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ON THE KADANE-DUNCAN CONJECTURE

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Abstract: Let X_1 and X_2 be two independent, nondegenerate and symmetric random variables with centers of symmetry μ_1 and μ_2 respectively. This note proves the Kadane-Duncan conjecture that X_1X_2 is symmetric if and only if $\mu_1\mu_2 = 0$.

Key words and phrases: Independent random variables, symmetric random variables, nondegenerate random variables, center of symmetry.

Let X_1 and X_2 be two independent and unskewed random variables (i.e. $E|X_i|^3 < \infty$ and $E(X_i - EX_i)^3 = 0$, $i = 1,2$). Kadane and Duncan (1980) proved that $W = X_1X_2$ is unskewed if and only if at least one of X_1 and X_2 is degenerate or at least one of $E(X_i)$ $(i = 1,2)$ is zero. They proposed the following conjecture:

Theorem (Kadane-Duncan). *Suppose X_1 and X_2 are independent and symmetric with centers of symmetry μ_1 and μ_2 respectively. Then $W = X_1X_2$ is symmetric if and only if either*
(i) *at least one of X_1 and X_2 is degenerate, or*
(ii) *at least one of μ_1 and μ_2 is zero.*

Chen and Slud (1984) and Hamedani and Walter (1985) proved independently that X_1X_2 is symmetric about $\mu_1\mu_2$ if and only if $\mu_1\mu_2 = 0$ under the condition that X_1 and X_2 are nondegenerate. This statement differs from the original Kadane-Duncan Conjecture by the additional condition that X_1X_2 is symmetric about $\mu_1\mu_2$. Chen and Slud (1984) note that one cannot assume that

$$(X_1 - \mu_1)(X_2 - \mu_2) + (X_1 - \mu_1) + (X_2 - \mu_2)$$

has to be symmetric about 0 based on the facts that $(X_1-\mu_1)(X_2-\mu_2)$, $(X_1-\mu_1)$ and $(X_2 - \mu_2)$ are all symmetric about 0, because Chen and Shepp (1983) find two random variables W and Z such that W and Z are symmetric about a and b, respectively, and $W + Z$ is symmetric about c, but $a + b \neq c$. Therefore, the proofs of the Kadane-Duncan Conjecture mentioned above are not complete.

In this note, we give a proof of the "necessity" part of the theorem, while the "sufficiency" part is obvious.

Frist we introduce functions f_a, $a \geq 0$ and g_b, $b > 0$ which are vital in our proof. The function f_a is defined on $R \times R$ by

$$\begin{aligned} f_a(u,v) &= \frac{u+v+uv-a}{1+|u+v+uv-a|} + \frac{u-v-uv-a}{1+|u-v-uv-a|} \\ &+ \frac{-u+v-uv-a}{1+|-u+v-uv-a|} + \frac{-u-v+uv-a}{1+|-u-v+uv-a|}, \quad u,v \in R. \end{aligned}$$

The function g_b is defined on $R \times R \times [0,+\infty)$ by

$$\begin{aligned} g_b(u,v,t) &= \frac{u+v+uv}{1+t|u+v+uv+b|} + \frac{u-v-uv}{1+t|u-v-uv+b|} \\ &+ \frac{-u+v-uv}{1+t|-u+v-uv+b|} + \frac{-u-v+uv}{1+t|-u-v+uv+b|}, \quad u,v \in R,\ t > 0. \end{aligned}$$

Note that $f_a(|u|,|v|) = f_a(u,v) = f_a(v,u)$, and $g_b(|u|,|v|,t) = g_b(u,v,t) = g_b(v,u,t)$. Moreover, they have the following properties.

Lemma 1. *For each $a \geq 0$, $f_a(u,v) < 0$, $f_a(0,v) \leq 0$, and $f_a(u,0) \leq 0$ for all $u \neq 0$ and $v \neq 0$.*

Lemma 2. *For each $b > 0$, $|g_b|$ is bounded.*

The proofs of Lemma 1 and Lemma 2 are given in the Appendix.

Proof of The Theorem. The "sufficiency" part is obvious.

We need only to show the necessity part of the theorem. For this, let $U = X_1 - \mu_1$, $V = X_2 - \mu_2$. Denote the cumulative distribution functions of U and V by F_1 and F_2, respectively. Note that U and V are symmetric about 0.

Suppose $\mu_1\mu_2 \neq 0$. Without loss of generality we assume $\mu_1 = \mu_2 = 1$. Then we can write

$$X_1X_2 = U + V + UV + 1.$$

Let $Y = U + V + UV$.

Suppose that neither F_1 nor F_2 is degenerate and suppose that Y is symmetric with the center of symmetry a. Then $Y - a$ is symmetric about 0; and so is

$$\frac{Y-a}{1+t|Y-a|} \quad \text{for every } t > 0.$$

Consequently,

$$E\left(\frac{Y-a}{1+t|Y-a|}\right) = 0 \text{ for all } t > 0. \tag{1}$$

Case 1. Suppose $a \geq 0$. Set $t = 1$. Then.

$$\begin{aligned} E\left(\frac{Y-a}{1+|Y-a|}\right) &= \iint_{u,v>0} f_a(u,v)dF_1(u)dF_2(v) \\ &\quad + \frac{1}{2}P(U=0)\int_{v>0} f_a(0,v)dF_2(v) \\ &\quad + \frac{1}{2}P(V=0)\int_{u>0} f_a(u,0)dF_1(u) \\ &\quad - P(U=0)P(V=0)\frac{a}{1+a}, \end{aligned} \tag{2}$$

where the function f_a is as in Lemma 1. Since we have assumed that neither F_1 nor F_2 is degenerate, by Lemma 1, we obtain

$$E\left(\frac{Y-a}{1+|Y-a|}\right) < 0, \tag{3}$$

a contradiction to (1).

Case 2. Suppose $a < 0$. Let $b = -a$ then $b > 0$. Denote the indicator function of set A by I_A. For g_b as in Lemma 2, we have

$$\begin{aligned} E\left(\frac{Y-a}{1+t|Y-a|}\right) &= E\left(\frac{Y}{1+t|Y+b|}\cdot((I_{[U>0,V>0]}+I_{[U>0,V<0]}+I_{[U<0,V>0]} \right. \\ &\quad + I_{[U<0,V<0]}) + (I_{[U=0,V>0]}+I_{[U=0,V<0]}) + (I_{[U>0,V=0]} \\ &\quad \left. + I_{[U<0,V=0]}) + I_{[U=0,V=0]})\right) - aE\left(\frac{1}{1+t|Y+b|}\right) \\ &= \iint_{u,v>0} g_b(u,v,t)dF_1(u)dF_2(v) \\ &\quad + \frac{1}{2}P(U=0)\int_{v>0} g_b(0,v,t)dF_2(v) \\ &\quad + \frac{1}{2}P(V=0)\int_{u>0} g_b(u,0,t)dF_1(u) \\ &\quad - aE\left(\frac{1}{1+t|Y+b|}\right) \\ &\triangleq H_1(t) + \frac{1}{2}P(U=0)H_2(t) \\ &\quad + \frac{1}{2}P(V=0)H_3(t) - aH_4(t), \end{aligned} \tag{4}$$

say. By Lemma 2, g_b is bounded. Since $\lim_{t\to 0^+} g_b(u,v,t) = 0$ for all u and v, an application of the dominated convergence theorem yields

$$\lim_{t\to 0^+} H_1(t) = \iint_{u,v>0} \lim_{t\to 0^+} g_b(u,v,t)dF_1(u)dF_2(v) = 0,$$

$$\lim_{t\to 0^+} H_2(t) = \int_{v>0} \lim_{t\to 0^+} g_b(0,v,t)dF_2(v) = 0,$$

and

$$\lim_{t\to 0^+} H_3(t) = \int_{u>0} \lim_{t\to 0^+} g_b(u,0,t)dF_1(u) = 0.$$

Applying the dominated convergence theorem to the bounded random variable $\frac{1}{1+t|Y+b|}$, we have

$$\lim_{t\to 0^+} H_4(t) = E\left(\lim_{t\to 0^+} \frac{1}{1+t|Y+b|}\right) = 1.$$

Then, taking limits on both sides of (4), we have

$$\begin{aligned}\lim_{t\to 0^+} E\left(\frac{Y-a}{1+t|Y-a|}\right) &= \lim_{t\to 0^+} H_1(t) + \frac{1}{2}P(U=0)\cdot \lim_{t\to 0^+} H_2(t)\\ &\quad +\frac{1}{2}P(V=0)\cdot \lim_{t\to 0^+} H_3(t) - a\cdot \lim_{t\to 0^+} H_4(t)\\ &= -a > 0. \qquad (5)\end{aligned}$$

This contradicts (1).

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Appendix: Proofs of the Lemmas

Proof of Lemma 1. Fix $a \geq 0$. We only show that $f_a(u,v) < 0$ for $u, v \geq 0$ here.

When $u+v+uv-a \leq 0$, obviously, $f_a(u,v) < 0$.

When $u+v+uv-a > 0$, we consider eight cases classified by the signs of $u-v-uv-a$, $-u+v-uv-a$ and $-u-v+uv-a$. Because at most one of $u-v-uv-a$, $-u+v-uv-a$ and $-u-v+uv-a$ can be nonnegative, we need only consider the following four cases.

(i) If

$$\begin{cases} u+v+uv-a > 0\\ u-v-uv-a < 0\\ -u+v-uv-a < 0\\ -u-v+uv-a \geq 0,\end{cases}$$

then

$$\begin{aligned} f_a(u,v) &\le \frac{2u}{(1+u+v+uv-a)(1+|u-v-uv-a|)} \\ &\quad + \frac{-2u}{(1+|-u+v-uv-a|)(1+|-u-v+uv-a|)} \\ &\le -2u(4v(1+uv)+4au)/\{(1+u+v+uv-a)(1+|u-v-uv-a|) \\ &\quad \cdot (1+|-u+v-uv-a|)(1+|-u-v+uv-a|)\} \\ &< 0. \end{aligned}$$

(ii) If

$$\begin{cases} u+v+uv-a>0 \\ u-v-uv-a<0 \\ -u+v-uv-a\ge 0 \\ -u-v+uv-a<0, \end{cases}$$

then $v>u$, so

$$\begin{aligned} f_a(u,v) &\le \frac{2(u-a)}{(1+u+v+uv-a)(1+|u-v-uv-a|)} \\ &\quad + \frac{-2(u+a)}{(1+|-u+v-uv-a|)(1+|-u-v+uv-a|)} \\ &\le -2u(4uv(1+v)+4au)/\{(1+u+v+uv-a)(1+|u-v-uv-a|) \\ &\quad \cdot (1+|-u+v-uv-a|)(1+|-u-v+uv-a|)\} \\ &< 0. \end{aligned}$$

(iii) If

$$\begin{cases} u+v+uv-a>0 \\ u-v-uv-a\ge 0 \\ -u+v-uv-a<0 \\ -u-v+uv-a<0, \end{cases}$$

then, similar to case (ii), $f_a(u,v)<0$.

(iv) If

$$\begin{cases} u+v+uv-a>0 \\ u-v-uv-a<0 \\ -u+v-uv-a<0 \\ -u-v+uv-a<0, \end{cases}$$

we consider the following three subcases.

–*Subcase 1.* Suppose $a > u$. Then

$$\begin{aligned} f_a(u,v) &\le \frac{u+v+uv-a}{1+u+v+uv-a} + \frac{u-v-uv-a}{1-(u-v-uv-a)} \\ &= \frac{-2(a-u)}{(1+v+uv)^2-(u-a)^2} \\ &< 0. \end{aligned}$$

Subcase 2. Suppose $a > uv$. Then

$$\begin{aligned} f_a(u,v) &\le \frac{u+v+uv-a}{1+u+v+uv-a} + \frac{-u-v+uv-a}{1-(-u-v+uv-a)} \\ &= \frac{-2(a-uv)}{(1+(u+v+uv-a))(1-(-u-v+uv-a))} \\ &< 0. \end{aligned}$$

Subcase 3. Suppose $a \le u$ and $a \le uv$. Since $f_a(u,v)$ is a symmetric function of u and v, we assume $u \le v$. Then

$$\begin{aligned} f_a(u,v) &= \frac{-2(a-u)}{(1+v+uv)^2-(u-a)^2} \\ &\quad + \frac{2(-u-a)-2(-u+v-uv-a)(-u-v+uv-a)}{(1+u+a)^2-v^2(1-u)^2} \\ &< \frac{2u}{(1+v+uv)^2-(u-a)^2} + \frac{-2u}{(1+u+a)^2-v^2(1-u)^2} \\ &= -4u\frac{(v^2-u^2)+(u^2v^2-a^2)+(v-u)+(uv-a)}{((1+v+uv)^2-(u-a)^2)((1+u+a)^2-v^2(1-u)^2)} \\ &\le 0. \end{aligned}$$

Therefore $f_a(u,v) < 0$ for $u, v > 0$.

Proof of Lemma 2. Let $b > 0$ be fixed. It is easily seen that $|g_b(\cdot,\cdot,t)| < 4/\min(1,\frac{1}{b})$ for $t \ge \min(1,\frac{1}{b})$. Since

$$g_b(|u|,|v|,t) = g_b(u,v,t) = g_b(v,u,t),$$

we need only show that $|g_b(u,v,t)|$ is bounded for $0 \le u \le v$ and for $0 < t < \min(1,\frac{1}{b})$.

For $u = 0$, and $0 \le v$,

$$|g_b(0,v,t)| = 2\left|\frac{v}{1+t(v+b)} + \frac{-v}{1+t|-v+b|}\right|$$

$$
\begin{aligned}
&= 2v\frac{t((v+b)-|v-b|)}{(1+t(v+b))(1+t|v-b|)} \\
&= \begin{cases} 2vt\dfrac{2b}{(1+t(v+b))(1+t|v-b|)} & \text{if } v \geq b \\ 2vt\dfrac{2v}{(1+t(v+b))(1+t|v-b|)} & \text{if } v < b \end{cases} \\
&\leq 4b.
\end{aligned}
$$

It remains to prove that $|g_b(u,v,t)|$ is bounded for $0 < u \leq v$, and for $0 < t < \min(1, \frac{1}{b})$. We consider the following eight cases classified by the signs of $u - v - uv + b$, $-u + v - uv + b$, and $-u - v + uv + b$.

(i) If

$$
\begin{cases} u - v - uv + b \geq 0 \\ -u + v - uv + b \geq 0 \\ -u - v + uv + b \geq 0, \end{cases}
$$

then, by adding the three inequalities together, we have $u + v + uv \leq 3b$. Thus

$$|g_b(u,v,t)| \leq 4(u+v+uv) \leq 12b.$$

(ii) If

$$
\begin{cases} u - v - uv + b \geq 0 \\ -u + v - uv + b \geq 0 \\ -u - v + uv + b < 0, \end{cases}
$$

then, by subtracting the third inequality from the first one, we have $u - uv > 0$, so $1 > v(\geq u > 0)$. Thus

$$|g_b(u,v,t)| \leq 4(u+v+uv) \leq 12.$$

(iii) If

$$
\begin{cases} u - v - uv + b \geq 0 \\ -u + v - uv + b < 0 \\ -u - v + uv + b \geq 0, \end{cases}
$$

then, by adding the first and the third inequalities together, we have $-v + b \geq 0$, so $b \geq v(\geq u > 0)$. Thus

$$|g_b(u,v,t)| \leq 4(u+v+uv) \leq 4(2b+b^2).$$

(iv) If

$$
\begin{cases} u - v - uv + b < 0 \\ -u + v - uv + b \geq 0 \\ -u - v + uv + b \geq 0, \end{cases}
$$

then, by adding the second and the third inequalities, we have $b \geq u(\geq 0)$. The second inequality implies $v - uv \geq -(b-u)$, and the third one implies $v - uv \leq b - u$, together we have $|v - uv| \leq b - u \leq b$. Thus

$$\begin{aligned}
|g_b(u,v,t)| &\leq \left|\frac{u+v+uv}{1+t(u+v+uv+b)} + \frac{u-v-uv}{1-t(u-v-uv+b)}\right| \\
&\quad +|u| + |v-uv| + |u| + |v-uv| \\
&\leq \frac{|2u + 2bt(-v-uv)|}{(1+t(u+v+uv+b))(1+t|u-v-uv+b|)} + 4b \\
&\leq 2u + \frac{2bt(v+uv)}{t(v+uv)} + 4b \\
&\leq 8b.
\end{aligned}$$

(v) If

$$\begin{cases} u - v - uv + b \geq 0 \\ -u + v - uv + b < 0 \\ -u - v + uv + b < 0, \end{cases}$$

then, by subtracting the second inequality from the first one, we have $u - v > 0$, impossible as assumed $0 < u \leq v$.

(vi) If

$$\begin{cases} u - v - uv + b < 0 \\ -u + v - uv + b \geq 0 \\ -u - v + uv + b < 0, \end{cases}$$

then, by subtracting the third inequality from the second one, we have $v - uv > 0$, and therefore $1 > u(> 0)$. Thus

$$\begin{aligned}
|g_b(u,v,t)| &\leq \left|\frac{u+v+uv}{1+t(u+v+uv+b)} + \frac{u-v-uv}{1-t(u-v-uv+b)}\right| \\
&\quad + \left|\frac{-u+v-uv}{1+t(-u+v-uv+b)} + \frac{-u-v+uv}{1-t(-u-v+uv+b)}\right| \\
&= \left|\frac{2u + 2bt(-v-uv)}{(1+t(u+v+uv+b))(1+t|u-v-uv+b|)}\right| \\
&\quad + \left|\frac{-2u + 2bt(-v+uv)}{(1+t|-u+v-uv+b|)(1-t(-u-v+uv+b))}\right| \\
&\leq 2u + \frac{2bt(v+uv)}{1+t(u+v+uv+b)} + 2u + \frac{2bt(v-uv)}{1-t(-u-v+uv+b)} \\
&\leq 2 + 2b + 2 + 2b\frac{t(v-uv)}{(1-tb)+tu+t(v-uv)}
\end{aligned}$$

$$
\begin{aligned}
&\leq 2+2b+2+2b\frac{t(v-uv)}{t(v-uv)} \\
&= 4+4b,
\end{aligned}
$$

as $0<u<1$ and $(1-tb)+tu>0$ in view of $0<t<\dfrac{1}{b}$.

(vii) If

$$
\begin{cases}
u-v-uv+b<0 \\
-u+v-uv+b<0 \\
-u-v+uv+b\geq 0,
\end{cases}
$$

then, by subtracting the second inequality from the third one, we have $uv-v>0$. Note also the assumptions that $0<t<\min(1,\frac{1}{b})$, and that $0<u\leq v$, we have

$$
\begin{aligned}
|g_b(u,v,t)| &= \left|\frac{2u+2bt(-v-uv)}{(1+t(u+v+uv+b))(1-t(u-v-uv+b))}\right. \\
&\quad \left.+\frac{-2u+2bt(v-uv)}{(1-t(-u+v-uv+b))(1+t(-u-v+uv+b))}\right| \\
&\leq 2u\left|\frac{1}{(1+t(u+v+uv+b))(1-t(u-v-uv+b))}\right. \\
&\quad \left.-\frac{1}{(1-t(-u+v-uv+b))(1+t(-u-v+uv+b))}\right| \\
&\quad +2b+\frac{2bt(uv-v)}{(1-t(-u+v-uv+b))(1+t|-u-v+uv+b|)} \\
&\leq 2u\left|\frac{(1+tv+tuv)^2-t^2(u+b)^2-(1-tv+tuv)^2+t^2(u-b)^2}{(1+t(u+v+uv+b))(1-t(u-v-uv+b))}\right| \\
&\quad +2b+2b\frac{t(uv-v)}{((1-bt)+tu+t(-v+uv))(1+t|-u-v+uv+b|)} \\
&\leq 2u\frac{|4tv(1+tuv)-4t^2ub|}{(1+t(u+v+uv+b))((1-tb)+t(-u+v)+tuv)}+4b \\
&\leq \frac{8tuv(1+tuv)}{(1+t(u+v+uv+b))tuv}+\frac{8t^2bu^2}{tuv}+4b \\
&\leq 8+8bt+4b \\
&\leq 8+12b.
\end{aligned}
$$

(viii) If

$$
\begin{cases}
u-v-uv+b<0 \\
-u+v-uv+b<0 \\
-u-v+uv+b<0,
\end{cases}
$$

then, by adding the second and the third inequalities, we have $b < u$ and from the second and the third inequalities, we have $|v - uv| < u - b \leq u$.

Subcases:

(a) If $2 \geq v(\geq u \geq 0)$, then

$$|g_b(u,v,t)| \leq 4(u + v + uv) \leq 32.$$

(b) If $v > 2$, then $-u - v + uv + b < 0$ implies $0 < u < \frac{1}{v-1} + 1 < 2$. Thus

$$\begin{aligned} |g_b(u,v,t)| &\leq \left| \frac{2u + 2bt(-v - uv)}{(1 + t(u + v + uv + b))(1 + t|u - v - uv + b|)} \right| \\ &\quad +(u + |v - uv|) + (u + |v - uv|) \\ &\leq \frac{2u + 2bt(v + uv)}{1 + t(u + v + uv + b)} + (2 + 2) + (2 + 2) \\ &< 12 + 2b. \end{aligned}$$

Therefore Lemma 2 holds.

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