Statistica Sinica **16**(2006), 1387-1408

# TWO-WAY HETEROSCEDASTIC ANOVA WHEN THE NUMBER OF LEVELS IS LARGE

Lan Wang and Michael G. Akritas

*University of Minnesota and The Penn State University*

*Abstract:* We consider testing for main treatment effects and interaction effects in crossed two-way layouts when one or both factors have large number of levels. Random errors are allowed to be nonnormal and heteroscedastic. In the heteroscedastic case, we propose new test statistics. The asymptotic distributions of our test statistics are derived under both the null hypothesis and local alternatives. The sample size per treatment combination can either be fixed or tend to infinity. Numerical simulations indicate that the proposed procedures have good power properties and maintain approximately the nominal $\alpha$-level with small sample sizes. A data set from a study evaluating forty varieties of winter wheat in a large-scale agricultural trial is analyzed.

*Key words and phrases:* Heteroscedasticity, large number of factor levels, local alternatives, projection method, quadratic forms, unbalanced designs.

## 1. Introduction

In many experiments, data are collected in the form of a crossed two-way layout. If we let $X_{ijk}$ denote the $k$th response associated with the $i$th level of factor $A$ and $j$th level of factor $B$, then the classical two-way ANOVA model specifies that

$$X_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ijk}, \tag{2.1}$$

where $i = 1, \ldots, a,\ j = 1, \ldots, b,\ k = 1, \ldots, n_{ij}$, and to be identifiable the parameters are restricted by conditions such as $\sum_{i=1}^{a} \alpha_i = \sum_{j=1}^{b} \beta_j = \sum_{i=1}^{a} \gamma_{ij} = \sum_{j=1}^{b} \gamma_{ij} = 0$. The classical ANOVA model assumes that the error terms $\epsilon_{ijk}$ are iid normal with mean 0, in which case the F statistics for testing the null hypotheses of no treatment effects or no interaction effects have certain optimality properties (cf., Arnold (1981, Chap. 7)).

The study of properties of F-tests under violation of the classical assumptions of normality and homoscedasticity has a long history. See for example Box (1954), Box and Anderson (1955), Scheffé (1959, Chap. 10), Miller (1986, Chap. 4). However, these studies are restricted to the case with small number of treatment levels. In this case, Arnold (1980) showed that the classical F-test is robust to normality if in addition the sample size per treatment level tends to infinity. Portnoy (1984, 1986) investigated the asymptotic behavior of $M$-estimators in the general linear model when the number of treatment levels goes to infinity with the sample size. Li, Lindsay and Waterman (2003) discussed adjusted maximun likelihood estimator for multistratum data, in which both the number of strata and the number of within-stratum replications go to $\infty$.

Recently, there has been some interest in investigating the behavior of testing procedures when the number of treatment levels is large, see Boos and Brownie (1995), Akritas and Arnold (2000), Bathke (2002), Akritas and Papadatos (2004). The results in the present paper can be applied in many disciplines. For example, in agricultural trials it is not uncommon to see large number of treatments (such as cultivars, pesticides, fertilizers) but limited replications per treatment combination. In a recent statewide agricultural study performed by Washington State University, 40 different varieties/lines of winter wheat are investigated. This data set will be analyzed in Section 5. Our tests can also be applied to certain types of microarray data, where one factor corresponds to a large number of genes. Dudoit, Yang, Callow and Speed (2002) described a replicated cDNA microarray experiment to compare the expression of genes in the livers of SR-BI transgenic mice with that of the corresponding wild-type mice. The data were summarized in a two-way layout, and our method can be applied to test for main and interaction effects.

The asymptotic theory of ANOVA test statistics when the number of levels tends to infinity is more complex than that when the levels are fixed and the sample sizes tend to infinity. For example, the test statistic for no main factor $A$ effects in the balanced case (so $n_{ij} = n$) is $MST_A/MSE$, where $MST_A = b(a-1)^{-1}\sum_i n(\overline{X}_{i\cdot\cdot} - \overline{X}_{\cdot\cdot\cdot})^2$ and $MSE = [ab(n-1)]^{-1}\sum_i\sum_j\sum_k(\overline{X}_{ijk} - \overline{X}_{ij\cdot})^2$. Thus it is easily guessed that its limiting distribution, as $n \to \infty$ and $a$, $b$ are fixed, is a constant multiple of a $\chi^2$ distribution. On the other hand, when $a \to \infty$ the degrees of freedom of both $MST_A$ and $MSE$ tend to infinity, and finding the limiting distribution requires that we study $a^{1/2}(MST_A/MSE - 1)$. Except for the one-way design, the aforementioned papers have considered only balanced homoscedastic models and showed that the usual $F$-procedure is asymptotically correct.

Hypothesis testing in an unbalanced design is much more complex than in balanced case. For the general situation, a closed-form expression for the test statistic may not be available (Arnold (1981)). Besides, one often needs to specify appropriate weights for the hypothesis. Different methods have been proposed (mostly for the homoscedastic case) and there still remain many controversies. Arnold (1981) reviewed five different methods and also discussed the problem of choosing appropriate weights for the hypotheses, see also discussions in Ananda and Weerahandi (1997), Rencher (2000) and Sahai and Ageel (2000). In this paper, we are interested in unbalanced heteroscedastic two-way ANOVA design when at least one of the factor levels tends to infinity. The cell sizes can be fixed or tend to infinity. For the homoscedastic model, we consider test statistics based on the method of unweighted means, originally proposed by Yates (1934), see also Searle (1971) and Hocking (1996). Since the method of unweighted means is valid only in the homoscedastic case, we propose and study a new statistic when the errors are heteroscedastic. One interesting aspect of the new statistic is that its limiting distribution does not depend on the fourth moment. Since higher moments are typically more difficult to estimate accurately, the new statistic is computationally competitive even when homoscedasticity can be ascertained.

We focus on the hypotheses of no main factor $A$ effects and no interaction effects:

$$H_0(A) : \alpha_i = 0,\ i = 1, \ldots, a, \text{ and } H_0(C) : \gamma_{ij} = 0, \quad i = 1, \ldots, a,\ j = 1, \ldots, b.$$

When both $a$ and $b$ tend to infinity, the problem of testing $H_0(B) : \beta_j = 0,\ j = 1, \ldots, b$, is symmetric to that of testing $H_0(A)$. When only $a$ tends to infinity but $b$ stays fixed, it can be shown that the test statistic for $H_0(B)$ has asymptotically a chi-square distribution. This result is different because the numerator has fixed degrees of freedom, and it will not be presented here. Finally, similar techniques apply for testing for no simple effects. For the sake of brevity, these results are not presented here but can be found in Wang (2003).

The rest of the paper is organized as the following. In Section 2, we present the main results for the homoscedastic and heteroscedastic cases, including a limiting result under local alternatives. The projection method is discussed in Section 3. Numerical results and a data analysis are given in Sections 4 and 5, respectively. Section 6 discusses conclusions and further work, while a sketch of the technical proofs is given in the Appendix.

## 2. Main Results

Throughout, the $X_{ijk}$ are assumed to be iid in each cell $(i, j)$. But the errors $\epsilon_{ijk}$ in (2.1) are not assumed to be iid. We write $\mathbf{X} = (X_{111}, \ldots, X_{11n_{11}}, \ldots, X_{1b1}, \ldots, X_{1bn_{1b}}, \ldots, X_{ab1}, \ldots, X_{abn_{11}})'$, $\mathbf{X}_i = (X_{i11}, \ldots, X_{i1n_{i1}}, \ldots, X_{ib1}, \ldots, X_{ibn_{ib}})'$, $\mathbf{X}_{ij} = (X_{ij1}, \ldots, X_{ijn_{ij}})'$, $\overline{X}_{ij.} = n_{ij}^{-1} \sum_{k=1}^{n_{ij}} X_{ijk}$, $\widetilde{X}_{i..} = b^{-1} \sum_{j=1}^{b} \overline{X}_{ij.}$, $\widetilde{X}_{...} = (ab)^{-1} \sum_{i=1}^{a} \sum_{j=1}^{b} \overline{X}_{ij}$ and $\widetilde{X}_{.j.} = a^{-1} \sum_{i=1}^{a} \overline{X}_{ij.}$, $N = \sum_i \sum_j n_{ij}$. If the cell sizes tend to infinity with $a$ or $b$, we can also write $n_{ij}(a, b)$ instead of $n_{ij}$, we set $n(a, b) = \min\{n_{ij}; i = 1, \ldots, a, j = 1, \ldots, b\}$, $\kappa(a, b) = \max\{n_{ij}; i = 1, \ldots, a, j =$

$1,\dots,b\}$, and assume that $n(a,b)\to\infty$ and $\kappa(a,b)/n(a,b)\le C<\infty$, for all $a,b$, for some $C\ge 1$. In the case when only $a\to\infty$, we also use the notations: $n_{ij}(a)$, $n(a)$ and $\kappa(a)$ without confusion. Finally, $\mathbf{1}_d$ and $\mathbf{I}_d$ denote the $d\times 1$ column vector of 1's and the $d$-dimensional identity matrix, respectively, $\mathbf{J}_d=\mathbf{1}_d\mathbf{1}_d'$ and $\mathbf{P}_d=\mathbf{I}_d-\mathbf{J}_d/d$.

### 2.1. Homoscedastic model

The test statistics for testing hypotheses $H_0(A)$ and $H_0(C)$ are

$$Q_A=\frac{MST_A}{MSE}\quad\text{and}\quad Q_C=\frac{MST_C}{MSE}, \tag{2.1}$$

respectively, where

$$MST_A=\frac{b}{\frac{a-1}{ab}\sum_{i=1}^a\sum_{j=1}^b\frac{1}{n_{ij}}}\sum_{i=1}^a\left(\widetilde{X}_{i..}-\widetilde{X}_{...}\right)^2,$$

$$MST_C=\frac{1}{\frac{(a-1)(b-1)}{ab}\sum_{i=1}^a\sum_{j=1}^b\frac{1}{n_{ij}}}\sum_{i=1}^a\sum_{j=1}^b\left(\overline{X}_{ij.}-\widetilde{X}_{i..}-\widetilde{X}_{.j.}+\widetilde{X}_{...}\right)^2,$$

$$MSE=\frac{1}{N-ab}\sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^{n_{ij}}\left(X_{ijk}-\overline{X}_{ij.}\right)^2.$$

In the balanced case, $Q_A$ and $Q_C$ are the same as the classical $F$ test statistics. In the unbalanced case, they correspond to the test statistics of Yates, also known as the method of unweighed, or harmonic means.

**Theorem 2.1.**(Balanced case) *Let* $\mathrm{Var}\,(X_{ijk})=\sigma^2>0$ *and assume that* $E(X_{ijk}^4)$ *are uniformly bounded.*

(a) *Suppose for all* $i,j$, $n_{ij}=n\ge 2$ *remains fixed.*
1. *Under* $H_0(A)$, $a^{1/2}(Q_A-1)\to N\left(0,2+2/(b(n-1))\right)$, *as* $a\to\infty$ *and* $b$ *is fixed,* $a^{1/2}(Q_A-1)\to N(0,2)$, *as* $a\to\infty$ *and* $b\to\infty$.
2. *Under* $H_0(C)$, $a^{1/2}(Q_C-1)\to N\left(0,2/(b-1)+2/(b(n-1))\right)$, *as* $a\to\infty$ *and* $b$ *is fixed,* $(ab)^{1/2}(Q_C-1)\to N\left(0,2+2/(n-1)\right)$, *as* $a\to\infty$ *and* $b\to\infty$.

(b) *Suppose for all* $i,j$, $n_{ij}=n=n(a)\to\infty$, *as* $a\to\infty$.
1. *Under* $H_0(A)$, $a^{1/2}(Q_A-1)\to N(0,2)$, *as* $a\to\infty$ *and* $b$ *is fixed,* $a^{1/2}(Q_A-1)\to N(0,2)$, *as* $a\to\infty$ *and* $b\to\infty$,
2. *Under* $H_0(C)$, $a^{1/2}(Q_C-1)\to N\left(0,2/(b-1)\right)$, *as* $a\to\infty$ *and* $b$ *is fixed,* $(ab)^{1/2}(Q_C-1)\to N(0,2)$, *as* $a\to\infty$ *and* $b\to\infty$.

**Remark 2.1.**

(a) If we assume all the errors $\epsilon_{ijk}$ are iid then the result of Theorem 2.1 requires only finite second moments.

(b) Results for the balanced case with fixed sample sizes when $a \to \infty$ and $b$ is fixed overlap with those of Akritas and Arnold (2000), but our assumptions are weaker.

**Theorem 2.2.**(Unbalanced case) *Let* $\mathrm{Var}\,(X_{ijk}) = \sigma^2 > 0$. *Assume that for some* $\delta > 0$, $E\left(|X_{ijk}|^{4+\delta}\right) < \infty$ *are uniformly bounded, and set* $\mu_4 = E(X_{ijk} - EX_{ijk})^4/\sigma^4$.

(a) *Suppose* $\limsup\,(ab)^{-1}\sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij}^{2+\delta} < \infty$ *for some* $\delta > 0$. *And write*

$$\frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij} \to b_1 \in (1,\infty),\quad \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b}\frac{1}{n_{ij}} \to b_2,\quad \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b}\frac{1}{n_{ij}^2} \to b_3,$$

$$\frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b}\frac{1}{n_{ij}^3} \to b_4,\quad \frac{1}{ab^2}\sum_{i=1}^{a}\Big(\sum_{j=1}^{b}\frac{1}{n_{ij}}\Big)^2 \to b_5,\ \textit{and}$$

$$\tau_1^2 = \tau^* + \frac{2bb_5}{b_2^2},\ \tau_2^2 = \tau^* + \frac{2(b_3b^2 - 2bb_3 + bb_5)}{b_2^2(b-1)^2},\ \tau_3^2 = \tau^* + \frac{2b_3}{b_2^2},$$

*where* $\tau^* = [(b_2-b_1)(b_1-1)^{-2} + b_4 b_2^{-2} + 2b_3 b_2^{-1}(b_1-1)^{-1}](\mu_4-3) + 2/(b_1-1)$.

1. *Under* $H_0(A)$, $a^{1/2}(Q_A - 1) \to N\left(0, \tau_1^2/b\right)$, *as* $a \to \infty$ *and* $b$ *is fixed,* $a^{1/2}(Q_A - 1) \to N\left(0, 2b_5/b_2^2\right)$, *as* $a \to \infty$ *and* $b \to \infty$.
2. *Under* $H_0(C)$, $a^{1/2}(Q_C - 1) \to N\left(0, \tau_2^2/b\right)$, *as* $a \to \infty$ *and* $b$ *is fixed,* $(ab)^{1/2}(Q_C - 1) \to N\left(0, \tau_3^2\right)$, *as* $a \to \infty$ *and* $b \to \infty$.

(b) *Suppose* $n_{ij}$ *also tends to infinity with* $a$ *or* $b$. *And write*

$$\frac{a}{\left(\sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij}^{-1}\right)^2}\sum_{i=1}^{a}\Big(\sum_{j=1}^{b} n_{ij}^{-1}\Big)^2 \to \tau_4^2,$$

$$\frac{ab}{\left(\sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij}^{-1}\right)^2}\sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij}^{-2} \to \tau_5^2.$$

1. *Under* $H_0(A)$, $a^{1/2}(Q_A - 1) \to N\left(0, 2\tau_4^2\right)$, *as* $a \to \infty$ *and* $b$ *is fixed,* $a^{1/2}(Q_A - 1) \to N\left(0, 2\tau_4^2\right)$, *as* $a \to \infty$ *and* $b \to \infty$.
2. *Under* $H_0(C)$, $a^{1/2}(Q_C - 1) \to N(0, 2\tau_4^2/(b-1)^2 + (2\tau_5^2/b)(1-(b-1)^{-2}))$, *as* $a \to \infty$ *and* $b$ *is fixed.* $(ab)^{1/2}(Q_C - 1) \to N\left(0, 2\tau_5^2\right)$, *as* $a \to \infty$ *and* $b \to \infty$.

**Remark 2.2.** The exact distribution of the unweighted means statistic in the unbalanced case is not known even under normality and homoscedasticity. Neter, Wasserman and Kutner (1985, p.753) indicate that the usual $F$ critical points give satisfactory approximation provided the ratios of sample sizes do not exceed 2, with most $n_{ij}$ agreeing more closely. Contrary to the balanced case, the presence of $\beta_2$ in the limiting distribution given in Theorem 2.2 for fixed sample sizes implies that the unweighted means procedure (i.e., using the unweighted means test statistic with critical points from the $F$ distribution) is not robust to departures from the normality assumption even under homoscedasticity. If the sample sizes also tend to infinity, the contribution of $\mu_4$ in the limiting distribution becomes negligible, but even then Theorem 2.2 shows that the procedure is not valid asymptotically. To see this, let $N = \sum_i \sum_j n_{ij}$, suppose that only $a \to \infty$ while $b$ remains fixed, and set $\overline{n} = \lim_{a\to\infty}(ab)^{-1}N$. Let $U_{ab}$ have an $F_{a-1,N-ab}$ distribution, the approximate distribution of the unweighted means statistic for no main factor $A$ effects under homoscedasticity and normality. It is easily verified that if $\overline{n} < \infty$, $a^{1/2}(U_{ab} - 1) \to N(0, 2 + 2/[b(\overline{n} - 1)])$ as $a \to \infty$, while if $N/a$ also tends to infinity with $a$ then $a^{1/2}(U_{ab} - 1) \to N(0, 2)$. It follows that the unweighted means procedure for testing for no main factor $A$ effects will be asymptotically valid if the limiting distribution of $a^{1/2}(MST_A/MSE - 1)$ is as above. Similarly the usual $F$-procedure for testing for no interaction effects will be asymptotically valid if the limiting distribution of $a^{1/2}(MST_C/MSE - 1)$ as $a \to \infty$ is $N(0, 2/(b-1) + 2/[b(\overline{n} - 1)])$ if $\overline{n} < \infty$, and is $N(0, 2/(b-1))$ if $N/a$ also tends to infinity with $a$. According to Theorem 2.2 the limiting distributions of $a^{1/2}(MST_A/MSE - 1)$ and $a^{1/2}(MST_C/MSE - 1)$ are different.

To appreciate the difference, and to see if Neter, Wasserman and Kutner's recommendation applies also to the case where one of the factors has many levels, we considered the test for no factor $A$ effects using normal observations (i.e., $\beta_2 = 3$) in a design with $a = 100$, $b = 4$, and $n_{ij} = 4$ for $i = 1, \ldots, 50$ and all $j$, and $n_{ij} = 8$ for $i = 51, \ldots, 100$ and all $j$. Then the 95th percentiles of the limiting distribution of $a^{1/2}(MST_A/MSE - 1)$ and the limit of its approximate distribution are $W_{1,0.05} = 2.51$ and $W_{2,0.05} = 2.38$, respectively. Use of $W_{1,0.05}$ results in an approximate rejection rate of 0.059 under the null hypothesis. Changing the sample size 8 to 12, 16 and 20, the rejection rates become 0.070, 0.079, and 0.085, respectively. Thus, Neter, Wasserman and Kutner's recommendation seems valid even with many levels, provided the normality assumption holds.

**Remark 2.3.**

(a) It can be shown that if $n_{ij} = n$, then in Theorem 2.2, $\tau_1^2 = 2(n-1)^{-1} + 2b$ and $\tau_2^2 = 2(n-1)^{-1} + 2b(b-1)^{-1}$, $2b_5 b_2^{-1} = 2$ and $\tau_3^2 = 2 + 2(n-1)^{-1}$. The results are therefore consistent with those in Theorem 2.1 if the design is balanced.

(b) We need not assume, as we do in Theorem 2.2, that the kurtosis $\mu_{4ij}$ is the same in all cells $(i,j)$. In this more general case, the test statistics will have more complex variance expressions, but the asymptotic normality still holds under similar assumptions.

### 2.2. Heteroscedastic model

In the balanced heteroscedastic case, we still have $E(MSE) = E(MST_A)$, $E(MSE) = E(MST_C)$, under $H_0(A)$, $H_0(C)$, respectively. Thus, we may still use $Q_A$, $Q_C$ as test statistics. This, however, is not true in the unbalanced case. A simple remedy is to replace $MSE$ by a different linear combination of the cell sample variances. This yields

$$T_A = (ab)^{-1/2} \sum_{i=1}^{a} \sum_{j=1}^{b} \left[ \left( \widetilde{X}_{i\cdot\cdot} - \widetilde{X}_{\cdot\cdot\cdot} \right)^2 - \frac{1}{b} \left( 1 - \frac{1}{a} \right) \frac{S_{ij}^2}{n_{ij}} \right], \tag{2.2}$$

$$T_C = (ab)^{-1/2} \sum_{i=1}^{a} \sum_{j=1}^{b} \left[ \left( \overline{X}_{ij\cdot} - \widetilde{X}_{i\cdot\cdot} - \widetilde{X}_{\cdot j\cdot} + \widetilde{X}_{\cdot\cdot\cdot} \right)^2 - \frac{(a-1)(b-1)}{ab} \frac{S_{ij}^2}{n_{ij}} \right], \tag{2.3}$$

where $S_{ij}^2 = (n_{ij}-1)^{-1} \sum_{k=1}^{n_{ij}} (X_{ijk} - \overline{X}_{ij\cdot})^2$, as test statistics for $H_0(A)$, $H_0(C)$, respectively.

In the balanced case, $T_A$ and $T_C$ are related to $Q_A$ and $Q_C$ by

$$T_A = (ab)^{-\frac{1}{2}} \frac{a-1}{n} (Q_A - 1) MSE, \quad T_C = (ab)^{-\frac{1}{2}} \frac{(a-1)(b-1)}{n} (Q_C - 1) MSE.$$

Theorems 2.3 deals with both the balanced and unbalanced case.

**Theorem 2.3.** *Let* $0 < \mathrm{Var}\,(X_{ijk}) = \sigma_{ij}^2 < \infty$. *Then,*

(a) *For* $n_{ij} \geq 2$ *fixed, suppose for some* $\delta > 0$, $\limsup (1/(ab)) \sum_{i=1}^{a} \sum_{j=1}^{b} E|X_{ij1} - E(X_{ij1})|^{4+\delta} < \infty$, *and write*

$$\frac{1}{ab} \sum_{i=1}^{a} \sum_{j=1}^{b} \frac{\sigma_{ij}^4}{n_{ij}(n_{ij}-1)} \to \phi^4, \quad \frac{1}{ab^2} \sum_{i=1}^{a} \sum_{j_1 \neq j_2} \frac{\sigma_{ij_1}^2}{n_{ij_1}} \frac{\sigma_{ij_2}^2}{n_{ij_2}} \to \eta^4.$$

1. *Under* $H_0(A)$, $T_A \to N\left(0, 2(\phi^4 + b\eta^4)/b^2\right)$, *as* $a \to \infty$ *and* $b$ *is fixed,* $b^{1/2} T_A \to N\left(0, 2\eta^4\right)$, *as* $a \to \infty$ *and* $b \to \infty$.
2. *Under* $H_0(C)$, $T_C \to N\left(0, 2(b-1)^2\phi^4/b^2 + 2\eta^4/b\right)$, *as* $a \to \infty$ *and* $b$ *is fixed,* $T_C \to N\left(0, 2\phi^4\right)$, *as* $a \to \infty$ *and* $b \to \infty$.

(b) *Suppose the* $n_{ij}$ *also tend to infinity with* $a$ *or* $b$, *and* $\limsup(1/(ab)) \sum_{i=1}^{a}$

$\sum_{j=1}^{b} E|X_{ij1} - E(X_{ij1})|^{4+\delta} < \infty$ *for some* $\delta > 0$. *And write*

$$\frac{n^2(a,b)}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b}\frac{\sigma_{ij}^4}{n_{ij}(a,b)(n_{ij}(a,b)-1)} \to \phi_1^4,$$

$$\frac{n^2(a,b)}{ab^2}\sum_{i=1}^{a}\sum_{j_1\neq j_2}\frac{\sigma_{ij_1}^2}{n_{ij_1}(a,b)}\frac{\sigma_{ij_2}^2}{n_{ij_2}(a,b)} \to \eta_1^4.$$

1. *Under* $H_0(A)$, $n(a,b)T_A \to N\left(0, 2(\phi_1^4 + b\eta_1^4)/b^2\right)$, *as* $a \to \infty$ *and* $b$ *is fixed,* $n(a,b)b^{1/2}T_A \to N\left(0, 2\eta_1^4\right)$, *as* $a \to \infty$ *and* $b \to \infty$.
2. *Under* $H_0(C)$, $n(a,b)T_C \to N\left(0, 2(b-1)^2\phi_1^4/b^2 + 2\eta_1^4/b\right)$ *as* $a \to \infty$ *and* $b$ *is fixed,* $n(a,b)T_C \to N\left(0, 2\phi_1^4\right)$, *as* $a \to \infty$ *and* $b \to \infty$.

**Remark 2.4.** To be able to apply the above theorem, we need to estimate $\phi^4$, $\eta^4$, $\phi_1^4$ and $\eta_1^4$. We estimate $\eta^4$ by $(ab^2)^{-1}\sum_{i=1}^{a}\sum_{j_1\neq j_2}^{b} n_{ij_1}^{-1}n_{ij_2}^{-1}s_{ij_1}^2 s_{ij_2}^2$, where $s_{ij}^2$ denotes the sample variance for cell (i,j). This estimator is consistent. For consistent estimation of $\eta^4$, we need to be able to estimate $\sigma_{ij}^4$ unbiasedly. We used a U-statistic to estimate $\sigma_{ij}^4$:

$$\binom{n_{ij}}{4}^{-1}\frac{1}{12}\sum\Big[(X_{ijk_1}-X_{ijk_2})^2(X_{ijk_3}-X_{ijk_4})^2$$
$$+(X_{ijk_1}-X_{ijk_3})^2(X_{ijk_2}-X_{ijk_4})^2+(X_{ijk_1}-X_{ijk_4})^2(X_{ijk_2}-X_{ijk_3})^2\Big],$$

where the sum is over all subsets of distinct values of $(k_1, k_2, k_3, k_4)$, $k_l \in [1, n_{ij}]$, $l = 1, \ldots, 4$. $\phi_1^4$ and $\eta_1^4$ are estimated similarly. For the consistency of $U$-statistics, we refer to Lee (1990).

We next investigate the asymptotic distributions of $T_A$ and $T_C$ under local alternatives. We consider only the case that $a \to \infty$ but $b$ remains fixed, and thus we denote the smallest cell size by $n(a)$. It is convenient to represent the random variables under local alternatives as simple translations of random variables that satisfy the null hypotheses. Thus, to test for the null hypothesis of no main row effect, we consider the local alternatives

$$Y_{ijk} = X_{ijk} + \alpha_i(a), \quad i = 1,\ldots,a,\ j = 1,\ldots,b,\ k = 1,\ldots,n_{ij}, \tag{2.4}$$

where $X_{ijk}$ is a sequence of random variables satisfying $H_0(A)$, and

$$\alpha_i(a) = a^{\frac{3}{4}}n(a)^{-\frac{1}{2}}\int_{\frac{i-1}{a}}^{\frac{i}{a}} g(t)dt, \tag{2.5}$$

where g is a continuous function on [0,1] such that $\int_0^1 g(t)dt = 0$.

To test for the null hypothesis of no interaction effect, we consider the local alternatives

$$Y_{ijk} = X_{ijk} + \gamma_{ij}(a), \quad i = 1, \ldots, a,\ j = 1, \ldots, b, k = 1, \ldots, n_{ij}, \tag{2.6}$$

where $X_{ijk}$ is a sequence of random variables satisfying $H_0(C)$, and

$$\gamma_{ij}(a) = a^{\frac{3}{4}} n(a)^{-\frac{1}{2}} \int_{\frac{i-1}{a}}^{\frac{i}{a}} g_j(t)dt, \tag{2.7}$$

where $g_j(t)$ are continuous on [0,1] such that $\sum_{j=1}^{b} g_j(t) = 0, \forall\, t$, and $\int_0^1 g_j(t)dt = 0, \forall\, j$.

Let $T_A(\mathbf{Y})$, $T_C(\mathbf{Y})$ be the statistics $T_A$, $T_C$ evaluated on the $Y_{ijk}$.

**Theorem 2.4.** *Assume* $\max\{\sigma_{ij}^2; 1 \le i \le a, 1 \le j \le b\} = o(a^{-1/2})$ *and set*

$$\theta_A^2 = \int_0^1 g^2(t)dt \quad \textit{and} \quad \theta_C^2 = \frac{1}{b}\sum_{j=1}^{b} \int_0^1 g_j^2(t)dt.$$

(a) *Suppose* $n_{ij} \ge 2$ *remains fixed and that* $X_{ijk}$ *satisfy the conditions of Theorem* 2.3(a).
1. *If the* $Y_{ijk}$ *are given in* (2.4) *with* $\alpha_i(a)$ *given by* (2.5), *then*

$$T_A(\mathbf{Y}) \to N\left(\frac{b^{\frac{1}{2}}\theta_A^2}{n(\infty)},\ \frac{2(\phi^4 + b\eta^4)}{b^2}\right) \quad \text{as } a \to \infty \text{ and } b \text{ is fixed.}$$

2. *If the* $Y_{ijk}$ *are given in* (2.6) *with* $\gamma_{ij}(a)$ *given by* (2.7), *then with* $n(\infty) = \lim n(a)$,

$$T_C(\mathbf{Y}) \to N\left(\frac{b^{\frac{1}{2}}\theta_C^2}{n(\infty)},\ \frac{2(b-1)^2\phi^4}{b^2} + \frac{2\eta^4}{b^2}\right) \quad \textit{as } a \to \infty \textit{ and } b \textit{ is fixed.}$$

(b) *Suppose* $n = n(a) \to \infty$ *as* $a \to \infty$ *and that the* $X_{ijk}$ *satisfy the conditions of Theorem* 2.3(b).
1. *If the* $Y_{ijk}$ *are given in* (2.4) *with* $\alpha_i(a)$ *given by* (2.5), *then*

$$n(a)T_A(\mathbf{Y}) \to N\left(b^{\frac{1}{2}}\theta_A^2,\ \frac{2(\phi_1^4 + b\eta_1^4)}{b^2}\right) \quad \textit{as } a \to \infty \textit{ and } b \textit{ is fixed.}$$

2. *If the* $Y_{ijk}$ *are given in* (2.6) *with* $\gamma_{ij}(a)$ *given by* (2.7), *then*

$$n(a)T_C(\mathbf{Y}) \to N\left(b^{\frac{1}{2}}\theta_C^2,\ \frac{2(b-1)^2\phi_1^4}{b^2} + \frac{2\eta_1^4}{b^2}\right) \quad \textit{as } a \to \infty \textit{ and } b \textit{ is fixed.}$$

## 3. Projection Method

In this section we apply Hájek's projection method to linearize our test statistics. If the statistic $S$ is based on independent random vectors $\mathbf{U}_1, \ldots, \mathbf{U}_n$ and has finite second moment, then its projection onto the class of random variables of the form $\sum_{i=1}^n g_i(\mathbf{U}_i)$, where $g_i$ are measurable with $Eg_i^2(\mathbf{U}_i) < \infty$, is given by $\hat{S} = \sum_{i=1}^n E(S|\mathbf{U}_i) - (n-1)ES$. See, for example, van der Vaart (1998, Chap. 11). If $\hat{S}$ is asymptotically equivalent to $S$, then it can be used for finding the asymptotic distribution of $S$. To achieve asymptotic equivalence, the space onto which we project has to be chosen appropriately. Akritas and Papadatos (2004) demonstrated that an appropriate space can be chosen for projecting quadratic forms that arise in one-way ANOVA designs. In this paper we show that the appropriate spaces on which to project the statistics $Q_A$ and $T_A$ is that of random variables of the form $\sum_{i=1}^a g_i(\mathbf{X}_i)$. The same space works also for $Q_C$ and $T_C$ if $b$ is fixed, but if $b \to \infty$ the appropriate space for projecting $Q_C$ and $T_C$ is that of $\sum_{i=1}^a \sum_{j=1}^b g_{ij}(\mathbf{X}_{ij})$.

The first lemma and proposition below consider the projection of $MST_A - MSE$ and $MST_C - MSE$. The projection of $T_A$ and $T_C$ is considered in the following lemma and proposition. Note that by Slutsky's theorem the asymptotic distribution of $Q_A - 1$ and $Q_C - 1$ follows from that of $MST_A - MSE$ and $MST_C - MSE$, respectively.

**Lemma 3.1.**

(a) *Under $H_0(A)$, the projection $\widehat{S}_A = \sum_{i=1}^a E\left(MST_A - MSE|\mathbf{X}_i\right)$ of $MST_A - MSE$ is given by*

$$\sum_{i=1}^a \sum_{j=1}^b \Big( \frac{1}{\sum_{i=1}^a \sum_{j=1}^b n_{ij}^{-1}} \frac{1}{n_{ij}^2} \mathbf{X}_{ij}' \mathbf{J}_{n_{ij}} \mathbf{X}_{ij} + \frac{1}{N-ab} \frac{1}{n_{ij}} \mathbf{X}_{ij}' \mathbf{J}_{n_{ij}} \mathbf{X}_{ij} - \frac{1}{N-ab} \mathbf{X}_{ij}' \mathbf{X}_{ij} \Big) + \frac{1}{\sum_{i=1}^a \sum_{j=1}^b n_{ij}^{-1}} \sum_{i=1}^a \sum_{j_1 \neq j_2}^b \frac{1}{n_{ij_1} n_{ij_2}} \mathbf{X}_{ij_1}' \mathbf{1}_{n_{ij_1}} \mathbf{1}_{n_{ij_2}}' \mathbf{X}_{ij_2}. \quad (3.1)$$

(b) *Under $H_0(C)$, the projection $\widehat{S}_{1C} = \sum_{i=1}^a E\left(MST_C - MSE|\mathbf{X}_i\right)$ of $MST_C - MSE$ is given by*

$$\sum_{i=1}^a \sum_{j=1}^b \Big( \frac{1}{\sum_{i=1}^a \sum_{j=1}^b n_{ij}^{-1}} \frac{1}{n_{ij}^2} \mathbf{X}_{ij}' \mathbf{J}_{n_{ij}} \mathbf{X}_{ij} + \frac{1}{N-ab} \frac{1}{n_{ij}} \mathbf{X}_{ij}' \mathbf{J}_{n_{ij}} \mathbf{X}_{ij} - \frac{1}{N-ab} \mathbf{X}_{ij}' \mathbf{X}_{ij} \Big) - \frac{1}{(b-1) \sum_{i=1}^a \sum_{j=1}^b n_{ij}^{-1}} \sum_{i=1}^a \sum_{j_1 \neq j_2}^b \frac{1}{n_{ij_1} n_{ij_2}} \mathbf{X}_{ij_1}' \mathbf{1}_{n_{ij_1}} \mathbf{1}_{n_{ij_2}}' \mathbf{X}_{ij_2},$$

*while that of $\widehat{S}_{2C} = \sum_{i=1}^a \sum_{j=1}^b E\left(MST_C - MSE|\mathbf{X}_{ij}\right)$ is given by*

$$\sum_{i=1}^{a}\sum_{j=1}^{b}\Big(\frac{1}{\sum_{i=1}^{a}\sum_{j=1}^{b}n_{ij}^{-1}}\frac{1}{n_{ij}^{2}}\mathbf{X}_{ij}'\mathbf{J}_{n_{ij}}\mathbf{X}_{ij}+\frac{1}{N-ab}\frac{1}{n_{ij}}\mathbf{X}_{ij}'\mathbf{J}_{n_{ij}}\mathbf{X}_{ij}-\frac{1}{N-ab}\mathbf{X}_{ij}'\mathbf{X}_{ij}\Big).$$

**Proposition 3.2.** *Under homoscedasticity, and with $n_{ij}$ either fixed or going to $\infty$,*

(a) *when $a \to \infty$, $b$ is fixed or both $a$ and $b$ go to $\infty$, $a^{1/2}\left(MST_A - MSE - \widehat{S}_A\right) \xrightarrow{p} 0$, under $H_0(A)$;*

(b) *under $H_0(C)$, when $a \to \infty$ and $b$ is fixed, $a^{1/2}\left(MST_C - MSE - \widehat{S}_{1C}\right) \xrightarrow{p} 0$, and when both $a$ and $b$ tend to infinity, $(ab)^{1/2}\left(MST_C - MSE - \widehat{S}_{2C}\right) \xrightarrow{p} 0$.*

**Lemma 3.3.**

(a) *Under $H_0(A)$, the projection $\widetilde{T}_A = \sum_{i=1}^{a} E(T_A|\mathbf{X}_i)$ of $T_A$ is*

$$(ab)^{-\frac{1}{2}}\sum_{i=1}^{a}\frac{a-1}{ab}\Big[\Big(\sum_{j=1}^{b}\overline{X}_{ij.}\Big)^2-\sum_{j=1}^{b}\frac{S_{ij}^2}{n_{ij}}\Big]. \tag{3.2}$$

(b) *Under $H_0(C)$, the projection $\widetilde{T}_{1C} = \sum_{i=1}^{a} E(T_C|\mathbf{X}_i)$ of $T_C$ is*

$$\frac{(a-1)(b-1)}{(ab)^{\frac{3}{2}}}\sum_{i=1}^{a}\sum_{j=1}^{b}\Big(\overline{X}_{ij.}^2-\frac{S_{ij}^2}{n_{ij}}\Big)-\frac{a-1}{(ab)^{\frac{3}{2}}}\sum_{i=1}^{a}\sum_{j_1\neq j_2}^{b}\overline{X}_{ij_1.}\overline{X}_{ij_2.}\ ,$$

*while that of $\widetilde{T}_{2C} = \sum_{i=1}^{a}\sum_{j=1}^{b} E(T_C|\mathbf{X}_{ij})$ is given by*

$$\frac{(a-1)(b-1)}{(ab)^{\frac{3}{2}}}\sum_{i=1}^{a}\sum_{j=1}^{b}\Big(\overline{X}_{ij.}^2-\frac{S_{ij}^2}{n_{ij}}\Big).$$

**Proposition 3.4.** *Assume $(1/(ab))\sum_{i=1}^{a}\sum_{j=1}^{b}\sigma_{ij}^2 \to \sigma^2 \in (0,\infty)$.*

(a) *Under $H_0(A)$, $n(a,b)b^{1/2}(T_A - \widetilde{T}_A) \xrightarrow{p} 0$, in all cases considered.*

(b) *Under $H_0(C)$, $n(a,b)(T_C - \widetilde{T}_{1C}) \xrightarrow{p} 0$ and $n(a,b)(T_C - \widetilde{T}_{2C}) \xrightarrow{p} 0$ when $b$ stays fixed and tends to infinity, respectively, in all cases.*

Propositions 3.2 and 3.4 are used in the Appendix for proving the main results.

## 4. Numerical Results

All simulations pertain to main row effects when only $a$ is large; similar results are obtained for interaction effects and also for both $a$ and $b$ large, but are not reported here. They are based on 5,000 runs. Software Matlab 6.1

is used to generate random data. The results reported in the tables pertain to balanced designs, with $n = 4$ and $b = 2$. In the following, "CF" denotes the classical F test, "BHOM", "BHET", "UBHOM", "UBHET" denote tests for the balanced homoscedastic, balanced heteroscedastic, unbalanced homoscedastic and unbalanced heteroscedastic situations, respectively.

Table 1 investigates the achieved levels using iid normal(0,1) and lognormal(0,1) random variables. In this case the statistics used in the "CF", "BHOM" and "BHET" procedures are equivalent (see also the comment preceding Theorem 2.3. Thus, the three procedures differ only because they use different cut-off points (which are asymptotically equivalent), and that explains the difference in the achieved $\alpha$-levels. Surprisingly, the level of the "BHET" procedure is quite accurate, though a bit on the conservative side, in this homoscedastic setting. In practice, of course, it is difficult to justify homoscedasticity with samples of size 4. Thus, given the results of Table 2 which used heteroscedastic errors, we can recommend the "BHET" procedure. In Table 2 the observations in cell (i, j) are generated from normal(0, 1) if $i \leq a/2$, and from normal(0, 25) otherwise.

Table 1. Estimated levels for nominal 0.05 level tests, homoscedastic errors.

| a | error | CF | BHOM | BHET |
|---|---|---|---|---|
| 20 | normal(0,1) | 0.055 | 0.084 | 0.088 |
| | lognormal(0,1) | 0.044 | 0.064 | 0.050 |
| 30 | normal(0,1) | 0.050 | 0.075 | 0.077 |
| | lognormal(0,1) | 0.042 | 0.060 | 0.040 |
| 40 | normal(0,1) | 0.049 | 0.072 | 0.073 |
| | lognormal(0,1) | 0.049 | 0.070 | 0.044 |

Table 2. Estimated levels for nominal 0.05 level tests, heteroscedastic errors.

| | CF | BHOM | BHET |
|---|---|---|---|
| a=20 | 0.095 | 0.125 | 0.076 |
| a=30 | 0.114 | 0.139 | 0.084 |
| a=40 | 0.107 | 0.131 | 0.075 |
| a=50 | 0.100 | 0.120 | 0.065 |

The effects of unequal variances can be more serious if the design is unbalanced. Some of our simulation results in heteroscedastic unbalanced case indicates that "CF" can be very liberal while the test based on our asymptotic results can be very accurate. In one such simulation with $a = 20$ and $b = 2$, we used sample sizes (7, 6, 7, 5, 5, 14, 14, 4, 7, 5, 5, 7, 4, 7, 6, 7, 5, 7, 5, 6), for $j = 1$, and (14, 16, 5, 16, 16, 7, 4, 14, 6, 7, 4, 7, 14, 6, 4, 6, 5, 7, 4, 6), for $j = 2$. The observations in cell $(i, j)$ are generated from a normal(0, $(1 + i/4 + j/4)^2$)

distribution. In this example, the level of "CF" is 0.1652, the level of "UBHOM" is 0.2528, while the level of test "UBHET" is 0.0806.

Tables 3 and 4 examine the power of the procedures in a balanced setting with a=20. In Table 3, the observations in the $(i, j)$th cell are generated from $i * \tau/a +$ normal$(0, 1)$, for $\tau = 0,\ 0.3,\ 0.6,\ 0.9,\ 1.2$. In Table 4, the observations in the $(i, j)$th cell are generated from $i * \tau/a +$ lognormal$(0, 1)$, for $\tau = 0,\ 1,\ 2,\ 3$. In both tables, $\tau = 0$ corresponds to the null hypothesis. In the normal setting of Table 3, "CF" is optimal so the increased power of our tests is due to their liberality. Table 4, however, indicates clearly that the power of "BHET" is higher than that of the others.

Table 3. Estimated power for nominal 0.05 level tests, normal(0,1) errors.

| $\tau$ | CF | BHOM | BHET |
|---|---|---|---|
| 0 | 0.048 | 0.076 | 0.078 |
| 0.3 | 0.070 | 0.110 | 0.114 |
| 0.6 | 0.163 | 0.224 | 0.226 |
| 0.9 | 0.403 | 0.485 | 0.489 |
| 1.2 | 0.694 | 0.761 | 0.763 |

Table 4. Estimated power for nominal 0.05 level tests, lognormal(0,1) errors.

| tao | CF | BHOM | BHET |
|---|---|---|---|
| 0 | 0.046 | 0.065 | 0.047 |
| 1 | 0.130 | 0.177 | 0.172 |
| 2 | 0.509 | 0.583 | 0.637 |
| 3 | 0.858 | 0.889 | 0.922 |

## 5. Data Analysis

In an agricultural trial performed in 2002 by the Washington State University Statewide Extension Uniform Cereal Variety Testing Program (http://variety.wsu.edu/), 40 different varieties/lines of winter wheat are studied in terms of yield. The growing conditions are classified into four classes by the amount of rainfall and soil condition. The number of observations is 4 for each variety/line in the first class, 5 in the second class, 6 in the third class, and 4 in the fourth class. Of interest is to test whether there is any difference among varieties/lines and whether there is interaction effect between varieties/lines and growing conditions. This is an unbalanced two-way layout with one factor having a large number of levels. The usual F test (CF) based on type III sum of squares gives a p-value of 0.069 for the variety/line main effect and a p-value of 1 for an interaction effect. "UBHOM" gives a p-value of 0.054 for the variety/line main effect and a p-value of 1 for an interaction effect, while "UBHET" gives a p-value of 0.089 for the variety/line main effect and a p-value of 1 for the interaction effect. Therefore, all approaches suggest a marginally strong variety/line effect, but there is no evidence suggesting an interaction effect.

To demonstrate the value of our nonparametric tests, we disturbed the data in both the mean values and the variances. We first disturbed the mean values by adding 5 (respectively 7) to the observations corresponding to varieties 15-26 for the treatment corresponding to the lowest (respectively second lowest) profile. As a result of this disturbance, the p-values for the variety/line main effect are 0.008, 0.002 and 0.007 for the CF, UBHOM and UBHET tests, respectively. Our heteroscedastic test distinguished itself when we also introduced a disturbance in the variances. To describe this, let $X_{ijk}$, $i = 1, \ldots, 40$, $j = 1, \ldots, 4$, denote the yield data after their means are disturbed as described above. Then the variance-disturbed data are $\widetilde{X}_{ijk} = \overline{X}_{ij.} + (X_{ijk} - \overline{X}_{ij.})\sigma_j$, where $\sigma_j = (1.5)^{-1}$, for $j = 1, 4$, and $\sigma_j = 1.5$, for $j = 2, 3$. As a result of this additional disturbance, the p-values for the variety/line main effect are 0.202, 0.204 and 0.012, for the CF, UBHOM and UBHET tests, respectively. Thus, even though the design is nearly balanced, this modest disturbance in the variances is enough to affect the classical F-test as well as the present test that is based on the assumption of homoscedasticity.

## 6. Some Extensions

Viewing the covariate in a one-way ANCOVA design as a factor with many levels, the present methodology (with $a \to \infty$ and $b$ fixed), can be applied to the yet unresolved problem of testing for covariate effects and for interaction effects between the factor and covariate in the nonlinear fully nonparametric ANCOVA model proposed by Akritas, Arnold and Du (2000) The extension is not trivial since the present methodology requires some replication in the cells while in typical ANCOVA designs there is only one observation per covariate value. As demonstrated in the regression setting of Wang, Akritas and Van Keilegom (2002), this difficulty can be circumvented by using smoothness assumptions and considering windows around each covariate value. This generates cells with replicated observations, but cells will have common observations which destroys the present independence assumption. The methodology with both $a$ and $b$ tending to infinity, can be applied to extend the lack-of-fit test of Wang, Akritas and Van Keilegom (2002) to regression designs with two covariates.

## Appendix. Proofs

Due to space limitation, we only give proofs for the results on testing for main row effects, and only when $a$ goes to $\infty$. Proofs for other cases are similar and can be found in Wang (2003). Under $H_0(A)$, we may assume $E(X_{ijk}) = 0$ without loss of generality. We may also assume $\mathrm{Var}\,(X_{ijk}) = 1$ in the homoscedastic setting.

**Proof of Theorem 2.1.** It is straightforward to show $MSE \xrightarrow{p} 1$ as $a \to \infty$ under the current moment assumptions, both when $n$ is fixed and $n = n(a) \to \infty$, so by Slutsky's theorem we may consider the numerator $a^{1/2}(MST_A - MSE)$ in both cases. By Proposition 3.2, it suffices to find the asymptotic distribution of $a^{1/2}\widehat{S}_A$. In the present balanced case, the expression in Lemma 3.1 is simplified to $a^{1/2}\widehat{S}_A = a^{-1/2}\sum_{i=1}^{a} Y_i$, where

$$Y_i = \frac{1}{b(n-1)}\sum_{j=1}^{b}\sum_{k_1\neq k_2}^{n} X_{ijk_1}X_{ijk_2} + \frac{1}{bn}\sum_{j_1\neq j_2}^{b}\sum_{k_1=1}^{n}\sum_{k_2=1}^{n} X_{ij_1k_1}X_{ij_2k_2} \tag{A.1}$$

are independent random variables. Clearly, $E(Y_i) = 0$, $E(Y_i^2) = 2+2(b(n-1))^{-1}$. Thus we have $\mathrm{Var}\,(a^{1/2}\widehat{S}_A) = 2 + 2/b(n-1)$. Under both conditions ($n$ fixed or $n \to \infty$), Liapounov's CLT can be applied to prove asymptotic normality. More specifically, we want to check that for some $\delta > 0$, $\sum_{i=1}^{a} E|a^{-1/2}Y_i|^{2+\delta} \to 0$ when $a \to \infty$. This is easily seen to be true when taking $\delta = 2$.

**Proof of Theorem 2.2.** (a) Using Proposition 3.2 and the fact that $MSE \xrightarrow{p} 1$, it suffices to show $a^{1/2}\widehat{S}_A \to N\left(0, \tau_1^2/b\right)$. Letting $c_1 = \left(\sum_{i=1}^{a}\sum_{j=1}^{b} 1/n_{ij}\right)^{-1}$ and $c_2 = (N-ab)^{-1}$, it follows that $a^{1/2}\widehat{S}_A = \sum_{i=1}^{a} Y_{a,i}$, where

$$\begin{aligned} Y_{a,i} = a^{\frac{1}{2}} \Bigg[ &\sum_{j=1}^{b}\sum_{k_1=1}^{n_{ij}}\sum_{k_2=1}^{n_{ij}} \left(\frac{c_1}{n_{ij}^2} + \frac{c_2}{n_{ij}}\right) X_{ijk_1}X_{ijk_2} - \sum_{j=1}^{b}\sum_{k=1}^{n_{ij}} c_2 X_{ijk}^2 \\ &+ c_1 \sum_{j_1\neq j_2}^{b}\sum_{k_1=1}^{n_{ij_1}}\sum_{k_2=1}^{n_{ij_2}} \frac{1}{n_{ij_1}n_{ij_2}} X_{ij_1k_1}X_{ij_2k_2}\Bigg] \end{aligned} \tag{A.2}$$

are independent random variables. Note that $\sum_{i=1}^{a} E(Y_{a,i}) = 0$. Straightforward but tedious calculations show that $Var\left(a^{1/2}\widehat{S}_A\right) \to \tau_1^2/b$. It remains to check Lyapounov's condition, i.e., that for some $\delta > 0$,

$$\sum_{i=1}^{a} E\Big[|Y_{a,i} - E(Y_{a,i})|^{2+\delta}\Big] = (abc_1)^{2+\delta}\frac{1}{a^{1+\frac{\delta}{2}}b^{2+\delta}}\sum_{i=1}^{a} R_{a,i} \to 0, \text{ as } a \to \infty, \tag{A.3}$$

where

$$\begin{aligned} R_{a,i} = E\Big| &\sum_{j=1}^{b}\sum_{k_1=1}^{n_{ij_1}}\sum_{k_2=1}^{n_{ij_2}} \Big(\frac{1}{n_{ij}^2} + \frac{c_2}{c_1}\frac{1}{n_{ij}}\Big) X_{ijk_1}X_{ijk_2} - \sum_{j=1}^{b}\sum_{k=1}^{n_{ij}} \frac{c_2}{c_1} X_{ijk}^2 \\ &+ \sum_{j_1\neq j_2}^{b}\sum_{k_1=1}^{n_{ij_1}}\sum_{k_2=1}^{n_{ij_2}} \frac{1}{n_{ij_1}n_{ij_2}} X_{ij_1k_1}X_{ij_2k_2} - \sum_{j=1}^{b}\sum_{k=1}^{n_{ij}} \Big(\frac{1}{n_{ij}^2} + \frac{c_2}{c_1 n_{ij}} - \frac{c_2}{c_1}\Big)\Big|^{2+\delta}. \end{aligned}$$

Note that $abc_1 \to 1/b_2$. (A.3) is proved by applying the following inequality repeatedly,

$$\Big|\sum_{i=1}^{m} z_i\Big|^p \le m^{p-1}\sum_{i=1}^{m}|z_i|^p, \quad m \ge 1, \quad p \ge 1. \tag{A.4}$$

More specifically, by (A.4),

$$\begin{aligned}
R_{a,i} &\le 3^{1+\delta}\Big\{E\Big|\sum_{j=1}^{b}\sum_{k=1}^{n_{ij}}\Big(\frac{1}{n_{ij}^2}+\frac{c_2}{c_1}\frac{1}{n_{ij}}-\frac{c_2}{c_1}\Big)(X_{ijk}^2-1)\Big|^{2+\delta}\\
&\quad +E\Big|\sum_{j=1}^{b}\sum_{k_1\neq k_2}^{n_{ij}}\Big(\frac{1}{n_{ij}^2}+\frac{c_2}{c_1}\frac{1}{n_{ij}}\Big)X_{ijk_1}X_{ijk_2}\Big|^{2+\delta}\\
&\quad +E\Big|\sum_{j_1\neq j_2}^{b}\sum_{k_1=1}^{n_{ij_1}}\sum_{k_2=1}^{n_{ij_2}}\frac{1}{n_{ij_1}n_{ij_2}}X_{ij_1k_1}X_{ij_2k_2}\Big|^{2+\delta}\Big\}\\
&= 3^{1+\delta}(D_1+D_2+D_3),
\end{aligned}$$

where the definition of $D_i$, $i=1,\ 2,\ 3$, is clear from the context. First, notice that $1/n_{ij}^2 \le 1$ and $|(1/n_{ij})(c_2/c_1)-c_2/c_1| = ((n_{ij}-1)/n_{ij})\cdot(\sum_{i=1}^{a}\sum_{j=1}^{b}1/n_{ij})/(N-ab) \le (n_{ij}-1)/n_{ij}\cdot(ab/(N-ab)) \le 1$, then apply (A.4) to get

$$\begin{aligned}
D_1 &\le b^{1+\delta}\sum_{j=1}^{b}E\Big|\Big(\frac{1}{n_{ij}^2}+\frac{c_2}{c_1}\frac{1}{n_{ij}}-\frac{c_2}{c_1}\Big)\sum_{k=1}^{n_{ij}}(X_{ijk}^2-1)\Big|^{2+\delta}\\
&\le b^{1+\delta}2^{2+\delta}\sum_{j=1}^{b}E\Big|\sum_{k=1}^{n_{ij}}(X_{ijk}^2-1)\Big|^{2+\delta}\\
&\le b^{1+\delta}2^{3+2\delta}\sum_{j=1}^{b}n_{ij}^{1+\delta}\sum_{k=1}^{n_{ij}}(E|X_{ijk}|^{4+2\delta}+1)\\
&= b^{1+\delta}2^{3+2\delta}\sum_{j=1}^{b}n_{ij}^{2+\delta}(E|X_{ij1}|^{4+2\delta}+1).
\end{aligned}$$

Similarly,

$$\begin{aligned}
D_2 &\le b^{1+\delta}\sum_{j=1}^{b}E\Big|\Big(\frac{1}{n_{ij}^2}+\frac{c_2}{c_1}\frac{1}{n_{ij}}\Big)\sum_{k_1\neq k_2}^{n_{ij}}X_{ijk_1}X_{ijk_2}\Big|^{2+\delta}\\
&\le b^{1+\delta}2^{2+\delta}\sum_{j=1}^{b}E\Big|\sum_{k_1\neq k_2}^{n_{ij}}X_{ijk_1}X_{ijk_2}\Big|^{2+\delta}.
\end{aligned}$$

We now apply a new inequality. For any $p \geq 2$, there exists a finite positive constant $A_p$ (depending only on $p$) such that for any i.i.d random variables $Z_1, \ldots, Z_n$ with $E(Z_i) = 0$,

$$E|Z_1 + \cdots + Z_n|^p \leq A_p \, n^{\frac{p}{2}} E|Z_1|^p. \tag{A.5}$$

This inequality can be proved by first using the Marcinkiewicz-Zygmund inequality (see Chow and Teicher (1997), pp. 386-387): For a sequence of independent random variables $V_1, \ldots, V_n$ with mean 0, there exists a finite positive constant $A_p$ depending only on $p$ such that

$$E\Big|\sum_{i=1}^{n} V_i\Big|^p \leq A_p E\Big|\Big(\sum_{i=1}^{n} V_i^2\Big)\Big|^{\frac{p}{2}}. \tag{A.6}$$

Then apply inequality (A.4). Thus, writing

$$\Big|\sum_{k_1 \neq k_2}^{n_{ij}} X_{ijk_1} X_{ijk_2}\Big|^{2+\delta} = \Big|\Big(\sum_{k=1}^{n_{ij}} X_{ijk}\Big)^2 - \sum_{k=1}^{n_{ij}} \left(X_{ijk}^2 - 1\right) - n_{ij}\Big|^{2+\delta}$$

and using inequality (A.4), (A.5) and Hölder's inequality, we have $E|\sum_{k_1 \neq k_2}^{n_{ij}} X_{ijk_1} X_{ijk_2}\|^{2+\delta} \leq 3^{1+\delta} D_\delta n_{ij}^{2+\delta} E|X_{ij1}|^{4+2\delta}$, where $D_\delta$ are finite positive constants depending only on $\delta$. We thus have

$$D_2 \leq b^{1+\delta} 2^{2+\delta} \sum_{j=1}^{b} 3^{1+\delta} D_\delta n_{ij}^{2+\delta} E|X_{ij1}|^{4+2\delta}.$$

Writing $D_3 = E\Big|\left(\sum_{j=1}^{b}\sum_{k=1}^{n_{ij}} X_{ijk}/n_{ij}\right)^2 - \sum_{j=1}^{b}\sum_{k_1=1}^{n_{ij}}\sum_{k_2=1}^{n_{ij}} X_{ijk_1}X_{ijk_2}/n_{ij}^2\Big|^{2+\delta}$ and applying inequalities (A.4) and (A.5) repeatedly, we have

$$D_3 \leq 2^{1+\delta} b^{1+\delta} G_\delta \sum_{j=1}^{b} \frac{E\left|X_{ij1}\right|^{4+2\delta}}{n_{ij}^{2+\delta}},$$

where $G_\delta$ are finite positive constants depending only on $\delta$. Combining the upper bounds we obtained above on $D_1$, $D_2$ and $D_3$, we have $R_{a,i} \leq H_\delta b^{1+\delta} \sum_{j=1}^{b} n_{ij}^{2+\delta}$ $E|X_{ij1}|^{4+2\delta}$, for some positive constant $H_\delta$ depending only on $\delta$. Thus (A.3) holds.

(b) Under the new set of conditions, we have Var $\left((ab)^{1/2}\widehat{S}_A\right) \to 2b\tau_4^2$. The asymptotic normality is proved by checking Lyapounov's condition in (A.3). The proof follows the same lines as in (a) except that now, since we have

$$\left|\frac{1}{n_{ij}^2(a)} + \frac{c_2}{c_1}\frac{1}{n_{ij}(a)} - \frac{c_2}{c_1}\right| \leq \frac{2}{n^2(a)}, \quad \left|\frac{1}{n_{ij}^2(a)} + \frac{c_2}{c_1}\frac{1}{n_{ij}(a)}\right| \leq \frac{2}{n^2(a)},$$

the upper bounds on $D_1$, $D_2$ and $D_3$ become

$$D_1 \leq b^{1+\delta} n(a)^{-4-2\delta} \kappa(a)^{2+\delta} 2^{3+2\delta} \sum_{j=1}^{b} (E|X_{ij1}|^{4+2\delta} + 1),$$

$$D_2 \leq b^{1+\delta} n(a)^{-4-2\delta} \kappa(a)^{2+\delta} 2^{2+\delta} \sum_{j=1}^{b} 3^{1+\delta} D_\delta E|X_{ij1}|^{4+2\delta},$$

$$D_3 \leq 2^{1+\delta} n(a)^{-2-\delta} b^{1+\delta} G_\delta \sum_{j=1}^{b} E\left|X_{ij1}\right|^{4+2\delta}.$$

Since $abc_1(a) \leq \kappa(a)$, (A.3) still holds.

**Proof of Theorem 2.3.** (a) For the case $n_{ij}$ are fixed, Proposition 3.4 indicates that the distributions of $T_A$ and $\widetilde{T}_A$ are asymptotically equivalent. $\widetilde{T}_A$ can be expressed as $\widetilde{T}_A = \sum_{i=1}^{a} \widetilde{T}_{Ai}$, where $\widetilde{T}_{Ai} = (ab)^{-1/2}((a-1)/(ab))[\sum_{j=1}^{b} \sum_{k_1 \neq k_2}^{n_{ij}} (X_{ijk_1} X_{ijk_2})/(n_{ij}(n_{ij}-1)) + \sum_{j_1 \neq j_2}^{b} \overline{X}_{ij_1.} \overline{X}_{ij_2.}]$ are independent with $E(\widetilde{T}_{Ai})$=0. The two terms in $\widetilde{T}_{Ai}$ are uncorrelated, and

$$E(\widetilde{T}_A^2) = \frac{2(a-1)^2}{(ab)^3} \left[ \sum_{i=1}^{a} \sum_{j=1}^{b} \frac{\sigma_{ij}^4}{n_{ij}(n_{ij}-1)} + \sum_{i=1}^{a} \sum_{j_1 \neq j_2}^{b} \frac{\sigma_{ij_1}^2}{n_{ij_1}} \frac{\sigma_{ij_2}^2}{n_{ij_2}} \right] \to \frac{2}{b^2}(\phi^4 + b\eta^4).$$

It remains to verify Lyapounov's condition. By inequality (A.4),

$$\begin{aligned}
&E\left|\widetilde{T}_{Ai}\right|^{2+\delta} \\
&\leq (ab)^{-1-\frac{\delta}{2}} \frac{1}{b^{2+\delta}} 2^{1+\delta} \Big[ E\Big| \sum_{j=1}^{b} \frac{\sum_{k_1 \neq k_2}^{n_{ij}} X_{ijk_1} X_{ijk_2}}{n_{ij}(n_{ij}-1)} \Big|^{2+\delta} + E\Big| \sum_{j_1 \neq j_2}^{b} \overline{X}_{ij_1.} \overline{X}_{ij_2.} \Big|^{2+\delta} \Big] \\
&\leq (ab)^{-1-\frac{\delta}{2}} \frac{1}{b^{2+\delta}} 2^{1+\delta} \Big[ b^{1+\delta} \sum_{j=1}^{b} \frac{3^{1+\delta} D_\delta E|X_{ij1}|^{4+2\delta}}{(n_{ij}-1)^{2+\delta}} + 2^{1+\delta} b^{1+\delta} G_\delta \sum_{j=1}^{b} \frac{E\left|X_{ij1}\right|^{4+2\delta}}{n_{ij}^{2+\delta}} \Big] \\
&\leq a^{-1-\frac{\delta}{2}} b^{-2-\frac{\delta}{2}} H_\delta \sum_{j=1}^{b} \frac{E|X_{ij1}|^{4+2\delta}}{(n_{ij}-1)^{2+\delta}},
\end{aligned}$$

where $D_\delta$, $G_\delta$ and $H_\delta$ are finite positive constants depending only on $\delta$, The second inequality uses (A.6) and (A.4), similar as in the proof of Theorem 2.2. Thus, $\sum_{i=1}^{a} E|\widetilde{T}_{Ai}|^{2+\delta} \to 0$ and Lyapounov's condition is satisfied.

(b) For the case that $n_{ij} \to \infty$, a similar calculation yields

$$Var\left(n(a)\widetilde{T}_A\right) \to \frac{2(\phi_1^4 + b\eta_1^4)}{b^2}.$$

Lyapounov's condition will be satisfied if there exists a $\delta > 0$, such that

$$L(a) = \sum_{i=1}^{a} E\left|n(a)\widetilde{T}_{Ai}\right|^{2+\delta} \to 0. \tag{A.7}$$

Similarly as in (a), we have

$$L(a) \leq \frac{n(a)^{2+\delta}}{(n(a)-1)^{2+\delta}} a^{-1-\frac{\delta}{2}} b^{-2-\frac{\delta}{2}} H_\delta \sum_{i=1}^{a}\sum_{j=1}^{b} E|X_{ij1}|^{4+2\delta} \to 0.$$

**Proof of Theorem 2.4.** Write

$$T_A(\mathbf{Y}) = T_A(\mathbf{X}) + (ab)^{-\frac{1}{2}} \sum_{i=1}^{a}\sum_{j=1}^{b} \alpha_i^2(a) + 2(ab)^{-\frac{1}{2}} \sum_{i=1}^{a}\sum_{j=1}^{b} \alpha_i(a)\widetilde{X}_{i..},$$

let $h(a) = a^{-1/2}\sum_{i=1}^{a}\alpha_i^2(a)$ and $H(a) = 2a^{-1/2}\sum_{i=1}^{a}\alpha_i(a)\widetilde{X}_{i..}$. In both cases, $n(a)h(a) = a\sum_{i=1}^{a}(\int_{(i-1)/a}^{i/a} g(t)dt)^2 = a^{-1}\sum_{i=1}^{a} g^2(\xi_{ia}) \to \int_0^1 g^2(t)dt = \theta_A^2$, where $\xi_{ia} \in [(i-1)/a, i/a]$. Note that $E(H(a)) = 0$. Then

$$\mathrm{Var}\,(H(a)) = 4a^{-1}\sum_{i=1}^{a}\alpha_i^2(a)\frac{1}{b^2}\sum_{j=1}^{b}\frac{\sigma_{ij}^2}{n_{ij}(a)} \leq \frac{4a^{-\frac{1}{2}} \max\limits_{\substack{1\leq i\leq a\\ 1\leq j\leq b}} \sigma_{ij}^2}{bn(a)} h(a) \to 0$$

as $a \to \infty$. Thus $H(a) \xrightarrow{p} 0$. The proof is finished by combining the results of Theorem 2.3. and Slutsky's Theorem.

**Proof of Lemma 3.1.**

$$\begin{aligned}
&MST_A - MSE\\
&= \sum_{i=1}^{a}\sum_{j=1}^{b}\Big(\frac{1}{\sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij}^{-1}}\frac{1}{n_{ij}^2}\mathbf{X}_{ij}'\mathbf{J}_{n_{ij}}\mathbf{X}_{ij} + \frac{1}{N-ab}\frac{1}{n_{ij}}\mathbf{X}_{ij}'\mathbf{J}_{n_{ij}}\mathbf{X}_{ij}\\
&\quad -\frac{1}{N-ab}\mathbf{X}_{ij}'\mathbf{X}_{ij}\Big) + \frac{a}{(a-1)\sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij}^{-1}} \sum_{i=1}^{a}\sum_{j_1\neq j_2}\frac{1}{n_{ij_1}n_{ij_2}}\mathbf{X}_{ij_1}'\mathbf{1}_{n_{ij_1}}\mathbf{1}_{n_{ij_2}}'\mathbf{X}_{ij_2}\\
&\quad -\frac{1}{(a-1)\sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij}^{-1}}\sum_{(i_1,j_1)}\sum_{\neq(i_2,j_2)}\frac{1}{n_{i_1j_1}n_{i_2j_2}}\mathbf{X}_{i_1j_1}'\mathbf{1}_{n_{i_1j_1}}\mathbf{1}_{n_{i_2j_2}}'\mathbf{X}_{i_2j_2}.
\end{aligned}$$

Using this expression and the fact that

$$\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n_{ij}}\left(\frac{1}{\sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij}^{-1}}\frac{1}{n_{ij}^2} + \frac{1}{N-ab}\frac{1}{n_{ij}} - \frac{1}{N-ab}\right) = 0,$$

we easily obtain the expression of projection $\sum_{i=1}^a E\left(MST_A - MSE|\mathbf{X}_i\right)$.

**Proof of Proposition 3.2.** From the proof of Lemma 3.1,

$$\begin{aligned}
&MST_A - MSE - \widehat{S}_A \\
&\quad = -\frac{1}{(a-1)\sum_{i=1}^a \sum_{j=1}^b n_{ij}^{-1}} \sum_{i_1\neq i_2}^{a} \sum_{j_1=1}^{b} \sum_{j_2=1}^{b} \sum_{k_1=1}^{n_{i_1j_1}} \sum_{k_2=1}^{n_{i_2j_2}} \frac{1}{n_{i_1j_1}n_{i_2j_2}} X_{i_1j_1k_1}X_{i_2j_2k_2}.
\end{aligned}$$

We have

$$\begin{aligned}
&aE\left(MST_A - MSE - \widehat{S}_A\right)^2 \\
&\quad = \frac{2a}{(a-1)^2(\sum_{i=1}^a \sum_{j=1}^b \frac{1}{n_{ij}})^2} \sum_{i_1\neq i_2}^{a} \sum_{j_1=1}^{b} \sum_{j_2=1}^{b} \sum_{k_1=1}^{n_{i_1j_1}} \sum_{k_2=1}^{n_{i_2j_2}} \frac{1}{n_{i_1j_1}^2 n_{i_2j_2}^2} \\
&\quad \leq \frac{2a}{(a-1)^2(\sum_{i=1}^a \sum_{j=1}^b \frac{1}{n_{ij}})^2} \Big\{ \sum_{i=1}^a \sum_{j=1}^b \frac{1}{n_{ij}} \Big\}^2 = \frac{2a}{(a-1)^2} \to 0,
\end{aligned}$$

as $a \to \infty$.

**Proof of Lemma 3.3.** The expression for the projection of the statistic $T_A$ is easily derived by rewriting $T_A$ as

$$\begin{aligned}
T_A = (ab)^{-\frac{1}{2}} \sum_{i=1}^a \frac{a-1}{ab} \Big[ \Big( \sum_{j=1}^b \overline{X}_{ij.} \Big)^2 - \sum_{j=1}^b \frac{S_{ij}^2}{n_{ij}} \Big] \\
-(ab)^{-\frac{3}{2}} \sum_{i_1\neq i_2}^a \Big( \sum_{j=1}^b \overline{X}_{i_1j.} \Big) \Big( \sum_{j=1}^b \overline{X}_{i_2j.} \Big).
\end{aligned}$$

**Proof of Proposition 3.4.** Write

$$T_A - \widetilde{T}_A = -(ab)^{-\frac{3}{2}} \sum_{i_1\neq i_2}^a \Big( \sum_{j=1}^b \overline{X}_{i_1j.} \Big) \Big( \sum_{j=1}^b \overline{X}_{i_2j.} \Big).$$

If $n_{ij}(a) \to \infty$ as $a \to \infty$, then

$$\begin{aligned}
E\left(b^{\frac{1}{2}} n(a)(T_A - \widetilde{T}_A)\right)^2 &= \frac{2n^2(a)}{a^3b^2} \sum_{i_1\neq i_2}^a E\Big( \sum_{j=1}^b \overline{X}_{i_1j.} \Big)^2 E\Big( \sum_{j=1}^b \overline{X}_{i_2j.} \Big)^2 \\
&\leq \frac{2M}{a} \Big( \frac{1}{ab} \sum_{i=1}^a \sum_{j=1}^b \sigma_{ij}^2 \Big)^2 \to 0,
\end{aligned}$$

where $M$ is a positive constant. For the case that $n_{ij} \geq 2$ is fixed, simply treat $n(a)$ as fixed in the above derivation.

## References


Akritas, M. and Arnold, S. (2000). Asymptotics for ANOVA when the number of levels is large. *J. Amer. Statist. Assoc.* **95,** 212-226.

Akritas, M., Arnold, S. and Du, Y. (2000). Nonparametric models and methods for nonlinear analysis of covariance. *Biometrica* **87,** 507-526.

Akritas and Papadatos (2004). Heteroscedastic one-way ANOVA and lack-of-fit test. *J. Amer. Statist. Assoc.* **99**, 368-382.

Ananda, M. M. A. and Weerahandi, S. (1997). Two-way ANOVA with unequal cell frequencies and unequal variances. *Statist. Sinica* **7**, 631-646.

Arnold, S. (1980). Asymptotic validity of F tests for the ordinary linear model and the multiple correlation model. *J. Amer. Statist. Assoc.* **75,** 890-894.

Arnold, S. (1981). *The Theory of Linear Models and Multivariate Analysis*. John Wiley, New York.

Bathke, A. (2002). ANOVA for a Large Number of Treatments. *Math. Methods Statist.* **11**, 118-132.

Boos, D. D. and Brownie, C. (1995). ANOVA and rank tests when the number of treatments is large. *Statist. Probab. Lett.* **23,** 183-191.

Box, G. E. P. (1954). Some theorems on quadratic forms applied in the study of analysis of variance problems, I. Effect of inequality of variance in the one-way classification. *Ann. Math. Statist.* **25**, 290-302.

Box, G. E. P. and Anderson, S. L. (1955). Permutation theory in the derivation of robust criteria and the study of departures from assumption. *J. Roy. Statist. Soc. Ser. B* **17**, 1-26.

Chow, Y. S. and Teicher, H. (1997). *Probability Theory: Independece, Interchangeability, Martingales.* 3rd edition. Springer-Verlag, New York.

Dudoit, S., Yang, Y. H., Callow, M. J. and Speed, T. P. (2002). Statistical methods for identifying differentially expressed genes in replicated cDNA microarray experiments. *Statist. Sinica* **12,** 111-139.

Hocking, R. R. (1996). *Methods and Applications of Linear Models.* John Wiley, New York.

Lee, A. J. (1990). *U-Statistics.* Marcel-Dekker, New York.

Li, H., Lindsay, B. G. and Waterman, R. P. (2003). Efficiency of projected score methods in rectangular array asymptotics. *J. Roy. Statist. Soc. Ser. B* **65,** 191-208.

Miller, R. G. Jr. (1986). *Beyond ANOVA: Basics of Applied Statistics.* John Wiley, New York. (Reprinted 1996, Chapman and Hall, New York.)

Neter, J., Wasserman, W. and Kutner, M. (1985). *Applied Linear Statistical Models.* Irwin, Illinois.

Portnoy, S. (1984). Asymptotic behavior of $M$-estimators of $p$ regression parameters when $p^2/n$ is large. I. Consistency. *Ann. Statist.* **12,** 1298-1309.

Portnoy, S. (1986). Asymptotic behavior of the empiric distribution of $M$-estimated residuals from a regression model with many parameters *Ann. Statist.* **14,** 1152-1170.

Rencher, A. C. (2000). *Linear Models in Statistics.* John Wiley, New York.

Sahai, H. and Ageel, M. (2000). *The Analysis of Variance: Fixed, Random, and Mixed Models.* Springer Verlag, New York.

Scheffé, H. (1959). *The Analysis of Variance.* John Wiley, New York.

Searle, S. R. (1971). *Linear Models.* John Wiley, New York.

van der Vaart, A. W. (1998). *Asymptotic Statistics*. Cambridge University Press, UK.

Wang, L. (2003). Testing in heteroscedastic ANOVA with large number of levels and in nonparametric ANCOVA. Ph.D. Thesis, the Pennsylvania State University.

Wang, L., Akritas, M. and Van Keilegom, I. (2002). Nonparametric Goodness-of-fit Test for Heteroscedastic Regression Models. Manuscript.

Yates, F. (1934). The analysis of multiple classifications with unequal numbers in the different classes. *J. Amer. Statist. Assoc.* **29,** 51-56.

385 Ford Hall, School of Statistics, University of Minnesota, 224 Church Street SE, Minneapolis, MN 55455, U.S.A.

E-mail: lan@stat.umn.edu

414 Thomas Building, Department of Statistics, the Penn State University, University Park, PA 16802, U.S.A.

E-mail: mga@stat.psu.edu

(Received January 2004; accepted August 2005)