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EDGEWORTH EXPANSIONS FOR SYMMETRIC STATISTICS WITH APPLICATIONS TO BOOTSTRAP METHODS

Tze Leung Lai and Julia Qizhi Wang

Stanford University and University of Minnesota

Abstract: Edgeworth expansions are developed for a general class of symmetric statistics. Applications of the results are given to obtain approximations to the sampling distributions of statistics in the random censorship model and of linear combinations of order statistics. In addition, Edgeworth expansions are also developed for the bootstrap distributions of these symmetric statistics, showing that the bootstrap approximations are accurate to the order of $O_p(n^{-1})$.

Key words and phrases: Efron-Stein ANOVA decomposition, asymptotic U-statistics, Edgeworth expansions, bootstrap, random censorship model, cumulative hazard function, log-rank statistics, linear combinations of order statistics.

1. Introduction

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random vectors. A statistic $S = S(X_1, \ldots, X_n)$ is said to be *symmetric* if it is invariant under permutation of the arguments. Assuming that $ES^2 < \infty$, let $\mu = ES$ and define

$$A(x_i) = E(S|X_i = x_i) - \mu,$$
$$B(x_i, x_j) = E(S|X_i = x_i, X_j = x_j) - E(S|X_i = x_i) - E(S|X_j = x_j) + \mu \quad (i \neq j)$$

etc. Then $B(x, y) = B(y, x)$, etc., and, as shown by Efron and Stein (1981), S has the ANOVA decomposition

$$\begin{aligned} S - \mu \ &= \sum_{i=1}^{n} A(X_i) + \sum_{1 \le i < j \le n} B(X_i, X_j) + \sum_{1 \le i < j < k \le n} C(X_i, X_j, X_k) \\ &\quad + \sum_{1 \le i < j < k < h \le n} D(X_i, X_j, X_k, X_h) + \cdots + R(X_1, \ldots, X_n), \end{aligned} \tag{1.1}$$

where all $2^n - 1$ random variables on the right hand side of (1.1) have mean 0 and are mutually uncorrelated with each other. In fact, $E\{B(X_1, X_2)|X_1\} = 0$, $E\{C(X_1, X_2, X_3)|X_1, X_2\} = 0$, etc.

Let α be a Borel function such that $E\alpha(X_1) = 0$. When $A = n^{-1/2}\alpha$ and all the other functions on the right hand side of (1.1) vanish, $S - \mu$ reduces to a normalized sum of i.i.d. zero-mean random variables, for which the Edgeworth expansion

$$P\{(S-\mu)/\sigma \leq z\} = \Phi(z) - n^{-1/2}\phi(z)P_1(z) - n^{-1}\phi(z)P_2(z) + o(n^{-1}) \tag{1.2}$$

holds under the assumption

$$E\alpha^2(X_1) = \sigma^2 > 0,\ E\alpha^4(X_1) < \infty \ \text{ and } \ \limsup_{|t|\to\infty} |Ee^{it\alpha(X_1)}| < 1. \tag{1.3}$$

Here, and in the sequel, we use $\phi(z)$ and $\Phi(z)$ to denote the density and distribution functions of the standard normal distribution, and $P_1(z), P_2(z)$ to denote polynomials in z. In addition to its obvious application as a more accurate approximation to $P\{(S-\mu)/\sigma \leq z\}$ than the crude normal approximation $\Phi(z)$, the Edgeworth expansion (1.2) has recently been used to show that $P\{(S-\mu)/\sigma \leq z\}$ can be alternatively approximated by Efron's (1979) bootstrap method with an error of the order $O_p(n^{-1})$, (cf. Hall (1986, 1988)).

When all functions except A and B on the right hand side of (1.1) vanish, $S-\mu$ reduces to a U-statistic of degree 2. In this case, Bickel, Götze and van Zwet (1986) established an Edgeworth expansion of the form (1.2) under the assumption that the functions $\alpha = n^{1/2}A$ and $\beta = n^{1/2}(n-1)B$ satisfy (1.3) and the following condition:

Condition (B). $E|\beta(X_1,X_2)|^r < \infty$ for some $r > 2$, and the linear operator L, mapping a function f (with $Ef^2(X_1) < \infty$) to the function Lf defined by $(Lf)(y) = E\{\beta(y,X_1)f(X_1)\}$, has at least K nonzero eigenvalues (with multiple eigenvalues repeated) such that $K > 4r/(r-2)$.

In this paper we show that the Edgeworth expansion (1.2) holds much more generally for symmetric statistics with $A \sim n^{-1/2}\alpha, B \sim n^{-3/2}\beta$ and $C \sim n^{-5/2}\gamma$ for some given Borel functions α, β, γ, and with the sum of the remaining terms in the ANOVA decomposition (1.1) having the order $O(n^{-1-\epsilon})$ for some $\epsilon > 0$. Our main result, which is stated in Section 2 and proved in Section 4, provides an Edgeworth expansion of the form (1.2) for a general class of statistics, which we call *asymptotic U-statistics*, and which are based on asymptotic modifications of (1.1) to make the decomposition more flexible and transparent in applications. In Section 3 we show that this main result yields Edgeworth expansions not only for the sampling distributions of a large variety of symmetric statistics but also for their bootstrap distributions. Of particular interest are the Edgeworth expansions in the random censorship model. Since censoring greatly complicates the distribution theory of the statistics, it has led to a heavy reliance on normal approximations. The Edgeworth expansions developed herein for these censored statistics not only provide more accurate approximations but also establish the bootstrap approach as a much more practical alternative that is accurate up to an $O_p(n^{-1})$ error.

2. Asymptotic U-statistics and Edgeworth Expansions

Let $X, X_1, \ldots, X_n$ be i.i.d. p-dimensional random vectors and let $U_n = U_n(X_1, \ldots, X_n)$ be a real-valued function of $X_1, \ldots, X_n$. We shall call U_n an *asymptotic U-statistic* if it has the decomposition

$$U_n = \sum_{i=1}^{n} \left\{ \frac{\alpha(X_i)}{\sqrt{n}} + \frac{\alpha'(X_i)}{n^{3/2}} \right\} + \sum_{1 \le i < j \le n} \frac{\beta(X_i, X_j)}{n^{3/2}} + \sum_{1 \le i < j < k \le n} \frac{\gamma(X_i, X_j, X_k)}{n^{5/2}} + R_n, \quad (2.1)$$

where $\alpha, \alpha', \beta, \gamma$ are nonrandom Borel functions which are invariant under permutation of the arguments and which satisfy assumptions (A2)–(A4) below, and the R_n are random variables satisfying (A1).

(A1) $P\{|R_n| \ge n^{-1-\epsilon}\} = o(n^{-1})$ for some $\epsilon > 0$,

(A2) $E\alpha(X) = E\alpha'(X) = 0$,

(A3) $E\{\beta(X_1, X_2)|X_1\} = 0$, $E\{\gamma(X_1, X_2, X_3)|X_1, X_2\} = 0$,

(A4) $E\{|\alpha'(X_1)|^3 + |\gamma(X_1, X_2, X_3)|^4\} < \infty$.

The main result in this section is an Edgeworth expansion of the form (1.2) for U_n under the assumption (1.3) on α. This result uses a more convenient reformulation of Condition (B) and also replaces it, in situations where it fails, by an assumption that is slightly stronger than the usual Cramér (strongly nonlattice) condition $\limsup_{|t| \to \infty} |Ee^{it\alpha(X_1)}| < 1$ in (1.3). To begin with, consider the linear operator L defined by $(Lf)(y) = E\{\beta(y, X)f(X)\}$ on the Hilbert space of Borel functions $f : R^p \to R$ with $\|f\|_2^2 = Ef^2(X) < \infty$. Then either L has infinitely many nonzero eigenvalues, or $P\{\beta(X_1, X_2) = 0\} = 0$ a.s. (which corresponds to the case of no nonzero eigenvalue), or there exists some positive integer K for which

$$\beta(X_1, X_2) = \sum_{\nu=1}^{K} \lambda_\nu \omega_\nu(X_1) \omega_\nu(X_2) \quad \text{a.s.}, \quad (2.2)$$

where the λ_ν are the nonzero eigenvalues of L and the ω_ν are the corresponding eigenfunctions which satisfy

$$E\omega_\nu(X) = 0, \quad E\omega_\nu^2(X) = 1, \quad E\{\omega_\nu(X)\omega_\ell(X)\} = 0 \text{ for } \ell \ne \nu, \quad (2.3)$$

(cf. (3.17) and (3.18) of Bickel, Götze and van Zwet (1986)). Note that (2.2) implies

$$\begin{aligned} \sum_{1 \le i < j \le n} \beta(X_i, X_j) &= \sum_{\nu=1}^{K} \lambda_\nu \left\{ \sum_{1 \le i < j \le n} \omega_\nu(X_i)\omega_\nu(X_j) \right\} \\ &= \sum_{\nu=1}^{K} (\lambda_\nu/2) \left\{ \left(\sum_{i=1}^{n} \omega_\nu(X_i) \right)^2 - \sum_{i=1}^{n} \omega_\nu^2(X_i) \right\}, \end{aligned} \quad (2.4)$$

and, therefore,

$$\sum_{i=1}^{n}\left\{\frac{\alpha(X_i)}{\sqrt{n}}+\frac{\alpha'(X_i)}{n^{3/2}}\right\}+\sum_{1\le i<j\le n}\frac{\beta(X_i,X_j)}{n^{3/2}}=\frac{1}{n^{3/2}}\sum_{i=1}^{n}\left\{\alpha'(X_i)-\frac{1}{2}\sum_{\nu=1}^{K}\lambda_\nu\omega_\nu^2(X_i)\right\}$$
$$+\sqrt{n}H\Big(n^{-1}\sum_{i=1}^{n}\alpha(X_i),n^{-1}\sum_{i=1}^{n}\omega_1(X_i),\dots,n^{-1}\sum_{i=1}^{n}\omega_K(X_i)\Big), \tag{2.5}$$

where $H(x,w_1,\dots,w_K)=x+\sum_{\nu=1}^{K}\lambda_\nu w_\nu^2/2$. The seminal work of Bhattacharya and Ghosh (1978) on Edgeworth expansions of smooth functions of sample mean vectors requires the joint Cramér condition

$$\limsup_{|t|+|s_1|+\dots+|s_K|\to\infty}\left|E\exp\left\{it\alpha(X)+i\sum_{\nu=1}^{K}s_\nu\omega_\nu(X)\right\}\right|<1, \tag{2.6}$$

when it is applied to the above function H. Bai and Rao (1991) recently established Edgeworth expansions under the conditional Cramér condition

$$\limsup_{|t|\to\infty}E|E(e^{it\alpha(X)}|\omega_1(X),\dots,\omega_K(X))|<1, \tag{2.7}$$

which can be used even when $\omega_\nu(X)$ is not strongly nonlattice. In Condition (D) below, we shall introduce a Cramér-type condition which is weaker than either (2.6) or (2.7).

As shown in Section 4 of Bickel, Götze and van Zwet (1986), one can check whether the number of nonzero eigenvalues of L satisfies Condition (B) without direct evaluation of these eigenvalues by checking Condition (C) below in the special case $\gamma=0$. In fact, for $\gamma=0$, Condition (B) is equivalent to Condition (C), cf. Lemma 4.1 of Bickel, Götze and van Zwet (1986).

Condition (C). $E|\beta(X_1,X_2)|^r<\infty$ for some $r>2$ and there exist K Borel functions $f_\nu:R^p\to R$ such that $K(r-2)>4r+(28r-40)I_{\{E|\gamma(X_1,X_2,X_3)|>0\}}$, $Ef_\nu^2(X_1)<\infty$ $(\nu=1,\dots,K)$, and the covariance matrix of $(W_1,\dots,W_K)$ is positive definite, where $W_\nu=(Lf_\nu)(X_1)$ and $(Lf)(y)=E\{\beta(y,X_2)f(X_2)\}$.

When condition (B) fails, the argument used in Section 3 of Bickel, Götze and van Zwet (1986) breaks down. However, since the representation (2.2) holds in this case, (2.5) suggests an alternative argument that involves a joint or conditional Cramér condition (2.6) or (2.7). Since what is actually involved in this argument is a representation of the form (2.2) without requiring the ω_ν to be eigenfunctions of L, we arrive at the following more general and convenient assumption.

Condition (D). There exist constants c_ν and Borel functions $g_\nu:R^p\to R$ such that $Eg_\nu(X)=0$, $E|g_\nu(X)|^r<\infty$ for some $r\ge 5$ and $\beta(X_1,X_2)=$

$\sum_{\nu=1}^{K} c_\nu g_\nu(X_1)g_\nu(X_2)$ a.s.; moreover, for some $0 < \delta < \min\{1, 2(1 - 11r^{-1}/3)\}$,

$$\limsup_{|t|\to\infty} \sup_{|s_1|+\cdots+|s_K|\le|t|^{-\delta}} \left| E\exp\left(it\left\{\alpha(X) + \sum_{\nu=1}^{K} s_\nu g_\nu(X)\right\}\right)\right| < 1.$$

Clearly, for $g_\nu = \omega_\nu$, the joint Cramér condition (2.6) implies the Cramér-type condition in (D), which is also weaker than the conditional Cramér condition (2.7) since

$$\left| E\exp\left\{it\alpha(X) + it\sum_{\nu=1}^{K} s_\nu g_\nu(X)\right\}\right| \le E|E(e^{it\alpha(X)}|g_1(X),\ldots,g_K(X))|.$$

On the other hand, the Cramér-type condition in (D) implies the condition $\limsup_{|t|\to\infty} |Ee^{it\alpha(X)}| < 1$ in (1.3), and is equivalent to the latter condition in the case where g_ν is a scalar multiple of α for every $\nu \in \{1,\ldots,K\}$.

Theorem 1. *Let U_n be an asymptotic U-statistic defined by* (2.1) *and* (A1)–(A4). *Suppose α satisfies* (1.3) *and either Condition (C) or (D) holds. Let $\sigma = (E\alpha^2(X))^{1/2}$ as in* (1.3) *and define*

$$\begin{aligned}
a_3 &= E\alpha^3(X),\ a_4 = E\alpha^4(X),\ a' = E\{\alpha(X)\alpha'(X)\},\ b = E\{\alpha(X_1)\alpha(X_2)\beta(X_1,X_2)\},\\
c &= E\{\alpha(X_1)\alpha(X_2)\alpha(X_3)\gamma(X_1,X_2,X_3)\},\ \kappa_3 = a_3 + 3b,\\
\kappa_4 &= a_4 - 3\sigma^4 + 4c + 12E\{\alpha^2(X_1)\alpha(X_2)\beta(X_1,X_2)\\
&\quad + \alpha(X_1)\alpha(X_2)\beta(X_1,X_3)\beta(X_2,X_3)\},\\
P_1(z) &= \kappa_3\sigma^{-3}(z^2-1)/6,\\
P_2(z) &= \left\{a' + \frac{E\beta^2(X_1,X_2)}{4}\right\}\frac{z}{\sigma^2} + \frac{\kappa_4}{24\sigma^4}(z^3-3z) + \frac{\kappa_3^2}{72\sigma^6}(z^5 - 10z^3 + 15z).
\end{aligned}$$

Then $P\{U_n/\sigma \le z\} = \Phi(z) - n^{-1/2}\phi(z)P_1(z) - n^{-1}\phi(z)P_2(z) + o(n^{-1})$, uniformly in $-\infty < z < \infty$.

In the remainder of this section we give three important examples of asymptotic U-statistics. These examples have motivated the preceding definition of asymptotic U-statistics and our development of Edgeworth expansions and bootstrap methods for them. They illustrate the wide applicability of Theorem 1 to give Edgeworth expansions for nonparametric statistics that are smooth functionals of empirical distribution functions, analogous to the Edgeworth expansions for smooth functions of sample mean vectors that have been developed by Bhattacharya and Ghosh (1978), Skovgaard (1981), Bai and Rao (1991) and others.

Example 1. Let $T_1, T_2, \ldots$ be i.i.d. random variables with a common continuous distribution function F. Consider the "random censorship model" in which the observations are $X_i = (T_i \wedge C_i, I_{\{T_i \le C_i\}})$, $i = 1,\ldots,n$, where $C_1, C_2, \ldots$ are i.i.d.

random variables that are independent of $T_1, T_2, \ldots$, and $\wedge$ denotes minimum. Let $\Lambda = -\log(1-F)$ be the cumulative hazard function. Let

$$\tilde{T}_j = T_j \wedge C_j, \quad Y_j(s) = I_{\{\tilde{T}_j \geq s\}}, \quad N_j(s) = I_{\{T_j \leq s, T_j \leq C_j\}}. \tag{2.8}$$

The Altschuler-Nelson estimate of $\Lambda(t)$ is given by

$$\hat{\Lambda}_n(t) = \sum_{i:\tilde{T}_i \leq t, T_i \leq C_i} \Big(\sum_{j=1}^n Y_j(\tilde{T}_i)\Big)^{-1} = \sum_{i=1}^n \int_{-\infty}^t \Big(\sum_{j=1}^n Y_j(s)\Big)^{-1} dN_i(s), \tag{2.9}$$

(cf. Fleming and Harrington (1991)). Define

$$p(s) = EY_1(s), \quad w_i(s) = Y_i(s) - p(s), \quad M_i(t) = N_i(t) - \int_{-\infty}^t Y_i(s) d\Lambda(s). \tag{2.10}$$

Then $\{M_i(s), -\infty < s < \infty\}$ is continuous-time martingale and

$$\sqrt{n}(\hat{\Lambda}_n(t) - \Lambda(t)) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \int_{-\infty}^t \frac{dM_i(s)}{n^{-1}\sum_{j=1}^n Y_j(s)} = \frac{1}{\sqrt{n}} \sum_{i=1}^n \int_{-\infty}^t \frac{dM_i(s)}{p(s) + n^{-1}\sum_{j=1}^n w_j(s)}.$$

Suppose that $p(t) > 0$ and let $U_n = \sqrt{n}(\hat{\Lambda}_n(t) - \Lambda(t))$. Expanding $f(x) = (p(s)+x)^{-1}$ around $x = 0$ by Taylor's theorem and making use of the identity

$$\sum_{j=1}^n w_j^2(s) = np(s)(1-p(s)) + (1-2p(s)) \sum_{j=1}^n w_j(s),$$

we can represent U_n in the form (2.1) with

$$\begin{aligned}
\alpha(X_i) &= \int_{-\infty}^t \frac{dM_i(s)}{p(s)}, \\
\alpha'(X_i) &= -\int_{-\infty}^t \frac{w_i(s)}{p^2(s)} dM_i(s) + \int_{-\infty}^t \frac{1-p(s)}{p^2(s)} dM_i(s), \\
\beta(X_i, X_j) &= -\int_{-\infty}^t \frac{1}{p^2(s)} \Big(w_i(s) dM_j(s) + w_j(s) dM_i(s)\Big), \\
\gamma(X_i, X_j, X_k) &= \int_{-\infty}^t \frac{2}{p^3(s)} \Big\{w_i(s) w_j(s) dM_k(s) + w_i(s) w_k(s) dM_j(x) \\
&\quad + w_j(s) w_k(s) dM_i(s)\Big\}, \\
R_n &= \sum_{i=1}^n \int_{-\infty}^t \Big\{\frac{(1-2p(s))\sum_{j=1}^n w_j(s)}{n^{5/2} p^3(s)} \\
&\quad - \frac{\{n^{-1}\sum_{j=1}^n w_j(s)\}^3}{\sqrt{n}\{p(s) + \theta_n(s) n^{-1} \sum_{j=1}^n w_j(s)\}^4}\Big\} dM_i(s),
\end{aligned}$$

where $\theta_n(s)$ lies between 0 and 1. An application of the exponential inequality for continuous-parameter martingales (cf. Shorack and Wellner (1986, page

899)) and exponential bounds for the empirical process $\sum_{j=1}^n w_j(s)$ can be used to show that $P\{|R_n| \geq n^{-1-\epsilon}\} = o(n^{-1})$ for $0 < \epsilon < 1/2$. Moreover, as will be shown below, Condition (C) is satisfied. Hence Theorem 1 is applicable to give an Edgeworth expansion of the form $P\{\sqrt{n}(\hat{\Lambda}_n(t) - \Lambda(t)) \leq \sigma z\} = \Phi(z) - n^{-1/2}\phi(z)P_1(z) - n^{-1}\phi(z)P_2(z) + o(n^{-1})$ when $\alpha(X_i) = \int_{-\infty}^t dM_i(s)/p(s)$ satisfies (1.3) and $F(t) > 0$, $p(t) > 0$.

To show that Condition (C) holds, let $f_\nu(X_j) = \int_{-\infty}^t h_\nu(s)dM_j(s)$ for $\nu = 1, 2, \ldots$, where $h_\nu : R \to R$ is a nonrandom Borel function that will be specified later. Then

$$\begin{aligned} W_\nu &= E\{\beta(X_1, X_2)f_\nu(X_2)|X_1\} \\ &= -\int_{-\infty}^t \frac{1}{p^2(s)}\Big\{w_1(s)h_\nu(s)p(s)d\Lambda(s) - p(s)\Big(\int_{-\infty}^s h_\nu(u)d\Lambda(u)\Big)dM_1(s)\Big\}, \end{aligned}$$

as can be shown by standard martingale and stochastic integral arguments (cf. Fleming and Harrington (1991, page 86)) and by noting that $E\{w_2(s)dM_2(u)\} = -I_{\{s\geq u\}}p(s)d\Lambda(u)$. Hence, for any $K \geq 1$ and constants $a_1, \ldots, a_K$, $\sum_{\nu=1}^K a_\nu W_\nu = 0$ a.s. implies that

$$\begin{aligned} &\int_{-\infty}^t (w_1(s)/p(s))\Big(\sum_{\nu=1}^K a_\nu h_\nu(s)\Big)d\Lambda(s) \\ &= \int_{-\infty}^t \Big\{\int_{-\infty}^s \sum_{\nu=1}^K a_\nu h_\nu(u)d\Lambda(u)\Big\}(p(s))^{-1}dM_1(s) \text{ a.s.} \end{aligned} \tag{2.11}$$

Since $E\{w_1(s)|I_{\{\tilde{T}_1\leq u\}}, I_{\{T_1\leq C_1\}}I_{\{\tilde{T}_1\leq u\}}, u \leq \tau\} = (p(s)/p(\tau))I_{\{\tilde{T}_1\geq\tau\}} - p(s)$ for $s > \tau$, it follows by taking conditional expectations on both sides of (2.11) that

$$\begin{aligned} &\int_{-\infty}^\tau \frac{w_1(s)}{p(s)}\Big(\sum_{\nu=1}^K a_\nu h_\nu(s)\Big)d\Lambda(s) + \frac{w_1(\tau)}{p(\tau)}\int_\tau^t \sum_{\nu=1}^K a_\nu h_\nu(s)d\Lambda(s) \\ &= \int_{-\infty}^\tau \Big\{\int_{-\infty}^s \sum_{\nu=1}^K a_\nu h_\nu(u)d\Lambda(u)\Big\}(p(s))^{-1}dM_1(s) \quad \text{a.s.} \end{aligned} \tag{2.12}$$

for all $\tau < t$. Letting $h_\nu = \Lambda^\nu$ and taking variances on both sides of (2.12) gives

$$\begin{aligned} &\int_{-\infty}^\tau\int_{-\infty}^\tau \Big(\frac{p(\max(s,u))}{p(s)p(u)} - 1\Big)\Big(\sum_{\nu=1}^K a_\nu\Lambda^\nu(s)\Big)\Big(\sum_{\nu=1}^K a_\nu\Lambda^\nu(u)\Big)d\Lambda(s)d\Lambda(u) \\ &+ \frac{1-p(\tau)}{p(\tau)}\Big\{\sum_{\nu=1}^K \frac{a_\nu}{\nu+1}(\Lambda^{\nu+1}(t) - \Lambda^{\nu+1}(\tau))\Big\}^2 \\ &+ 2\Big\{\sum_{\nu=1}^K \frac{a_\nu}{\nu+1}(\Lambda^{\nu+1}(t) - \Lambda^{\nu+1}(\tau))\Big\}\Big\{\int_{-\infty}^\tau \sum_{\nu=1}^K \frac{1-p(s)}{p(s)}a_\nu\Lambda^\nu(s)d\Lambda(s)\Big\} \\ = &\int_{-\infty}^\tau \Big(\sum_{\nu=1}^K \frac{a_\nu}{\nu+1}\Lambda^{\nu+1}(s)\Big)^2\frac{d\Lambda(s)}{p(s)} \quad \text{for all } \tau < t. \end{aligned}$$

Since Λ is continuous, this implies that $a_1 = \cdots = a_K = 0$. Note that $\beta(X_1, X_2)$ is a bounded random variable since $p(t) > 0$. Hence Condition (C) holds, and the linear operator L defined by $(Lf)(y) = E\{\beta(y, X_2)f(X_2)\}$ has infinitely many nonzero eigenvalues.

Example 2. Suppose that in the random censorship model of Example 1, the T_i are related to covariates Z_i via the linear regression model $T_i = \rho Z_i + \epsilon_i$, where $(Z_1, \epsilon_1), \ldots, (Z_n, \epsilon_n)$ are i.i.d. random vectors such that Z_1 is bounded and independent of ϵ_1, which is assumed to have a continuous distribution function H. Letting $e_i(a) = \tilde{T}_i - aZ_i$ and $\#_i(a) = \sum_{j=1}^n I_{\{e_j(a) \geq e_i(a)\}}$, modified log-rank statistics of the form

$$S_n(a) = \sum_{1 \leq i \leq n, T_i \leq C_i \wedge (\tau + aZ_i)} \Big\{ Z_i - \sum_{j=1}^n Z_j I_{\{e_j(a) \geq e_i(a)\}} / \#_i(a) \Big\} \tag{2.13}$$

have been studied by Tsiatis (1990) in connection with testing $H_0 : \rho = \rho_0$ on the basis of the test statistic $S_n(\rho_0)$ and estimating ρ via the estimating equation $S_n(a) = 0$. When $\tau = \infty$, (2.13) corresponds to the usual log-rank statistic. Tsiatis *op. cit.* chooses τ so that $p(\tau) > 0$, where

$$\begin{aligned}
&p(s) = P\{\epsilon_1 \wedge (C_1 - \rho Z_1) \geq s\} = EY_1(s),\\
&Y_i(s) = I_{\{\epsilon_i \wedge (C_i - \rho Z_i) \geq s\}}, \quad w_i(s) = Y_i(s) - p(s),\\
&M_i(s) = I_{\{\epsilon_i \leq s \wedge (C_i - \rho Z_i)\}} - \int_{-\infty}^t Y_i(s) d\Lambda(s), \quad \Lambda = -\log(1 - H).
\end{aligned}$$

A basic tool in Tsiatis' analysis to use the stochastic integral representation

$$S_n(\rho) = \sum_{i=1}^n \int_{-\infty}^{\tau} \Big\{ Z_i - \frac{n^{-1} \sum_{j=1}^n Z_j Y_j(s)}{p(s) + n^{-1} \sum_{j=1}^n w_j(s)} \Big\} dM_i(s), \tag{2.14}$$

with martingale integrators M_i and left continuous integrands. By an argument similar to that in Example 1, it can be shown that $n^{-1/2} S_n(\rho)$ is an asymptotic U-statistic with $X_i = (Z_i, \tilde{T}_i, I_{\{T_i \leq C_i\}})$ and

$$\begin{aligned}
\alpha(X_i) &= (Z_i - EZ_1) \int_{-\infty}^{\tau} dM_i(s),\\
\alpha'(X_i) &= (EZ_1) \int_{-\infty}^{\tau} \frac{w_i(s)}{p(s)} dM_i(s) - \int_{-\infty}^{\tau} \Big\{ \frac{Z_i Y_i(s)}{p(s)} - EZ_1 \Big\} dM_i(s)\\
&\quad - (EZ_1) \int_{-\infty}^{\tau} \frac{1 - p(s)}{p(s)} dM_i(s),\\
\beta(X_i, X_j) &= \sum_{(\mu,\nu) \in \{(i,j),(j,i)\}} \int_{-\infty}^{\tau} \Big\{ \frac{(EZ_1) w_\mu(s)}{p(s)} - \frac{Z_\mu Y_\mu(s)}{p(s)} + EZ_1 \Big\} dM_\nu(s),\\
\gamma(X_i, X_j, X_k) &= \sum_{\pi} \int_{-\infty}^{\tau} \Big\{ (Z_{\pi(j)} Y_{\pi(j)}(s) - p(s) EZ_1) \frac{w_{\pi(k)}(s)}{p^2(s)}
\end{aligned}$$

$$-\frac{EZ_1}{p^2(s)} w_{\pi(j)}(s) w_{\pi(k)}(s) \Big\} dM_{\pi(i)}(s),$$

where $\sum_\pi$ denotes summation over all six permutations of $\{i, j, k\}$; moreover, Condition (C) is satisfied.

Example 3. Let $X_1, \ldots, X_n$ be i.i.d. random variables with a common continuous distribution function F such that $\int_{-\infty}^{\infty} |x|^3 dF(x) < \infty$. Let $F_n = n^{-1} \sum_{i=1}^n I_{\{X_i \leq x\}}$ denote the empirical distribution function, and let $X_{(1)} \leq \cdots \leq X_{(n)}$ denote the order statistics. Let $\psi : [0,1] \to R$ be four times continuously differentiable. As shown by Moore (1968), the linear combination

$$S_n = \sum_{i=1}^n \psi(i/n) X_{(i)} = n \int_{-\infty}^{\infty} x\psi(F_n(x)) dF_n(x)$$

of order statistics is asymptotically normal with asymptotic variance $n\sigma^2$, where

$$\sigma^2 = 2 \iint_{s<t} \psi(F(s))\psi(F(t))F(s)(1 - F(t)) ds\, dt.$$

Let $G(u) = F^{-1}(u)$ $(= \sup\{t : F(t) = u\})$, $F_n^*(u) = F_n\ (F^{-1}(u))$ and

$$\begin{aligned} Z_n &= n^{-1/2} \Big\{ S_n - n \int_{-\infty}^{\infty} x\psi(F(x)) dF(x) \Big\} \\ &= \sqrt{n} \Big\{ \int_0^1 G(u)\psi(F_n^*(u)) dF_n^*(u) - \int_0^1 G(u)\psi(u) du \Big\}. \end{aligned}$$

Note that F_n^* is the empirical distribution function of the i.i.d. uniform random variables $F(X_1), \ldots, F(X_n)$. As shown by Moore (1968, pages 264–265), $Z_n = I_{1n} + I_{2n} + I_{3n}$, where

$$\begin{aligned} I_{1n} &= -\sqrt{n} \int_0^1 \psi(u)(F_n^*(u) - u) dG(u), \\ I_{2n} &= \sqrt{n} \int_0^1 G(u) \Big\{ \frac{\psi''(u)}{2} (F_n^*(u) - u)^2 + \frac{\psi'''(u)}{3!} (F_n^*(u) - u)^3 \\ &\quad + O(|F_n^*(u) - u|^4) \Big\} \Big\{ du + d(F_n^*(u) - u) \Big\}, \qquad (2.15) \\ I_{3n} &= -\frac{\sqrt{n}}{2} \int_0^1 (F_n^*(u) - u)^2 \Big\{ \psi'(u) dG(u) + G(u)\psi''(u) du \Big\} \\ &\quad + \frac{1}{2\sqrt{n}} \int_0^1 G(u)\psi'(u) dF_n^*(u). \end{aligned}$$

Let $w_i(u) = I_{\{F(X_i) \leq u\}} - u$ and note that

$$(F_n^*(u) - u)^2 = n^{-2} \Big\{ \sum_{j=1}^n w_j^2(u) + 2 \sum_{1 \leq i < j \leq n} w_i(u) w_j(u) \Big\}$$

$$= n^{-1}u(1-u) + n^{-2}(1-2u)\sum_{j=1}^{n} w_j(u) + 2n^{-2}\sum_{1\le i<j\le n} w_i(u)w_j(u),$$

$$(F_n^*(u)-u)^3 = n^{-3}\Big\{\sum_{j=1}^{n} w_j^3(u) + 3\sum_{1\le i\ne j\le n} w_i^2(u)w_j(u) + 6\sum_{1\le i<j\le k\le n} w_i(u)w_j(u)w_k(u)\Big\}.$$

Using these representations in (2.15), it can be shown that

$$U_n \triangleq Z_n + \frac{1}{2\sqrt{n}}\Big\{\int_0^1 u(1-u)\psi'(u)dG(u) - \int_0^1 G(u)\psi'(u)du\Big\}$$

is an asymptotic U-statistic with

$$\alpha(X_i) = -\int_0^1 \psi(u)w_i(u)dG(u) + \frac{1}{2}\int_0^1 G(u)\psi'(u)dw_i(u),$$

$$\alpha'(X_i) = \frac{1}{2}\Big\{\int_0^1 G(u)\psi''(u)u(1-u)dw_i(u) - \int_0^1 (1-2u)w_i(u)\psi'(u)dG(u)\Big\},$$

$$\beta(X_i,X_j) = -\int_0^1 w_i(u)w_j(u)\psi'(u)dG(u),$$

$$\gamma(X_i,X_j,X_k) = \int_0^1 G(u)\psi'''(u)w_i(u)w_j(u)w_k(u)du + \frac{1}{2}\sum_{\pi}\int_0^1 G(u)\psi''(u)w_{\pi(i)}(u)w_{\pi(j)}(u)dw_{\pi(k)}(u),$$

and that Condition (C) is satisfied, where $\sum_\pi$ denotes summation over all six permutations of $\{i,j,k\}$.

3. Edgeworth Expansions of Bootstrap Distributions

For statistics which can be expressed as smooth functions of multivariate sample means, it is known that Efron's (1979) bootstrap method provides an empirical Edgeworth expansion, with an $O_p(n^{-1})$ error, of the sampling distribution, (cf. Singh (1981), Beran (1982), Abramovitch and Singh (1985) and Hall (1986, 1988)). The following theorem, which will be proved in Section 4, shows that this result can be extended to asymptotic U-statistics.

Theorem 2. *With the same notation and assumptions as in Theorem 1, let H denote the distribution of X_1 and $\hat{H}_n(A) = n^{-1}\sum_{i=1}^n I_{\{X_i\in A\}}$ denote the empirical distribution, and let $X_1^*,\ldots,X_n^*$ be i.i.d. with common distribution $\hat{H}_n$. Suppose that there exist functions $\hat{\alpha}_n, \hat{A}_n, \hat{\beta}_n, \hat{\gamma}_n$, depending on $\hat{H}_n$ and invariant under* *permutation of the arguments, such that*

$$n^{-1}\sum_{i=1}^{n}|\hat{A}_n(X_i)|^3 + n^{-3}\sum_{1\le i<j<k\le n}|\hat{\gamma}_n(X_i,X_j,X_k)|^4 = O_p(1), \tag{3.1}$$

$$\begin{aligned}\sum_{i=1}^{n}\hat{\alpha}_n(X_i) &= \sum_{i=1}^{n}\hat{A}_n(X_i) = 0 = \sum_{i=1}^{n}\hat{\beta}_n(y_1,X_i)\\ &= \sum_{i=1}^{n}\hat{\gamma}_n(y_1,y_2,X_i), \quad \textit{for any } \ y_1,y_2\in S(H),\end{aligned} \tag{3.2}$$

$$\sup_{x\in S(H)}\frac{|\hat{\alpha}_n(x)-\alpha(x)|}{1+|\alpha(x)|} + \sup_{x,y\in S(H)}|\hat{\beta}_n(x,y)-\beta(x,y)| = O_p(n^{-1/2}), \tag{3.3}$$

where $S(H)$ denotes the support of H. Let

$$\begin{aligned}U_n^* &= \sum_{i=1}^{n}\left\{\frac{\hat{\alpha}_n(X_i^*)}{\sqrt{n}} + \frac{\hat{A}_n(X_i^*)}{n^{3/2}}\right\} + \sum_{1\le i<j\le n}\frac{\hat{\beta}_n(X_i^*,X_j^*)}{n^{3/2}}\\ &\quad + n^{-5/2}\sum_{1\le i<j<k\le n}\hat{\gamma}_n(X_i^*,X_j^*,X_k^*) + R_n^*,\end{aligned} \tag{3.4}$$

where $nP\{|\tilde{R}_n^*|\ge n^{-1-\epsilon}|\hat{H}_n\}\xrightarrow{P} 0$ for some $\epsilon>0$. Let $\hat{\sigma}_n^2 = E\{\hat{\alpha}_n^2(X_1^*)|\hat{H}_n\}$. Then

$$P\{U_n^*\le\hat{\sigma}_n z|\hat{H}_n\} = \Phi(z) - n^{-1/2}\phi(z)P_1(z) + O_p(n^{-1}), \quad \textit{uniformly in} \quad -\infty<z<\infty.$$

Consequently, $\sup_z |P\{U_n/\sigma\le z\} - P\{U_n^*\le\hat{\sigma}_n z|\hat{H}_n\}| = O_p(n^{-1})$.

As an illustration of the applications of Theorem 2, the following corollary shows that $P\{\sqrt{n}(\hat{\Lambda}_n(t)-\Lambda(t))/\sigma\le z\}$ in Example 1 can be approximated by the bootstrap estimate with an error of the order $O_p(n^{-1})$. First we review the bootstrap method for randomly censored data $X_i=(T_i\wedge C_i, I_{\{T_i\le C_i\}})$, $i=1,\ldots,n$. Let $\hat{F}_n$ and $\hat{G}_n$ be the Kaplan-Meier (1958) estimates of the distribution functions F and G of T_1 and C_1, respectively. Generating independent T_i^* from $\hat{F}_n$ and C_i^* from $\hat{G}_n$ gives the bootstrap sample $X_i^*=(T_i^*\wedge C_i^*, I_{\{T_i^*\le C_i^*\}})$, $i=1,\ldots,n$. In the random censorship model, this is equivalent to taking i.i.d. observations from the empirical distribution of $X_1,\ldots,X_n$, as shown by Efron (1982).

Corollary. *With the same notation and assumptions as in Example 1, suppose that $F(t)>0$, $p(t)>0$ and that α satisfies (1.3). Let $\hat{H}_n$ put weight $1/n$ on each of the bivariate vectors $X_i=(T_i\wedge C_i, I_{\{T_i\le C_i\}})$, $i=1,\ldots,n$, and let $X_1^*,\ldots,X_n^*$ be i.i.d. with common distribution $\hat{H}_n$. Let*

$$\tilde{T}_i^* = T_i^*\wedge C_i^*, \quad Y_i^*(s) = I_{\{\tilde{T}_i^*\ge s\}}, \quad \Lambda_n^*(t) = \sum_{i:\tilde{T}_i^*\le t, T_i^*\le C_i^*}\left(\sum_{j=1}^{n}Y_j^*(\tilde{T}_i^*)\right)^{-1},$$

and $\hat{\sigma}_n^2 = n\sum_{i:\tilde{T}_i\le t, T_i\le C_i}(\sum_{j=1}^n Y_j(\tilde{T}_i))^{-1}\{1-(\sum_{j=1}^n Y_j(\tilde{T}_i))^{-1}\}$. *Then*

$$n\sup_z\left|P\left\{\sqrt{n}(\Lambda_n^*(t)-\hat{\Lambda}_n(t))\le\hat{\sigma}_n z|\hat{H}_n\right\}-P\left\{\sqrt{n}(\hat{\Lambda}_n(t)-\Lambda_n(t))\le\sigma z\right\}\right|=O_p(1).$$

Proof. Let $\hat{p}(s)=n^{-1}\sum_{i=1}^n Y_i(s)$, $N_i^*(s)=I_{\{T_i^*\le s, T_i^*\le C_i^*\}}$, $w_i^*(s)=Y_i^*(s)-\hat{p}(s)$, and $M_i^*(t)=N_i^*(t)-\int_{-\infty}^t Y_i^*(s)(1-\Delta\hat{\Lambda}_n(s))d\hat{\Lambda}_n(s)$, where $\Delta\hat{\Lambda}_n(s)=\hat{\Lambda}_n(s)-\hat{\Lambda}_n(s-)$. Then the same argument as that in Example 1 shows that $U_n^*=\sqrt{n}(\Lambda_n^*(t)-\hat{\Lambda}_n(t))$ has the representation (3.4) with

$$\hat{\alpha}_n(X_i^*)=\int_{-\infty}^t\frac{dM_i^*(s)}{\hat{p}(s)},\quad \hat{\beta}_n(X_i^*,X_j^*)=-\int_{-\infty}^t\frac{w_i^*(s)dM_j^*(s)+w_j^*(s)dM_i^*(s)}{\hat{p}^2(s)},$$

$$R_n^*=\sum_{i=1}^n\int_{-\infty}^t\left\{\frac{(1-2\hat{p}(s))\sum_{j=1}^n w_j^*(s)}{n^{5/2}\hat{p}^3(s)}-\frac{\{n^{-1}\sum_{j=1}^n w_j^*(s)\}^3}{\sqrt{n}\{\hat{p}(s)+\theta_n(s)n^{-1}\sum_{j=1}^n w_j^*(s)\}^4}\right\}dM_i^*(s),$$

where $0\le\theta_n(s)\le 1$, and with $\hat{A}_n,\hat{\gamma}_n$ bounded in absolute values by some nonrandom constant C on the event $\Omega_n=\{\hat{p}(t)>\frac{1}{2}p(t)\}$. Since $P(\Omega_n)\to 1$ and $\sup_{s\le t}|\hat{p}(s)-p(s)|=O_p(n^{-1/2})$, the conditions of Theorem 2 are satisfied and therefore we can apply Theorem 2 to obtain the desired conclusion, noting that

$$E(\hat{\alpha}^2(X_1^*)|\hat{H}_n) = \int_{-\infty}^t\frac{1-\Delta\hat{\Lambda}_n(s)}{\hat{p}(s)}d\hat{\Lambda}_n(s),$$
$$\Delta\hat{\Lambda}_n(s) = \left(\sum_{j=1}^n Y_j(s)\right)^{-1}\quad\text{at } s=\tilde{T}_i \text{ with } T_i\le C_i.$$

While Example 1 provides an Edgeworth correction to the normal approximation $\Phi(z)$ for the probability $P\{\sqrt{n}(\hat{\Lambda}_n(t)-\Lambda(t))/\sigma\le z\}$, the above corollary shows that comparable accuracy can be achieved by using the bootstrap approximation, which can be evaluated by simulation without assuming any knowleddge about the underlying distribution functions F and G of T_1 and C_1. Tables 1 and 2 below report some numerical results comparing the normal, Edgeworth and bootstrap approximations to this probability in the case of exponential T_1 and C_1, with respective density functions $\lambda_1 e^{-\lambda_1 x}$ and $\lambda_2 e^{-\lambda_2 x}(x>0)$. Here $\Lambda(t)=\lambda_1 t$ and $p(t)=e^{-(\lambda_1+\lambda_2)t}$. In addition, the "exact" value of $P\{\sqrt{n}(\hat{\Lambda}_n(t)-\Lambda(t))/\sigma\le z\}$ is computed by the Monte Carlo method using 100,000 simulations. The Edgeworth approximation is the one-term Edgeworth correction $\text{EDG}(z)=\Phi(z)-n^{-1/2}\phi(z)P_1(z)$ to the normal approximation $\Phi(z)$, which is accurate to the order of $O(n^{-1})$ by Theorem 1.

Each bootstrap approximation in Tables 1 and 2 is based on (i) a single random sample of n observations $X_i=(T_i\wedge C_i, I_{\{T_i\le C_i\}})$, $i=1,\ldots,n$, giving the empirical distribution $\hat{H}_n$, and (ii) 10,000 bootsrtap samples for the evaluation of $P\{\sqrt{n}(\Lambda_n^*(t) - \hat{\Lambda}_n(t))/\hat{\sigma}_n \le z\}$ by simulation. Instead of the $\hat{\sigma}_n^2$ defined in the above corollary, we use here the following simpler version:

$$\hat{\sigma}_n^2 = I_{\{\sum_{j=1}^n Y_j(t)=0\}} + n \sum_{i:\tilde{T}_i \le t, T_i \le C_i} \left(\sum_{j=1}^n Y_j(\tilde{T}_i)\right)^{-1} I_{\{\sum_{j=1}^n Y_j(t)\ge 1\}}.$$

This differs from the $\hat{\sigma}_n^2$ in the corollary by at most $O_p(n^{-1})$, and has the advantage of being always positive. As the proof of Theorem 2 shows, an $O_p(n^{-1})$ modification of $\hat{\sigma}_n$ does not change the conclusion of the theorem.

Table 1. Values of $P_z = P\{\sqrt{n}(\hat{\Lambda}_n(0.4) - \Lambda(0.4))/\sigma \le z\}$ and of the normal approximation $\Phi(z)$, Edgeworth approximation $\mathrm{EDG}(z)$, and bootstrap approximation $\mathrm{BOOT}(z)$, for exponential T_i (with $\lambda_1 = 0.6$) and C_i (with $\lambda_2 = 0.4$).

	$n = 20$				$n = 60$			
z	P_z	$\Phi(z)$	$\mathrm{EDG}(z)$	$\mathrm{BOOT}(z)$	P_z	$\Phi(z)$	$\mathrm{EDG}(z)$	$\mathrm{BOOT}(z)$
−0.5	0.373	0.309	0.315	0.387	0.327	0.309	0.312	0.322
−0.25	0.419	0.401	0.411	0.402	0.432	0.401	0.407	0.433
−0.1	0.494	0.460	0.470	0.469	0.486	0.460	0.466	0.479
0	0.564	0.500	0.510	0.586	0.531	0.500	0.506	0.536
0.1	0.605	0.540	0.550	0.617	0.568	0.540	0.546	0.561
0.25	0.629	0.599	0.608	0.643	0.625	0.599	0.604	0.622
0.5	0.724	0.692	0.698	0.698	0.711	0.692	0.695	0.703

Table 2. Values of $P_z = P\{\sqrt{n}(\hat{\Lambda}_n(0.4) - \Lambda(0.4))/\sigma \le z\}$ and of the normal approximation $\Phi(z)$, Edgeworth approximation $\mathrm{EDG}(z)$, and bootstrap approximation $\mathrm{BOOT}(z)$, for exponential T_i (with $\lambda_1 = 0.2$) and C_i (with $\lambda_2 = 0.8$).

	$n = 60$				$n = 200$			
z	P_z	$\Phi(z)$	$\mathrm{EDG}(z)$	$\mathrm{BOOT}(z)$	P_z	$\Phi(z)$	$\mathrm{EDG}(z)$	$\mathrm{BOOT}(z)$
−0.5	0.347	0.309	0.321	0.382	0.323	0.309	0.315	0.325
−0.25	0.433	0.401	0.418	0.416	0.420	0.401	0.411	0.422
−0.1	0.483	0.460	0.479	0.464	0.481	0.460	0.470	0.480
0	0.545	0.500	0.519	0.547	0.521	0.500	0.510	0.524
0.1	0.595	0.540	0.558	0.606	0.559	0.540	0.550	0.563
0.25	0.630	0.599	0.616	0.639	0.619	0.599	0.608	0.623
0.5	0.717	0.692	0.704	0.699	0.706	0.692	0.698	0.709

The censoring probability $P\{C_i > T_i\}$ is 40% in Table 1 and 80% in Table 2. The tables show consistent improvement of the bootstrap and Edgeworth approximations over the normal approximations, and the improvement is particularly apparent when there is non-negligible discrepancy between the exact value and the normal approximation, e.g., at $z = 0$ where $\Phi(z) = 0.5$ while the exact value ranges from 0.52 to 0.56 in the four cases. The bootstrap method even outperforms the Edgeworth approximation in most cases.

4. Proof of Theorems 1 and 2

The following two lemmas are basic to the subsequent proofs.

Lemma 1. *Let $X_1, \ldots, X_n$ be i.i.d. random vectors and let $k \geq 2$. Suppose that $E|\psi(X_1, \ldots, X_k)|^r < \infty$ for some $r \geq 2$ and $E\{\psi(X_1, \ldots, X_k)|X_i, i \in I\} = 0$ for any proper subset I of $\{1, \ldots, k\}$. Then there exist absolute constants $A_{k,r}$ and $B_{k,r}$, depending only on k and r, such that for all $n \geq k$,*

$$E\left|\sum_{1\leq i_1<\cdots<i_k\leq n} \psi(X_{i_1}, \ldots, X_{i_k})\right|^r \leq A_{k,r} n^{kr/2} E|\psi(X_1, \ldots, X_k)|^r,$$

$$E\left|\sum_{i=1}^{m} \sum_{i<j_1<\cdots<j_{k-1}\leq n} \psi(X_i, X_{j_1}, \ldots, X_{j_{k-1}})\right|^r \leq B_{k,r}(mn^{k-1})^{r/2} E|\psi(X_1, \ldots, X_k)|^r$$

for all $1 \leq m \leq n - k + 1.$

Proof. The case $k = 2$ and $r = 3$ has been established by Callaert and Janssen (1978). A straightforward extension of their argument, making use of the moment bounds of Dharmadhikari, Fabian and Jogdeo (1968) for martingales, can be used to prove the lemma by induction.

Lemma 2. (Esseen's smoothing inequality, cf. Feller (1971)). *Let F_n be a probability distribution function and G_n be a function of bounded variation on the real line with respective characteristic functions f_n and g_n such that $g_n(0) = 1$ $(= f_n(0))$ and $g_n'(0) = f_n'(0) = 0$. Suppose that $F_n - G_n$ vanishes at $\pm\infty$ and that G_n has a bounded derivative. Then for every $T > 0$,*

$$\sup_{-\infty<z<\infty} |F_n(z) - G_n(z)| \leq \frac{1}{\pi}\int_{-T}^{T}\left|\frac{f_n(t) - g_n(t)}{t}\right| dt + \frac{24}{\pi T}\sup_{-\infty<z<\infty}|G_n'(z)|. \quad (4.1)$$

To prove Theorem 1, let $U_n' = U_n - R_n$. Since $P\{|R_n| \geq n^{-1-\epsilon}\} = o(n^{-1})$ by (A1) and since

$$\sup_{|t|\leq n^{-1-\epsilon}, -\infty<z<\infty} \left\{|\Phi(z+t) - \Phi(z)| + |\phi(z+t)P_j(z+t) - \phi(z)P_j(z)|\right\} = o(n^{-1}),$$

for $j = 1, 2$, it suffices to show the validity of the Edgeworth expansion with U_n' in place of U_n. We shall therefore apply (4.1) with $T = n \log n$ and

$$F_n(z) = P\{U_n'/\sigma \leq z\}, \quad G_n(z) = \Phi(z) - n^{-1/2}\phi(z)P_1(z) - n^{-1}\phi(z)P_2(z). \quad (4.2)$$

Since $g_n(t) = e^{-t^2/2}\{1 + n^{-1/2}q_1(t) + n^{-1}q_2(t)\}$ for some polynomials q_1, q_2,

$$\int_{|t|\geq n^\delta} |t|^{-1}|g_n(t)|dt = o(n^{-1}), \quad \text{for any } \delta > 0. \tag{4.3}$$

Let $r > 2$ be the same as that in Condition (C) or Condition (D) when (C) or (D) holds. In view of (4.1) and (4.3), Theorem 1 will follow if it can be shown that

$$\int_{|t|\leq n^\rho} \frac{|f_n(t) - g_n(t)|}{|t|} dt = o(n^{-1}), \tag{4.4}$$

$$\int_{n^\rho \leq |t| \leq n^{(r-1)/r}(\log n)^{-1}} |t^{-1} f_n(t)|dt = o(n^{-1}), \tag{4.5}$$

$$\int_{n^{(r-1)/r}(\log n)^{-1} \leq |t| \leq n \log n} |t^{-1} f_n(t)|dt = o(n^{-1}), \tag{4.6}$$

where $0 < \rho < 1/4$ will be specified later. Throughout the sequel we shall let $\mathrm{i} = \sqrt{-1}, \psi(t) = Ee^{\mathrm{i}t\alpha(X)}$, $A_n(X) = \alpha(X) + n^{-1}\alpha'(X)$ and $\psi_n(t) = Ee^{\mathrm{i}tA_n(X)}$. Note that $\psi_n(t) \to \psi(t)$ uniformly in $|t| \leq n(\log n)^{-1}$.

Proof of (4.4) under the assumptions of Theorem 1. We shall modify the arguments in Section 2 of Bickel, Götze and van Zwet (1986), which will be denoted by BGZ for brevity. Take $2 < s \leq \min(3, r)$ and choose $0 < \rho < 1/4$ such that $s/2 - \rho(s-1) > 1 - 2\rho$. In view of (A1)–(A4) and Lemma 1,

$$\begin{aligned} & t^2 E\left(n^{-5/2} \sum_{1<i<j<k\leq n} \gamma(X_i, X_j, X_k)\right)^2 + |t|^s E\left|n^{-3/2} \sum_{1\leq i<j\leq n} \beta(X_i, X_j)\right|^s \\ = \ & o(n^{-(1+2\rho)}|t|), \\ & t^2 E\left\{\left|n^{-5/2} \sum_{1\leq i<j<k\leq n} \gamma(X_i, X_j, X_k)\right|\left|n^{-3/2} \sum_{1\leq i<j\leq n} \beta(X_i, X_j)\right|\right\} \\ \leq \ & n^{-4}t^2 \left(E^{1/2}\left|\sum_{1\leq i<j<k\leq n} \gamma(X_i, X_j, X_k)\right|^2\right)\left(E^{1/2}\left|\sum_{1\leq i<j\leq n} \beta(X_i, X_j)\right|^2\right) \\ = \ & o(n^{-5/4}|t|), \end{aligned}$$

uniformly in $|t| \leq n^\rho$. Combining these with (2.1) and the Taylor expansions $e^{\mathrm{i}u} = 1 + \mathrm{i}u + O(u^2)$, $e^{\mathrm{i}u} = 1 + O(|u|)$ and $e^{\mathrm{i}u} = 1 + \mathrm{i}u - u^2/2 + O(|u|^s)$ as $u \to 0$ (cf. (2.7) of BGZ) yields

$$\begin{aligned} & Ee^{\mathrm{i}tU_n'/\sigma} \\ = \ & E\Bigg\{\left(1 + \frac{\mathrm{i}t}{\sigma n^{5/2}} \sum_{1\leq i<j<k\leq n} \gamma(X_i, X_j, X_k)\right) \exp\left(\frac{\mathrm{i}t}{\sigma\sqrt{n}} \sum_{i=1}^n A_n(X_i)\right. \\ & \left.\left. + \frac{\mathrm{i}t}{\sigma n^{3/2}} \sum_{1\leq i<j\leq n} \beta(X_i, X_j)\right)\right\} + o(n^{-(1+2\rho)}|t|) \end{aligned}$$

$$
\begin{aligned}
&= E\Big\{\Big(1+\frac{\mathrm{i}t}{\sigma n^{3/2}}\sum_{1\le i<j\le n}\beta(X_i,X_j) \\
&\quad -\frac{t^2}{2\sigma^2 n^3}\Big(\sum_{1\le i<j\le n}\beta(X_i,X_j)\Big)^2\Big)\exp\Big(\frac{\mathrm{i}t}{\sigma\sqrt{n}}\sum_{i=1}^n A_n(X_i)\Big)\Big\} \\
&\quad +\frac{\mathrm{i}t}{\sigma n^{5/2}}E\Big\{\Big(\sum_{1\le i<j<k\le n}\gamma(X_i,X_j,X_k)\Big)\exp\Big(\frac{\mathrm{i}t}{\sigma\sqrt{n}}\sum_{i=1}^n A_n(X_i)\Big)\Big\}+o(n^{-(1+2\rho)}|t|) \\
&= \psi_n^n\Big(\frac{t}{\sigma\sqrt{n}}\Big)+\frac{\mathrm{i}t}{\sigma n^{3/2}}\binom{n}{2}\psi_n^{n-2}\Big(\frac{t}{\sigma\sqrt{n}}\Big) \\
&\quad \cdot E\Big\{\beta(X_1,X_2)\exp\Big(\frac{\mathrm{i}t}{\sigma\sqrt{n}}(A_n(X_1)+A_n(X_2))\Big)\Big\} \\
&\quad -\frac{t^2}{2\sigma^2 n^3}\binom{n}{2}\psi_n^{n-2}\Big(\frac{t}{\sigma\sqrt{n}}\Big)E\Big\{\beta^2(X_1,X_2)\exp\Big(\frac{\mathrm{i}t}{\sigma\sqrt{n}}(A_n(X_1)+A_n(X_2))\Big)\Big\} \\
&\quad -\frac{3t^2}{\sigma^2 n^3}\binom{n}{3}\psi_n^{n-3}\Big(\frac{t}{\sigma\sqrt{n}}\Big)E\Big\{\beta(X_1,X_3)\beta(X_2,X_3)\exp\Big(\frac{\mathrm{i}t}{\sigma\sqrt{n}}\sum_{i=1}^3 A_n(X_i)\Big)\Big\} \\
&\quad -\frac{3t^2}{\sigma^2 n^3}\binom{n}{4}\psi_n^{n-4}\Big(\frac{t}{\sigma\sqrt{n}}\Big)\Big[E\Big\{\beta(X_1,X_2)\exp\Big(\frac{\mathrm{i}t}{\sigma\sqrt{n}}(A_n(X_1)+A_n(X_2))\Big)\Big\}\Big]^2 \\
&\quad +\frac{\mathrm{i}t}{\sigma n^{5/2}}\binom{n}{3}\psi_n^{n-3}\Big(\frac{t}{\sigma\sqrt{n}}\Big)E\Big\{\gamma(X_1,X_2,X_3)\exp\Big(\frac{\mathrm{i}t}{\sigma\sqrt{n}}\sum_{i=1}^3 A_n(X_i)\Big)\Big\} \\
&\quad +o(n^{-(1+2\rho)}|t|). \qquad (4.7)
\end{aligned}
$$

Applying Taylor's expansions for $\psi_n(t/\sigma\sqrt{n})$ and for $e^{\mathrm{i}u}$ to the above expression then shows that $f_n(t)=Ee^{\mathrm{i}tU_n'/\sigma}$ is equal to

$$
\begin{aligned}
&e^{-t^2/2}-n^{-1/2}\mathrm{i}t^3e^{-t^2/2}(a_3/6+b/2)/\sigma^3 \\
&\quad -\frac{t^2e^{-t^2/2}}{n\sigma^2}\Big\{a'+\frac{E\beta^2(X_1,X_2)}{4}\Big\}+\frac{t^4e^{-t^2/2}}{n\sigma^4}\frac{\kappa_4}{24}-\frac{t^6e^{-t^2/2}}{n\sigma^6}\Big(\frac{a_3^2}{72}+\frac{a_3b}{12}+\frac{b^2}{8}\Big) \\
&\quad +o(n^{-(1+2\rho)}|t|),
\end{aligned}
$$

uniformly in $|t|\le n^\rho$. Hence $f_n(t)=g_n(t)+o(n^{-(1+2\rho)}|t|)$ uniformlly in $|t|\le n^\rho$, implying (4.4).

Proof of (4.5) under the assumptions of Theorem 1. Following BGZ, we shall decompose the range of integration in (4.5) into two parts: $n^{(r-1)/r}(\log n)^{-1}\ge|t|\ge\epsilon\sqrt{n}$ and $\epsilon\sqrt{n}>|t|\ge n^\rho$, where $\epsilon>0$ will be specified later. By (1.3), there exists $0<\eta_\epsilon<1$ such that $\sup_{|u|\ge\epsilon}|\psi(u)|\le\eta_\epsilon$. For $m<n$, let

$$
W_{m,n}=n^{-1/2}\sum_{i=1}^n A_n(X_i)+n^{-3/2}\sum_{m+1\le i<j\le n}\beta(X_i,X_j)+n^{-5/2}\sum_{m+1\le i<j<k\le n}\beta(X_i,X_j,X_k). \qquad (4.8)
$$

For $n^{(r-1)/r}(\log n)^{-1} \geq |t| \geq \epsilon\sqrt{n}$, apply Lemma 1 with $m \sim (2r\log n)/|\log \eta_\epsilon|$ to get

$$t^2 E\left(n^{-5/2}\sum_{i=1}^{m}\sum_{i\leq j<k\leq n}\gamma(X_i,X_j,X_k)\right)^2 + |t|^r E\left|n^{-3/2}\sum_{i=1}^{m}\sum_{j=i+1}^{n}\beta(X_i,X_j)\right|^r$$
$$= O(n^{-1}(\log n)^{-r/2}), \tag{4.9}$$

$$E\left\{\left|tn^{-5/2}\sum_{i=1}^{m}\sum_{i<j<k\leq n}\gamma(X_i,X_j,X_k)\right|\left|tn^{-3/2}\sum_{i=1}^{m}\sum_{j=i+1}^{n}\beta(X_i,X_j)\right|^{3r/4}\right\}$$
$$\leq |t|^{1+3r/4}n^{-5/2-9r/8}\left\{E\left(\sum_{i=1}^{m}\sum_{i<j<k\leq n}\gamma(X_i,X_j,X_k)\right)^4\right\}^{1/4}$$
$$\cdot\left\{E\left|\sum_{i=1}^{m}\sum_{j=i+1}^{n}\beta(X_i,X_j)\right|^r\right\}^{3/4}$$
$$= O(n^{-5/4-1/r}(\log n)^{-1/2-3r/8}), \tag{4.10}$$

using Hölder's inequality. Let H be the greatest integer $< r$, and let h be the greatest integer $< 3r/4$. Combining (2.1) with (4.9), (4.10) and using Taylor's expansions for e^{iu} as in the first two equalities in (4.7) yields

$$f_n(t) = E\left\{\sum_{\nu=0}^{H}\frac{1}{\nu!}\left(\frac{it}{\sigma n^{3/2}}\sum_{i=1}^{m}\sum_{j=i+1}^{n}\beta(X_i,X_j)\right)^\nu e^{itW_{m,n}/\sigma}\right\}$$
$$+\frac{it}{\sigma n^{5/2}}E\left\{\left(\sum_{i=1}^{m}\sum_{i<j<k\leq n}\gamma(X_i,X_j,X_k)\right)\sum_{\nu=0}^{h}\frac{1}{\nu!}\left(\frac{it}{\sigma n^{3/2}}\sum_{i=1}^{m}\sum_{j=i+1}^{n}\beta(X_i,X_j)\right)^\nu\right.$$
$$\left.\times\, e^{itW_{m,n}/\sigma}\right\} + O(n^{-1}(\log n)^{-r/2}). \tag{4.11}$$

In view of (4.8), $|E(e^{itW_{m,n}/\sigma}|X_{m+1},\ldots,X_n)| \leq |E\exp(itn^{-1/2}\sum_{i=1}^{m}A_n(X_i)/\sigma)| = |\psi_n^m(n^{-1/2}t/\sigma)|$, and therefore $|Ee^{itW_{m,n}/\sigma}| \leq |\psi_n(t/\sigma\sqrt{n})|^m$. Likewise, conditioning on $X_m, X_{m+1},\ldots,X_n$ can be used to show that

$$\left|E\beta(X_m,X_{m+1})e^{itW_{m,n}/\sigma}\right| \leq \left|E\exp\left(itn^{-1/2}\sum_{i=1}^{m-1}A_n(X_i)/\sigma\right)\right|(E|\beta(X_m,X_{m+1})|),$$

and, therefore, by symmetry,

$$\sum_{i=1}^{m}\sum_{j=m+1}^{n}\left|E\beta(X_i,X_j)e^{itW_{m,n}/\sigma}\right| \leq m(n-m)|\psi_n(t/\sigma\sqrt{n})|^{m-1}E|\beta(X_1,X_2)|.$$

Repeating this argument for the other terms of (4.11) shows that

$$|f_n(t)| = o(n^r|\psi_n(t/\sigma\sqrt{n})|^{m-H}) + O(n^{-1}(\log n)^{-r/2}), \tag{4.12}$$

uniformly in $\epsilon\sqrt{n} \le t \le n^{(r-1)/r}(\log n)^{-1}$. Since $\psi_n(t/\sigma\sqrt{n}) = \psi(t/\sigma\sqrt{n}) + o(1)$ uniformly in $|t| \le n^{(r-1)/r}(\log n)^{-1}$ and since $\sup_{|u|\ge\epsilon}|\psi(u)| \le \eta_\epsilon$ while $m \sim (2r\log n)/|\log\eta_\epsilon|$, (4.12) implies that $|f_n(t)| = O(n^{-1}(\log n)^{-r/2})$ uniformly in $\epsilon\sqrt{n} \le |t| \le n^{(r-1)/r}(\log n)^{-1}$.

For $n^\rho \le |t| \le \epsilon\sqrt{n}$, take $2 < s \le \min(3,r)$ and apply Lemma 1 with $m \sim (9n\log n)/t^2$ to show that

$$\begin{aligned} & t^2 E\Big(n^{-5/2}\sum_{i=1}^{m}\sum_{i<j<k\le n}\gamma(X_i,X_j,X_k)\Big)^2 + |t|^s E\Big|n^{-3/2}\sum_{i=1}^{m}\sum_{j=i+1}^{n}\beta(X_i,X_j)\Big|^s \\ = \; & O((n^{-1}\log n)^{s/2}), \\ & t^2\Bigg(E^{1/2}\Big\{\Big|n^{-5/2}\sum_{i=1}^{m}\sum_{i<j<k\le n}\gamma(X_i,X_j,X_k)\Big|^2\Big\}\Bigg) \\ & \cdot\Bigg(E^{1/2}\Big\{\Big|n^{-3/2}\sum_{i=1}^{m}\sum_{j=i+1}^{n}\beta(X_i,X_j)\Big|^2\Big\}\Bigg) \\ = \; & O(n^{-3/2}\log n). \end{aligned}$$

Therefore, proceeding as in the first two equalities of (4.7). we obtain (4.11) with $H = 2$, $h = 0$ and with the $O(n^{-1}(\log n)^{-r/2})$ term there replaced by $O((n^{-1}\log n)^{s/2})$. Hence, analogous to (4.12), we now have

$$|f_n(t)| = o(n|\psi_n(t/\sigma\sqrt{n})|^m) + O((n^{-1}\log n)^{s/2}). \tag{4.13}$$

Choose ϵ sufficiently small so that $|\psi_n(t/\sigma\sqrt{n})| \le 1 - t^2/(3n)$ for all $|t| \le \epsilon\sqrt{n}$ and $n \ge n_0$ (sufficiently large). Hence, for $|t| \le \epsilon\sqrt{n}$ and $n \ge n_0$,

$$|\psi_n(t/\sigma\sqrt{n})|^m \le \exp\{-mt^2/(3n)\} = \exp\{-(3+o(1))\log n\};$$

so (4.13) implies that $|f_n(t)| = O((n^{-1}\log n)^{s/2})$ for $n^\rho \le |t| \le \epsilon\sqrt{n}$. Hence (4.5) follows.

Proof of (4.6) under the assumptions of Therorem 1 when Condition (D) holds. For $n\log n \ge |t| \ge n^{(r-1)/r}(\log n)^{-1}$, we shall apply Lemma 1 with $m \sim (\log n)^2$. Instead of (4.8), we use here

$$\tilde{W}_{m,n} = n^{-1/2}\sum_{i=1}^{n}A_n(X_i) + n^{-3/2}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\beta(X_i,X_j) + n^{-5/2}\sum_{m+1\le i<j<k\le n}\gamma(X_i,X_j,X_k). \tag{4.14}$$

Making use of (2.1), Lemma 1 and arguments similar to (4.7) and (4.9), we obtain, for $n^{(r-1)/r}(\log n)^{-1} \le |t| \le n\log n$,

$$f_n(t) = E\Bigg\{\Bigg[1 + \frac{it}{\sigma n^{5/2}}\sum_{i=1}^{m}\sum_{i<j<k\le n}\gamma(X_i,X_j,X_k)$$

$$-\frac{t^2}{2\sigma^2 n^5}\Big(\sum_{i=1}^{m}\sum_{i<j<k\le n}\gamma(X_i,X_j,X_k)\Big)^2\Big]\exp\Big(\frac{it}{\sigma}\tilde{W}_{m,n}\Big)\Big\}+o(n^{-(1+\theta)})$$
$$= E\Big\{Q_{m,n}(t,X_{m-2},X_{m-1},\ldots,X_{m+4})\exp(it\tilde{W}_{m,n}/\sigma)\Big\}+o(n^{-(1+\theta)}) \quad (4.15)$$

for some $\theta > 0$ and some nonrandom function $Q_{m,n}$ of the indicated arguments, where the last equality follows by symmetry, e.g.,

$$E\Big\{\sum_{i=1}^{m}\sum_{i<j<k\le n}\gamma(X_i,X_j,X_k)e^{it\tilde{W}_{m,n}/\sigma}\Big\}$$
$$= \binom{m}{3}E\Big\{\gamma(X_{m-2},X_{m-1},X_m)e^{it\tilde{W}_{m,n}/\sigma}\Big\}$$
$$+ (n-m)\binom{m}{2}E\Big\{\gamma(X_{m-1},X_m,X_{m+1})e^{it\tilde{W}_{m,n}/\sigma}\Big\}$$
$$+ m\binom{n-m}{2}E\Big\{\gamma(X_m,X_{m+1},X_{m+2})e^{it\tilde{W}_{m,n}/\sigma}\Big\}.$$

Since $E\gamma^4(X_1,X_2,X_3) < \infty$, these symmetry and combinatorial arguments also yield

$$\{EQ^4_{m,n}(t,X_{m-2},\ldots,X_{m+4})\}^{1/4} = O((mn^2|t|/n^{5/2})^2). \quad (4.16)$$

From (4.15) it follows that

$$|f_n(t)| \le E\{|Q_{m,n}(t,X_{m-2},\ldots,X_{m+4})||E(^{it\tilde{W}_{m,n}/\sigma}|X_{m-2},\ldots,X_n)|\} + o(n^{-1-\theta}). \quad (4.17)$$

By Condition (D), $\sum_{i=1}^{m-3}\sum_{j=m-2}^{n}\beta(X_i,X_j) = \sum_{\nu=1}^{K}c_\nu(\sum_{i=1}^{m-3}g_\nu(X_i))(\sum_{j=m-2}^{n} g_\nu(X_j))$. Let $\Omega_n = \{\sum_{\nu=1}^{K}|c_\nu n^{-1}\sum_{j=m-2}^{n}g_\nu(X_j)| \le (t/\sigma\sqrt{n})^{-\delta}\}$, where $0<\delta<1$ is the same as that given in Condition (D), and let Ω_n^c denote the complement of Ω_n. Since $(t/\sqrt{n})^{-\delta} \ge n^{-\delta/2}(\log n)^{-\delta}$ and $E|g_\nu(X)|^r < \infty$ with $r \ge 5$,

$$P(\Omega_n^c) \le P\Big\{\sum_{\nu=1}^{K}|c_\nu|\Big|\sum_{j=m-2}^{n}g_\nu(X_j)\Big| \ge \sigma^{-\delta}n^{1-\delta/2}(\log n)^{-\delta}\Big\}$$
$$= O(n^{-((1-\delta/2)r-1)}(\log n)^{r\delta}), \quad (4.18)$$

by the tail probability bounds (3.2)–(3.4) of Chow and Lai (1975). Since $E|\sum_{i=1}^{m-3}\sum_{j=i+1}^{m-3}\beta(X_i,X_j)|^5 \le Cm^5$ for some constant C by Lemma 1, it follows from (4.14) and arguments similar to (4.7) that

$$\Big|E(e^{it\tilde{W}_{m,n}/\sigma}|X_{m-2},\ldots,X_n)\Big|$$
$$\le \Big|E\Big(\exp\Big\{\frac{it}{\sigma\sqrt{n}}\Big[\sum_{i=1}^{m-3}A_n(X_i)+\sum_{\nu=1}^{K}\Big(\sum_{i=1}^{m-3}g_\nu(X_i)\Big)\Big(c_\nu n^{-1}\sum_{j=m-2}^{n}g_\nu(X_j)\Big)\Big]$$

$$
\begin{aligned}
&+ \frac{it}{\sigma n^{3/2}} \sum_{i=1}^{m-3} \sum_{j=i+1}^{m-3} \beta(X_i, X_j)\Big\} \Big| X_{m-2}, \ldots, X_n\Big)\Big| \\
\leq\ & \Big| E\Big[\Big\{1 + \sum_{\ell=1}^{4} \frac{1}{\ell!}\Big(\frac{it}{\sigma n^{3/2}} \sum_{i=1}^{m-3} \sum_{j=i+1}^{m-3} \beta(X_i, X_j)\Big)^{\ell}\Big\} \exp\Big\{\frac{it}{\sigma\sqrt{n}}\Big[\sum_{i=1}^{m-3} A_n(X_i) \\
&+ \sum_{\nu=1}^{K}\Big(\sum_{i=1}^{m-3} g_\nu(X_i)\Big)\Big(c_\nu n^{-1} \sum_{j=m-2}^{n} g_\nu(X_j)\Big)\Big]\Big\}\Big| X_{m-2}, \ldots, X_n\Big]\Big| + \frac{Cm^5|t|^5}{\sigma^5 n^{15/2}} \\
=\ & \Big| \phi_n^{m-3} + \frac{it}{\sigma n^{3/2}} \binom{m}{2} \phi_n^{m-5} \xi_n + \cdots + \frac{t^4}{24\sigma^4 n^6}\Big\{\binom{m}{2} \phi_n^{m-5} \zeta_{n,2} \\
&+ \binom{m}{3} \phi_n^{m-6} \zeta_{n,3} + \cdots + \binom{m}{8} \phi_n^{m-11} \zeta_{n,8}\Big\}\Big| + C(m|t|n^{-3/2}/\sigma)^5, \qquad (4.19)
\end{aligned}
$$

where

$$
\begin{aligned}
\phi_n \ =\ & E\Big(\exp\Big\{\frac{it\alpha'(X)}{\sigma n^{3/2}} + \frac{it}{\sigma\sqrt{n}}\Big[\alpha(X) \\
&+ \sum_{\nu=1}^{K} g_\nu(X)\Big(c_\nu n^{-1} \sum_{j=m-2}^{n} g_\nu(X_j)\Big)\Big]\Big\}\Big| \sum_{j=m-2}^{n} g_\nu(X_j)\Big), \qquad (4.20)
\end{aligned}
$$

and $\xi_n, \ldots, \zeta_{n,2}, \ldots, \zeta_{n,8}$ are bounded random variables, e.g., $|\xi_n| \leq E|\beta(X_1, X_2)|$, $|\zeta_{n,2}| \leq E\beta^4(X_1, X_2), |\zeta_{n,8}| \leq (E|\beta(X_1, X_2)|)^4$. In view of Condition (D), there exists $0 < \eta < 1$ such that $|\phi_n| \leq \eta$ on Ω_n for all sufficiently large n (with $n^{(r-1)/r}$ $(\log n)^{-1} \leq |t| \leq n \log n$), noting that $\sup_{|t| \leq n \log n} |t| E|\alpha'(X)|/n^{3/2} = o(1)$. Therefore it follows from (4.19) that

$$
E(|E(e^{it\tilde{W}_{m,n}/\sigma} | X_{m-2}, \ldots, X_n)|^{4/3} I_{\Omega_n}) = \{O(\eta^m) + O((m|t|n^{-3/2})^5)\}^{4/3}. \qquad (4.21)
$$

From (4.17), it follows by Hölder's inequality that

$$
\begin{aligned}
|f_n(t)| \ \leq\ & \Big\{EQ^4_{m,n}(t, X_{m-2}, \ldots, X_{m+4})\Big\}^{1/4} \Big\{E(I_{\Omega_n^c}) \\
&+ E(|E(e^{it\tilde{W}_{m,n}/\sigma} | X_{m-2}, \ldots, X_n)|^{4/3} I_{\Omega_n})\Big\}^{3/4}. \qquad (4.22)
\end{aligned}
$$

Since $m \sim (\log n)^2, |t| \leq n \log n$ and $\frac{3}{4}((1 - \delta/2)r - 1) > \frac{3}{4}(\frac{11}{3} - 1) = 2$, combining (4.22) with (4.16), (4.18) and (4.21) yields $|f_n(t)| = o(n^{-1-\theta})$ uniformly in $n \log n \geq |t| \geq n^{(r-1)/r}$, for sufficiently small $\theta > 0$.

Proof of (4.6) under the assumptions of Theorem 1 when Condition (C) holds. Let

$$
\begin{aligned}
q \ &=\ 2 \ \text{ if } \ E|\gamma(X_1, X_2, X_3)| = 0, \\
&=\ 4K(r+1)/\{(K-8)r\} \ \text{ if } \ E|\gamma(X_1, X_2, X_3)| > 0. \qquad (4.23)
\end{aligned}
$$

Define $\tilde{W}_{m,n}$ as in (4.14), where m is so chosen that $m-3=2M$ is even and

$$Mt^2 \sim n^{2+2q/K}(\log n)^{2+40/K}. \tag{4.24}$$

Since $n\log n \ge |t| \ge n^{(r-1)/r}(\log n)^{-1}$, (4.24) implies that

$$n^{2q/K}(\log n)^{40/K} \le M \le n^{2q/K+2/r}(\log n)^{4+40/K}. \tag{4.25}$$

From (4.23) and the assumption that $K(r-2)>4r$ if $E|\gamma(X_1,X_2,X_3)|=0$ and $K(r-2)>32r-40$ if $E|\gamma(X_1,X_2,X_3)|>0$, it follows that $2q/K+2/r<1$. Moreover, for the case $E|\gamma(X_1,X_2,X_3)|>0$, we obtain by Lemma 1 and (4.24) that

$$|t|^3E|n^{-5/2}\sum_{i=1}^{m}\sum_{i<j<k\le n}\gamma(X_i,X_j,X_k)|^3 = O(n^{3(1+q/K)-9/2}(\log n)^{3+60/K}),$$

noting that $9/2-3(1+q/K)>1$ by (4.23) since $K>(32r-40)/(r-2)$. Hence (4.15) still holds for sufficiently small $\theta>0$ and with $Q_{m,n}$ satisfying (4.16). Moreover, in the case $E|\gamma(X_1,X_2,X_3)|=0$ (i.e., $\gamma(X_1,X_2,X_3)=0$ a.s.), (4.15) trivially holds with $Q_{m,n}=1$.

Recalling that $m-3=2M$, we obtain from (4.14), (4.15) and (4.16) (in which we use $EQ_{m,n}^2 \le (EQ_{m,n}^4)^{1/2}$) by an argument similar to (3.4) of BGZ that

$$\begin{aligned}
&|f_n(t)|^2+o(n^{-2-2\theta})\\
\le\ & 2\Big(E\{|Q_{m,n}(t,X_{m-2},\dots,X_{m+4})||E(e^{it\tilde{W}_{m,n}/\sigma}|X_{m-2},\dots,X_n)|\}\Big)^2\\
\le\ & 2\Big\{EQ_{m,n}^2(t,X_{m-2},\dots,X_{m+4})\Big\}E\Bigg|E\Bigg(\exp\Bigg\{\frac{it}{\sigma}\Bigg(\frac{1}{\sqrt{n}}\sum_{i=1}^{m-3}A_n(X_i)\Bigg)\\
&+n^{-3/2}\sum_{i=1}^{m-4}\sum_{j=i+1}^{m-3}\beta(X_i,X_j)+n^{-3/2}\sum_{i=1}^{m-3}\sum_{j=m-2}^{n}\beta(X_i,X_j)\Bigg\}\Bigg|X_{m-2},\dots,X_n\Bigg)\Bigg|^2\\
=\ & O(M^4t^4/n^2)E\Bigg(\exp\Bigg\{\frac{it}{\sigma\sqrt{n}}\sum_{i=1}^{2M}(A_n(X_i)-A_n(Y_i))\\
&+\frac{it}{\sigma n^{3/2}}\sum_{i=1}^{2M-1}\sum_{j=i+1}^{2M}(\beta(X_i,X_j)-\beta(Y_i,Y_j))\\
&+\frac{it}{\sigma n^{3/2}}\sum_{i=1}^{2M}\sum_{j=2M+1}^{n}(\beta(X_i,X_j)-\beta(Y_i,Y_j))\Bigg\}\Bigg),\\
\le\ & O(M^4t^4/n^2)E\Bigg(\Bigg|E\exp\Bigg\{\frac{it}{\sigma n^{3/2}}\sum_{i=1}^{2M}(\beta(X_i,X)\\
&-\beta(Y_i,X))|X_1,Y_1,\dots,X_{2M},Y_{2M}\Bigg\}\Bigg|^{n-2M}\Bigg),
\end{aligned} \tag{4.26}$$

where $Y, Y_1, \ldots, Y_n$ are i.i.d. random variables that are independent of $X, X_1, \ldots, X_n$ and such that Y has the same distribution as X. Let $n' = 2M + [(n-2M)/2]$, where $[x]$ denotes the integer part of x. Since $n - 2M \geq 2[(n-2M)/2] = 2(n' - 2M)$, it follows from (4.26) that

$$\begin{aligned}
&|f_n(t)|^2 + o(n^{-2-2\theta}) \\
&\leq O\Big(\frac{M^4t^4}{n^2}\Big) E\bigg(\bigg|E\bigg(\exp\bigg\{\frac{it}{\sigma n^{3/2}}\sum_{i=1}^{2M}(\beta(X_i, X) \\
&\quad - \beta(Y_i, X))\bigg\}\bigg|X_1, Y_1, \ldots, X_{2M}, Y_{2M}\bigg)\bigg|^{2(n'-2M)}\bigg) \\
&= O\Big(\frac{M^4t^4}{n^2}\Big) E\bigg(\exp\bigg\{\frac{it}{\sigma n^{3/2}}\sum_{i=2}^{2M}\sum_{j=2M+1}^{n'}\Big(\beta(X_i, X_j) \\
&\quad - \beta(X_i, Y_j) - \beta(Y_i, X_j) + \beta(Y_i, Y_j)\Big)\bigg\}\bigg) \\
&= O(M^4t^4/n^2) E\exp\bigg\{itn^{-3/2}\sum_{i=1}^{2M}\sum_{j=2M+1}^{n'} v(X_i, Y_i; X_j, Y_j)\bigg\}, \qquad (4.27)
\end{aligned}$$

where $v(x, y; X, Y) = \{\beta(x, X) - \beta(x, Y) - \beta(y, X) + \beta(y, Y)\}/\sigma$. Moreover, the factor $O(M^4t^4/n^2)$ in (4.26) and (4.27) can be replaced by 1 if $\gamma(X_1, X_2, X_3) = 0$ a.s.

Since Condition (C) holds, we can use exactly the same argument as that in BGZ, pages 1473–1477, to show that

$$\begin{aligned}
&E\exp\bigg\{itn^{-3/2}\sum_{i=1}^{2M}\sum_{j=2M+1}^{n'} v(X_i, Y_i; X_j, Y_j)\bigg\} \\
&\leq \bigg[1 - \frac{1}{6}\frac{t^2M}{n^3}n^{-2q/K}(\log n)^{-40/K}\bigg]^{n'-2M} + O\Big(n^{-q}(\log n)^{-20} + M^{-K/2}\Big), \qquad (4.28)
\end{aligned}$$

analogous to the upper bound at the top of page 1477 of BGZ. Since $n' - 2M \sim n/2$, it follows from (4.24), (4.25), (4.27) and (4.28) that

$$\begin{aligned}
&|f_n(t)|^2 + o(n^{-2-2\theta}) \\
&\leq \bigg\{1 + O\Big(\frac{M^4t^4}{n^2}\Big) I_{\{E|\gamma(X_1,X_2,X_3)|>0\}}\bigg\}\bigg\{\exp\Big(-\frac{1+o(1)}{12}(\log n)^2\Big) + O(n^{-q}(\log n)^{-20})\bigg\} \\
&= o(n^{-2}(\log n)^{-3}), \qquad (4.29)
\end{aligned}$$

noting that $M^4t^4/n^2 = O(n^{2+8q/K+4/r}(\log n)^{12+160/K})$ and that in the case $E|\gamma(X_1, X_2, X_3)| > 0$, $q - (2 + 8q/K + 4/r) = q(K-8)/K - 2 - 4/r = 2$ by (4.23) while $160/K < 160(r-2)/(32r-40) < 160/32 = 5$. (For the case $\gamma(X_1, X_2, X_3) = 0$ a.s., $q (= 2)$ is the same as that used in BGZ.) From (4.29),

$|f_n(t)| = o(n^{-1}(\log n)^{-3/2})$ uniformly in $n^{(r-1)/r}(\log n)^{-1} \leq |t| \leq n \log n$, proving (4.6).

Proof of Theorem 2. We shall apply (4.1) with $T = n \log n$ (or $T = dn$ for some $d > 0$), $\hat{F}_n(z) = P\{U_n^* - R_n^* \leq \hat{\sigma}_n z | \hat{H}_n\}$ and $G_n(z) = \Phi(z) - n^{-1/2}\phi(z)P_1(z)$. We can proceed in the same way as in the proof of (4.4) above to show that $\int_{|t| \leq n^\rho} |t|^{-1} |\hat{f}_n(t) - g_n(t)| dt = O_p(n^{-1})$ in this case, replacing E by $E(\cdot|\hat{H}_n)$ and $o(n^{-b}), O(n^{-b})$ by $o_p(n^{-b}), O_p(n^{-b})$, etc. From (3.3) it follows that

$$\int \hat{\alpha}_n^2(x)d\hat{H}_n(x) - \sigma^2 = \int (\hat{\alpha}_n^2(x) - \alpha^2(x))d\hat{H}_n(x) + \int \alpha^2(x)d\hat{H}_n(x) - \sigma^2$$
$$= \int \left\{2\alpha(x)(1+|\alpha(x)|)\frac{\hat{\alpha}_n(x) - \alpha(x)}{1+|\alpha(x)|} + (1+|\alpha(x)|)^2\left(\frac{\hat{\alpha}_n(x) - \alpha(x)}{1+|\alpha(x)|}\right)^2\right\}d\hat{H}_n(x)$$
$$+ \frac{1}{n}\sum_{i=1}^{n}(\alpha^2(X_i) - \sigma^2) = O_p\left(\frac{1}{\sqrt{n}}\right), \tag{4.30}$$

since $\int \alpha^2(x)d\hat{H}_n(x) = n^{-1}\sum_{i=1}^n \alpha^2(X_i) = O_p(1)$. Similarly,

$$\int \hat{\alpha}_n^3(x)d\hat{H}_n(x) = a_3 + O_p(n^{-1/2}),$$
$$\iint \hat{\alpha}_n(x)\hat{\alpha}_n(y)\hat{\beta}_n(x,y)d\hat{H}_n(x)d\hat{H}_n(y) = b + O_p(n^{-1/2}).$$

To prove $\int_{n^\rho \leq |t| \leq n^{(r-1)/r}(\log n)^{-1}} |t^{-1}\hat{f}_n(t)|dt = o_p(n^{-1})$, we make use of the following result of Abramovitch and Singh (1985, page 129) on the empirical characteristic function $\hat{\psi}(t) = \int e^{\mathrm{i}t\alpha(x)}d\hat{H}_n(x)$:

$$\sup_{|t| \leq n^a} \left|\hat{\psi}(t/\sqrt{n}\sigma) - \psi(t/\sqrt{n}\sigma)\right| \to 0 \text{ a.s. for any } a > 0. \tag{4.31}$$

Let $\hat{\psi}_n(t) = \int \exp\{\mathrm{i}t\alpha(c) + \mathrm{i}t(\hat{\alpha}_n(x) - \alpha(x) + n^{-1}\hat{A}_n(x))\}d\hat{H}_n(x)$. Since $\int |\hat{A}_n(x)| d\hat{H}_n(x) = O_p(1)$ by (3.1) and since $\int |\hat{\alpha}_n(x) - \alpha(x)|d\hat{H}_n(x) = O_p(n^{-1/2})$ by an argument similar to (4.30), it follows that

$$\sup_{|t| \leq n^{(r-1)/r}} \left|\hat{\psi}_n(t/\sqrt{n}\sigma) - \hat{\psi}(t/\sqrt{n}\sigma)\right| \xrightarrow{P} 0. \tag{4.32}$$

Combining (4.31) and (4.32) yields $\hat{\psi}_n(t/\sqrt{n}\sigma) = \psi(t/\sqrt{n}\sigma) + o_p(1)$, uniformly in $|t| \leq n^{(r-1)/r}$ and we can therefore repeat the same proof as that of (4.5).

To prove $\int_{n^{(r-1)/r}(\log n)^{-1} \leq |t| \leq n \log n} |t^{-1}\hat{f}_n(t)|dt = o_p(n^{-1})$ when Condition (C) holds, define the linear operator $\hat{L}$ by $(\hat{L}f)(y) = \int \hat{\beta}_n(y,x)f(x)d\hat{H}_n(x)$, in analogy with the linear operator L in Condition (C). Let $W_\nu^* = (\hat{L}f_\nu)(X_1^*)$ and define

$$\hat{V} = \left(E_{\hat{H}_n}(W_\mu^* W_\nu^*)/\{E_{\hat{H}_n}(W_\mu^{*2})E_{\hat{H}_n}(W_\nu^{*2})\}^{1/2}\right)_{1 \leq \mu,\nu \leq K},$$

where $E_{\hat{H}_n}$ denotes expectation under the distribution $\hat{H}_n$. Let V denote the correlation matrix of the random variables $W_\nu = f_\nu(X_1)$. By (3.3) and an argument similar to that used in (4.30), $\lambda_{\min}(\hat{V}) = \lambda_{\min}(V) + O_p(n^{-1/2})$, where $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue of a symmetric matrix. Ordering the absolute values of the eigenvalues $\hat{\lambda}_i$ of $\hat{L}$ as $|\hat{\lambda}_1| \geq |\hat{\lambda}_2| \geq \cdots$, it then follows that $|\hat{\lambda}_K|^2 \geq K^{-1}\{\lambda_{\min}(V) + O_p(n^{-1/2})\}$, (cf. (4.6) of BGZ). Moreover, $E(|\hat{\beta}_n(X_1^*, X_2^*)|^r|\hat{H}_n) = O_p(1)$ by (3.3) and Condition (C). Hence we can use the arguments of BGZ to complete the proof, after some modifications similar to those introduced in the proof of (4.6) under Condition (C).

Finally suppose that Condition (D) holds. For $d > 0$, let

$$\Delta_{n,d} = \Big\{ \sup_{n^{(r-1)/r}(\log n)^{-1} \leq |t| \leq dn} |\hat{f}_n(t)| \leq n^{-1}(\log n)^{-4} \Big\}.$$

It will be shown that given any $\epsilon > 0$, there exists $d_\epsilon > 0$ such that

$$P(\Delta_{n,d_\epsilon}) \geq 1 - \epsilon \quad \text{for all large } n. \tag{4.33}$$

By (4.1), on the event Δ_{n,d_ϵ},

$$\begin{aligned} \sup_z |\hat{F}_n(z) - G_n(z)| \leq \pi^{-1} \int_{|t| \leq n^{(r-1)/r}(\log n)^{-1}} |t|^{-1} |\hat{f}_n(t) - g_n(t)| dt \\ + n^{-1}(\log n)^{-3} + 8(d_\epsilon n)^{-1} \sup_z |G_n'(z)| \end{aligned} \tag{4.34}$$

for all large n. Since the first term on the right hand side of (4.34) has been shown to be $O_p(n^{-1})$, (4.33) and (4.34) imply the desired conclusion $\sup_z n|\hat{F}_n(z) - G_n(z)| = O_p(1)$.

In view of Condition (D), there exists $0 < \eta < 1$ such that

$$\sup_{|s_1|+\cdots+|s_K| \leq |\tau|^{-\delta}} |\Psi(\tau, \tau s_1, \ldots, \tau s_K)| \leq \eta \quad \text{for all large } |\tau|, \tag{4.35}$$

where $\Psi(\tau, u_1, \ldots, u_K) = \int \exp\{\mathrm{i}\tau\alpha(x) + \mathrm{i}\sum_{\nu=1}^K u_\nu g_\nu(x)\} dH(x)$ is the characteristic function of $(\alpha(X), g_1(X), \ldots, g_K(X))$. The empirical characteristic function $\hat{\Psi}(\tau, u_1, \ldots, u_K) = \int \exp\{\mathrm{i}\tau\alpha(x) + \mathrm{i}\sum_{\nu=1}^K u_\nu g_\nu(x)\} d\hat{H}_n(x)$ satisfies

$$\sup_{\tau^2+u_1^2+\cdots+u_K^2 \leq n^a} \Big| \hat{\Psi}(\tau, u_1, \ldots, u_K) - \Psi(\tau, u_1, \ldots, u_K) \Big| \to 0 \text{ a.s.} \tag{4.36}$$

for any $a > 0$, (cf. Abramovitch and Singh (1985)). Let

$$\hat{\phi}_n(\tau, y_{m-2}, \ldots, y_n) = \int \exp\left(\mathrm{i}\tau \Big\{ \hat{\alpha}_n(x) + n^{-1}\hat{A}_n(x) + n^{-1} \sum_{j=m-2}^n \hat{\beta}_n(x, y_j) \Big\} \right) d\hat{H}_n(x).$$

By (3.3) and (3.1),

$$\sup_{x\in S(H),y_1\in S(H),\ldots,y_n\in S(H)} n^{-1}\sum_{j=1}^{n}|\hat{\beta}_n(x,y_j)-\beta(x,y_j)|$$
$$+\int|\hat{\alpha}_n(x)-\alpha(x)|d\hat{H}_n(x)+n^{-1}\int|\hat{A}_n(x)|d\hat{H}_n(x)=O_p(n^{-1/2}).$$

Since $n^{-1}\sum_{j=m-2}^{n}\beta(x,y_j)=\sum_{\nu=1}^{K}g_\nu(x)(c_\nu n^{-1}\sum_{j=m-2}^{n}g_\nu(y_j))$, it then follows that there exists for any $\epsilon>0$ sufficiently small $d_\epsilon>0$ such that for all large n,

$$P\left\{\sup_{|\tau|\le d_\epsilon\sqrt{n}/\sigma,y_{m-2}\in S(H),\ldots,y_n\in S(H)}\left|\hat{\phi}_n(\tau,y_{m-2},\ldots,y_n)\right.\right.$$
$$\left.\left.-\hat{\Psi}\left(\tau,\tau c_1 n^{-1}\sum_{j=m-2}^{n}g_1(y_j),\ldots,\tau c_K n^{-1}\sum_{j=m-2}^{n}g_K(y_j)\right)\right|\le(1-\eta)/3\right\}\ge 1-\epsilon/3. \quad (4.37)$$

Combining (4.37) with (4.35) and (4.36) yields

$$P\left\{\sup_{|t|\le d_\epsilon n,(y_{m-2},\ldots,y_n)\in S_{n,t}}|\hat{\phi}_n(t/\sigma\sqrt{n},y_{m-2},\ldots,y_n)|\le\eta+(1-\eta)/2\right\}\ge 1-\epsilon/2 \quad (4.38)$$

for all large n, where $S_{n,t}=\{(y_{m-2},\ldots,y_n):y_j\in S(H),\sum_{\nu=1}^{K}|c_\nu n^{-1}\sum_{j=m-2}^{n}g_\nu(y_j)|\le(t/\sigma\sqrt{n})^{-\delta}\}$. To obtain (4.33) from (4.38), we can proceed as in the proof of (4.6) under Condition (D), replacing the ϕ_n defined in (4.20) by $\hat{\phi}_n(t/\sigma\sqrt{n},X^*_{m-2},\ldots,X^*_n)$.

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Department of Statistics, Stanford University, Stanford, CA 94305, U.S.A.
Division of Biostatistics, School of Public Health, University of Minnesota, Minneapolis, MN 55455, U.S.A.

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