem appropriate for modeling.

6. Discussion

In the proposed test, Theorems 1 and 2 require the covariance matrices $\hat{\mathbf{\Sigma}}$, $\hat{\mathbf{\Sigma}}_{11}$, and $\hat{\mathbf{\Sigma}}_{22}$ to be positive definite. In our simulation studies and data applications, we checked the eigenvalues of the estimates of these matrices for all lags considered and all subblock sizes used, and we found all eigenvalues positive.

We do not make the assumption that the random field is Gaussian. If it is, then weak stationary under the null hypothesis implies strong stationarity and the asymptotic results in Guan, Sherman, and Calvin (2004) and Li, Genton, and Sherman (2007, 2008a,b) apply. In particular, (2.5), (2.6), and (2.9) can be simplified as in Guan, Sherman, and Calvin (2004) and Li, Genton, and Sherman (2007, 2008a,b). However we still need to show the joint asymptotic normality of $\widehat{\mathbf{G}}_{n,1}$ and $\widehat{\mathbf{G}}_{n,2}$, as in A.2 of the Appendix.

One issue in performing our test is how to determine the subblock size. For a spatio-temporal random field, if the temporal structure of the random field is

AR(1), we use the subblock of size $l(n) = [2\gamma/(1-\gamma^2)]^{2/3}(3n/2)^{1/3}$, where n is the number of time points and γ is the estimate of the AR(1) coefficient (as noted in Li, Genton, and Sherman (2007)). For more general random fields, we can resort to model-free approaches such as those in Lahiri (2003). Alternatively, we may use ideas as in Shao and Li (2009) who used inconsistent estimators to normalize out the asymptotic covariance matrix, so as to be free of the parameter for the subblock size. This procedure may, however, suffer from a loss of power.

Many data sets do not come from a stationary process. However, common practice is to fit stationary (in fact, isotropic) models to them without thoroughly checking the assumptions. In fact one finds only a limited number of nonstationary covariance models. On the other hand, many global data sets exhibit stationarity with respect to longitude and time in the covariance structure, although they have strong nonstationarity with respect to latitude. Our motivation is to provide a formal statistical testing procedure of the stationarity assumption, in particular, the axial symmetry assumption and the stationarity assumption in time. If we fail to reject the latter, we would recommend using the stationary models to fit the data. If we do reject the stationarity assumption, we need to develop a nonstationary model that can properly describe the nonstationarity in the data.

Even if we reject the null hypothesis in our test, it is possible that the process is still weakly stationary but some of the moment or mixing conditions required for the asymptotic normality do not hold. This type of problem is common in the test based on asymptotic results and similar problems exist in the tests proposed by Guan, Sherman, and Calvin (2004), Li, Genton, and Sherman (2007, 2008a,b). At least for the data sets considered here, we found the moment and mixing conditions required for Theorem 2 to be reasonable.

There may be situations where our test has little power in detecting nonstationarity in space, for example, if the nonstationarity varies slowly for short distances in space. For testing stationarity in time of a spatio-temporal random field, in some situations, especially for a random field that does not induce space-time interactions in the covariance structure (i.e. spatio-temporal separable), the proposed test may not detect nonstationarity in time. Then we may need to split the temporal domain as we split the spatial domain to detect nonstationarity in time and may need to adjust the theory accordingly.

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Appendix: Proof of Theorem 1

We first show that the asymptotic normality of $\sqrt{|D_{n,i}|}\ (\widehat{\mathbf{G}}_{n,i} - \mathbf{G})^T$, for each i, holds under the conditions given in Section 2.1. Then we show the joint asymptotic normality of $(\sqrt{|D_{n,1}|}(\widehat{\mathbf{G}}_{n,1} - \mathbf{G}), \sqrt{|D_{n,2}|}(\widehat{\mathbf{G}}_{n,2} - \mathbf{G}))^T$.

A.1 Asymptotic normality of $\sqrt{|D_{n,1}|}(\widehat{\mathbf{G}}_{n,1} - \mathbf{G})^T$: If (2.5) holds, then the existence of the limit of the covariance of $\widehat{\mathbf{G}}_{n,1}$ is trivial. For a lag $\mathbf{h} \in \Lambda$, we let $\sigma_1^2 = \lim_{n\to\infty} |D_{n,1}|\mathrm{Cov}\{\widehat{\mathbf{G}}_{n,1}(\mathbf{h}) - \mathbf{G}(\mathbf{h}), \widehat{\mathbf{G}}_{n,1}(\mathbf{h}) - \mathbf{G}(\mathbf{h})\}$.

Let $S_{n,1} \equiv \sqrt{|D_{n,1}|}(\widehat{\mathbf{G}}_{n,1}(\mathbf{h}) - \mathbf{G}(\mathbf{h}))$. We show $S_{n,1} \xrightarrow{d} N(0, \sigma_1^2)$. Let $l(n) = n^\alpha$, $m(n) = n^\alpha - n^\eta$ for some $\frac{4}{2+\epsilon} < \eta < \alpha < 1$. Divide $D_{n,1}$ into nonoverlapping $l(n) \times l(n)$ subsquares, $D^j_{l(n),1}$, $j = 1, \ldots, k_n$. Within each subsquare, we further get $D^j_{m(n),1}$, which shares the same center as $D^j_{l(n),1}$. Therefore we have $d(D^j_{m(n),1}, D^{j'}_{m(n),1}) \geq n^\eta$, $j \neq j'$. Now for a particular lag $\mathbf{h}$, let $\widehat{\mathbf{G}}^j_{m(n),1}(\mathbf{h})$ denote the statistic obtained from $D^j_{m(n),1}$. Let $s_{n,1} = \sum_{j=1}^{k_n} s^j_{n,1}/\sqrt{k_n}$ and $s'_{n,1} = \sum_{j=1}^{k_n}(s^j_{n,1})'/\sqrt{k_n}$, where $s^j_{n,1} = m(n)\{\widehat{\mathbf{G}}^j_{m(n),1}(\mathbf{h}) - \mathbf{G}(\mathbf{h})\}$ and $(s^j_{n,1})'$ have the same marginal distribution as $s^j_{n,1}$ but are independent.

Let $\phi_{n,1}(x)$ and $\phi'_{n,1}(x)$ be the characteristic functions of $s_{n,1}$ and $s'_{n,1}$, respectively. We use three steps to complete the proof:

S1. $S_{n,1} - s_{n,1} \xrightarrow{p} 0$;

S2. $\phi'_{n,1}(x) - \phi_{n,1}(x) \longrightarrow 0$;

S3. $s'_{n,1} \xrightarrow{d} N(0, \sigma_1^2)$.

Proof of S1. It suffices to show $\mathrm{E}(S_{n,1} - s_{n,1})^2 \to 0$ as $n \to 0$. Let $D_1^{m(n)}$ denote the union of all $D^j_{m(n),1}$. Simple algebra shows that $s_{n,1}$ can be written as $\sqrt{|D_1^{m(n)}|}\{\widehat{\mathbf{G}}_{D_1^{m(n)}}(\mathbf{h}) - \mathbf{G}(\mathbf{h})\}$. Given (2.2), it can be shown that $|D_1^{m(n)}|/|D_{n,1}|$

$\to 1$. Then using the moment condition in (2.5), we get $\mathrm{E}(S_{n,1} - s_{n,1})^2 \to 0$ as $n \to 0$.

Proof of S2. Let ι be the imaginary unit. We set $U_i = \exp\big(\iota x(s^i_{n,1}/\sqrt{k_n})\big)$, $X_k = \prod_{i=1}^k U_i$, and $Y_k = U_{k+1}$. Due to Lemma A.2 (b) of Ekström (2008), we have $|\mathrm{Cov}(X_k, Y_k)| \le c \cdot \alpha(n^2, n^\eta)$ for some constant $c > 0$. We apply a similar telescope argument as in p.338 of Ibragimov and Linnik (1971). Suppose the characteristic function of $s^i_{n,1}/\sqrt{k_n}$ is given by $\phi^{(i)}_{n,1}$ (since we do not assume strong stationarity, $s^i_{n,1}/\sqrt{k_n}$ may not have the same distribution). Then

$$
\begin{aligned}
&|\phi_{n,1}(x) - \prod_{i=1}^{k_n} \phi^{(i)}_{n,1}(x)| = |E(X_{k_n}) - \prod_{i=1}^{k_n} \phi^{(i)}_{n,1}(x)| \\
&= |E(X_{k_n}) - \phi^{(k_n)}_{n,1}(x) E(X_{k_n-1}) + \phi^{(k_n)}_{n,1}(x) E(X_{k_n-1}) - \prod_{i=1}^{k_n} \phi^{(i)}_{n,1}(x)| \\
&\le |E(X_{k_n}) - \phi^{(k_n)}_{n,1}(x) E(X_{k_n-1})| + |E(X_{k_n-1}) - \prod_{i=1}^{k_n-1} \phi^{(i)}_{n,1}(x)| \\
&= |\mathrm{Cov}(X_{k_n-1}, Y_{k_n-1})| + |E(X_{k_n-1}) - \prod_{i=1}^{k_n-1} \phi^{(i)}_{n,1}(x)| \\
&\le c \cdot \alpha(n^2, n^\eta) + |E(X_{k_n-1}) - \prod_{i=1}^{k_n-1} \phi^{(i)}_{n,1}(x)| \\
&\le \cdots \le k_n \cdot c \cdot \alpha(n^2, n^\eta) \to 0 \text{ as } n \to 0. \qquad (6.1)
\end{aligned}
$$

Note that (6.1) holds due to (2.3). Since $(s^i_{n,1})'$ has the same marginal distribution as $s^i_{n,1}$ but is independent of it, (6.1) implies $|\phi_{n,1}(x) - \phi'_{n,1}(x)| \to 0$ as $n \to 0$.

Proof of S3. This follows from the Lyapounov Theorem.

Proof of the joint normality. The joint normality of the vector across several spatial lags can be shown using the Cramér-Wold device.

A.2 Joint asymptotic normality of $(\sqrt{|D_{n,1}|}(\widehat{\mathbf{G}}_{n,1} - \mathbf{G}), \sqrt{|D_{n,2}|}(\widehat{\mathbf{G}}_{n,2} - \mathbf{G}))^T$: The basic idea of the proof is similar to the one in A.1. If (2.5) holds, then the existence of the limit of the covariance between $\widehat{\mathbf{G}}_{n,i}$ and $\widehat{\mathbf{G}}_{n,j}$ $(i, j = 1, 2)$ is trivial. For a lag $\mathbf{h} \in \Lambda$, we let $\sigma_i^2 = \lim_{n\to\infty} |D_{n,i}| \mathrm{Var}\{\widehat{\mathbf{G}}_{n,i}(\mathbf{h}) - \mathbf{G}(\mathbf{h})\}$ for $i = 1, 2$, and $\sigma_{ij} = \lim_{n\to\infty} \sqrt{|D_{n,i}||D_{n,j}|} \mathrm{Cov}\{\widehat{\mathbf{G}}_{n,i}(\mathbf{h}) - \mathbf{G}(\mathbf{h}), \widehat{\mathbf{G}}_{n,j}(\mathbf{h}) - \mathbf{G}(\mathbf{h})\}$ for $i \neq j$.

Let a and b be two arbitrary real numbers such that $|a| \le 1$, $|b| \le 1$ and at least one of them is not zero. Set

$$S_n = a\sqrt{|D_{n,1}|}\{\widehat{\mathbf{G}}_{n,1}(\mathbf{h}) - \mathbf{G}(\mathbf{h})\} + b\sqrt{|D_{n,2}|}\{\widehat{\mathbf{G}}_{n,2}(\mathbf{h}) - \mathbf{G}(\mathbf{h})\}$$
$$\equiv aS_{n,1}(\mathbf{h}) + bS_{n,2}(\mathbf{h}).$$

Also let $s_n = as_{n,1}(\mathbf{h}) + bs_{n,2}(\mathbf{h})$ and $s_n^{'} = as_{n,1}^{'}(\mathbf{h}) + bs_{n,2}^{'}(\mathbf{h})$. We further assume that $(s_{n,1}^j, s_{n,2}^j)$ and $((s_{n,1}^j)^{'}, (s_{n,2}^j)^{'})$ have the same distribution for each j and $(s_{n,1}^j)^{'}$ and $(s_{n,2}^j)^{'}$ are independent for each j. Moreover, we assume $(s_{n,1}^j)^{'}$ and $(s_{n,2}^{j'})^{'}$ are independent for $j \neq j^{'}$.

Let $\phi_n(x)$ and $\phi_n^{'}(x)$ be the characteristic functions of s_n and $s_n^{'}$, respectively. We follow three steps to complete the proof:

S4. $S_n - s_n \xrightarrow{p} 0$;

S5. $\phi_n^{'}(x) - \phi_n(x) \longrightarrow 0$;

S6. $s_n^{'} \xrightarrow{d} N(0, \sigma^2)$ with $\sigma^2 = a^2\sigma_1^2 + b^2\sigma_2^2 + 2ab\sigma_{12}$.

Proof of S4. Due to the result in S1, it is trivial to show $S_n - s_n \xrightarrow{p} 0$.
Proof of S5. Due to the result in S2 and a similar telescope argument, we can show that

$$|\phi_n^{'}(x) - \phi_n(x)| \to 0 \text{ as } n \to 0.$$

Proof of S6. This again follows from the Lyapounov Theorem.
Proof of the joint normality. This also follows from the Cramér-Wold device.

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