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EFFECT OF MEASUREMENT ERROR ON MONITORING MULTIVARIATE PROCESS VARIABILITY

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Abstract: The effect of measurement error on the performance of two control chart schemes, derived from the sample generalized variance and an unbiased likelihood ratio test for monitoring multivariate process variability, is investigated. It is demonstrated that the performance of the sample generalized variance chart based on surrogate quality characterisics is not proportionally invariant to changes in the covariance matrix of the true quality characteristics, even though it is proportionally invariant when no measurement error exists. Further, it is shown that for the sample generalized variance chart, the power to detect a change in the covariance matrix under measurement error could be larger than that under no measurement error. On the other hand, the control chart obtained from the unbiased likelihood ratio test does not seem to have this undesirable property. For this chart, under certain assumptions, measurement error results in reduced power to detect changes in the covariance matrix of the true quality characteristics. Bivariate examples of both charts are considered to demonstrate the possible scenarios.

Key words and phrases: Control limits, likelihood ratio test chart, measurement error, multivariate process variability, proportionally invariant, sample generalized variance chart.

1. Introduction

Many applications in industrial quality control concern the monitoring of several correlated quality characteristics. Suppose that we can observe a random sample of $p\times 1$ vectors $X_1, \ldots, X_n$, $n \geq p$, from a process at regular time intervals. Here each of the p elements in the vector X represents an observation of one of p quality characteristics being monitored. From now on we assume that the $X_i, i = 1, \ldots, n$, follow a $p-$dimensional normal random vector with mean vector μ_x and positive definite covariance matrix Σ_x.

In practice, the true quality characteristics that we are interested in monitoring are not easy to observe, but we instead observe surrogates of them which are the true quality characteristics plus measurement errors. In the following discussion, we assume the measurement error model

$$Y_i = X_i + \varepsilon_i, \qquad i = 1, \ldots, n, \tag{1}$$

where ε_i are $p \times 1$ normal random vectors, independent of the X_i, with mean vector $\mathbf{0}$ and known covariance matrix Σ_ε. Here Σ_ε is assumed to be a positive semidefinite matrix, i.e., some elements of X_i are observed exactly and the rest are not. It is also assumed that Σ_ε does not change during the monitoring period. The Y_i, usually obtained from some physical measurements, are surrogate quality characteristics which are independent and identically distributed normal random vectors with mean vector μ_x and covariance matrix $\Sigma_x + \Sigma_\varepsilon$. When $p = 1$, Kanazuka (1986) used the model at (1) to study the effect of measurement error on the performance of an $\bar{X} - R$ chart. Mittag (1995) and Mittag and Stemann (1998) investigated how the measurement error affects the $\bar{X} - S$ chart under (1). Linna and Woodall (2001) assumed a linear relationship between the surrogate and the true quality characteristics to study the effect of measurement error on the performance of $\bar{X}$ and S^2 charts. As for $p \geq 2$, Linna, Woodall and Busby (2001) considered a multivariate linear relationship between the surrogate and true quality characteristics to show that the usual χ^2 chart is not directionally invariant to shifts in the mean vector of the true quality characteristics.

In this paper, we consider two control charts derived, respectively, from the sample generalized variance and an unbiased likelihood ratio test for monitoring multivariate process variability under model (1). The sole purpose is to study the effect of measurement error on the performance of these two charts in detecting shifts in the covariance matrix of the true quality characteristics. It parallels the paper of Linna, Woodall and Busby (2001), who investigated the performance of multivariate control charts for monitoring the process mean of the true quality characteristics in the presence of measurement error. As for the comparison of the signal detecting capability of the sample generalized variance and the unbiased likelihood ratio test charts for monitoring process variability when no measurement error exists, one may refer to Chan and Zhang (2001). The unbiased likelihood ratio test chart usually gives better results, especially when the determinants of the in-control and out-of-control covariance matrices are not appreciably different.

It is shown that the sample generalized variance chart based on surrogate quality characteristics does not possess the property of proportional invariance with respect to the true quality characteristics. Namely, the power of the sample generalized variance chart under measurement error is not a function of the ratio of out-of-control to in-control generalized variances of the true quality characteristics, even though it is when no measurement error exists. Further, for a given value of the ratio of out-of-control to in-control generalized variances of the true quality characteristics, the power to dectect shifts in the covariance matrix of the true quality characteristics in the presence of measurement error could be larger than that when no measurement error exists. This counter-intuitive phenomenon contradicts the perception that measurement error diminishes the power of control methods to detect process shifts, and indicates that the sample generalized variance chart is not an appealing control chart scheme for monitoring multivariate process variability in the presence of measurement error. As for the control chart derived from the unbiased likelihood ratio test, measurement error makes the power to detect shifts in the covariance matrix of the true quality characteristics harder to calculate. When the out-of-control covariance matrix of the true quality characteristics is proportional to its in-control covariance matrix, or the measurement error covariance matrix is proportional to the in-control covariance matrix of the true quality characteristics, it is shown that the power in the presence of measurement error is always less than or equal to that when no measurement error exists. Two bivariate examples of both control charts considered in this paper are presented as illustrations.

2. The Sample Generalized Variance Chart

The sample generalized variance chart is a commonly used control chart for monitoring multivariate process variability. Under the measurement error model (1), the sample generalized variance is defined by $|S_y|$, where $|S_y|$ is the determinant of $S_y = \sum_{i=1}^{n}(Y_i - \bar{Y})(Y_i - \bar{Y})'/(n-1)$. When the process is in control, it is assumed that the covariance matrix of X_i, Σ_x, is Σ_0, known. Thus, after an appropriate transformation, we can take $\Sigma_0 = I_p$, a $p \times p$ identity matrix.

For $p = 2$, if the covariance matrix of X_i does not change, $2(n-1)(|S_y|/|\Sigma_0 + \Sigma_\varepsilon|)^{1/2}$ has a chi-square distribution with $2(n-2)$ degrees of freedom (see, for example, Anderson (1984, p.264). Consequently, the control limits for the $|S_y|^{1/2}$ chart are

$$\mathrm{UCL} = \frac{|\Sigma_0 + \Sigma_\varepsilon|^{\frac{1}{2}}}{2(n-1)}\chi^2_{2(n-2),\frac{\alpha}{2}} \text{ and } \mathrm{LCL} = \frac{|\Sigma_0 + \Sigma_\varepsilon|^{\frac{1}{2}}}{2(n-1)}\chi^2_{2(n-2),1-\frac{\alpha}{2}}, \tag{2}$$

where $\chi^2_{\gamma,\nu}$ represents the upper ν quantile of a χ^2_γ distribution. An out-of-control signal is generated on this chart if the computed value of $|S_y|^{\frac{1}{2}}$ falls outside the interval defined by the UCL and LCL. If the covariance matrix of X_i changes from $\Sigma_x = \Sigma_0$ to $\Sigma_x = \Sigma_1$, then the probability of a signal on this chart is

$$\begin{aligned} &P_r\left(|S_y|^{\frac{1}{2}} < \frac{|\Sigma_0+\Sigma_\varepsilon|^{\frac{1}{2}}}{2(n-1)}\chi^2_{2(n-2),1-\frac{\alpha}{2}} \text{ or } |S_y|^{\frac{1}{2}} > \frac{|\Sigma_0+\Sigma_\varepsilon|^{\frac{1}{2}}}{2(n-1)}\chi^2_{2(n-2),\frac{\alpha}{2}} |\Sigma_x = \Sigma_1\right) \\ &= 1 - P_r\left(\chi^2_{2(n-2)} < \Big(\frac{|\Sigma_1+\Sigma_\varepsilon|}{|\Sigma_0+\Sigma_\varepsilon|}\Big)^{-\frac{1}{2}}\chi^2_{2(n-2),\frac{\alpha}{2}}\right) \\ &\quad + P_r\left(\chi^2_{2(n-2)} < (\frac{|\Sigma_1+\Sigma_\varepsilon|}{|\Sigma_0+\Sigma_\varepsilon|})^{-\frac{1}{2}}\chi^2_{2(n-2),1-\frac{\alpha}{2}}\right). \end{aligned} \tag{3}$$

For $p \geq 3$, if the covariance matrix of X_i does not change, the determinant $|S_y|$ is distributed as $|\Sigma_0+\Sigma_\varepsilon|(\prod_{i=1}^p \chi^2_{n-i})/(n-1)^p$, where the p chi-square distributions in the product are independent. As a result, the usual control limits for the $|S_y|$ chart are

$$\mathrm{UCL}=E_0(|S_y|)+z_{\frac{\alpha}{2}}\sqrt{\mathrm{Var}_0(|S_y|)} \text{ and } \mathrm{LCL}=E_0(|S_y|)-z_{1-\frac{\alpha}{2}}\sqrt{\mathrm{Var}_0(|S_y|)}, \quad (4)$$

where z_ν represents the upper ν quantile of a standard normal distribution and $E_0(|S_y|)$ and $Var_0(|S_y|)$ are, respectively, the expectation and variance of $|S_y|$ when $\Sigma_x = \Sigma_0$,

$$E_0(|S_y|) = \frac{|\Sigma_0+\Sigma_\varepsilon|\prod_{i=1}^p(n-i)}{(n-1)^p},$$
$$\mathrm{Var}_0(|S_y|) = \frac{|\Sigma_0+\Sigma_\varepsilon|^2[\prod_{i=1}^p(n-i+2)-\prod_{i=1}^p(n-i)]\prod_{i=1}^p(n-i)}{(n-1)^{2p}}.$$

Basically, the control limits in (4) are obtained by treating $|S_y|$ as a normal random variable. Thus, the value of $z_{\frac{\alpha}{2}}$ is usually taken to be 3 to adopt a six-sigma length interval. If the covariance matrix of X_i changes from $\Sigma_x = \Sigma_0$ to $\Sigma_x = \Sigma_1$, the power of the control chart (4) is

$$\begin{aligned}
&P_r\Big(|S_y| < E_0(|S_y|) - z_{1-\frac{\alpha}{2}}\sqrt{\mathrm{Var}_0(|S_y|)} \\
&\qquad \text{or } |S_y| > E_0(|S_y|) + z_{\frac{\alpha}{2}}\sqrt{\mathrm{Var}_0(|S_y|)}|\Sigma_x = \Sigma_1\Big) \\
&= 1-P_r\Bigg(\prod_{i=1}^p \chi^2_{n-i} < \Big(\frac{|\Sigma_1+\Sigma_\varepsilon|}{|\Sigma_0+\Sigma_\varepsilon|}\Big)^{-1}\Big\{\prod_{i=1}^p(n-i) \\
&\qquad\qquad + z_{\frac{\alpha}{2}}\sqrt{\Big[\prod_{i=1}^p(n-i+2)-\prod_{i=1}^p(n-i)\Big]\prod_{i=1}^p(n-i)}\Big\}\Bigg) \\
&\quad + P_r\Bigg(\prod_{i=1}^p \chi^2_{n-i} < \Big(\frac{|\Sigma_1+\Sigma_\varepsilon|}{|\Sigma_0+\Sigma_\varepsilon|}\Big)^{-1}\Big\{\prod_{i=1}^p(n-i) \\
&\qquad\qquad - z_{1-\frac{\alpha}{2}}\sqrt{\Big[\prod_{i=1}^p(n-i+2)-\prod_{i=1}^p(n-i)\Big]\prod_{i=1}^p(n-i)}\Big\}\Bigg). \quad (5)
\end{aligned}$$

Note that the signal probabilities in (3) and (5) only depend on, and are increasing functions of the ratio of, the generalized variances $|\Sigma_1+\Sigma_\varepsilon|/|\Sigma_0+\Sigma_\varepsilon|$. For convenience, we define

$$r_\varepsilon = \frac{|\Sigma_1+\Sigma_\varepsilon|}{|\Sigma_0+\Sigma_\varepsilon|}. \quad (6)$$

If there is no measurement error at (1), the ratio r_ε reduces to $r_0 = |\Sigma_1|/|\Sigma_0|$, the ratio of the out-of-control to in-control generalized variances of the true quality characteristics. For a given value of r_0, the probability of a signal on the control chart (2) or (4) is a constant in the absence of measurement error. In other words, the power of the control chart based on the sample generalized variance is proportionally invariant to changes in the covariance matrix of the true quality characteristics when no measurement error exists. One way to assess the performance of the control chart (2) or (4) under measurement error is to identify the minimum and maximum values of r_ε achieved for a given value of r_0. Using these values, one can compute the minimum and maximum signal probabilities on the control chart to detect changes in the covariance matrix Σ_x of a particular quantity.

As shown in the Appendix, for a given value of r_0, the value of r_ε is bounded below but not above under (1) for any positive definite covariance matrix Σ_ε. Using this result in the signal probability (3) or (5), one can conclude that for a given value of r_0, the probability of a signal on the control chart (2) or (4) can be arbitrarily close to 1 for any positive definite Σ_ε. This implies that the probability of a signal on the control chart (2) or (4) in the presence of measurement error could be larger than the counterpart probability when no measurement error exists. However, it is known that the true quality characteristics without measurement errors contain more precise information about their own covariance matrix than the surrogate quality characteristics with measurement errors. This undesirable property reveals that the sample generalized variance chart may not be suitable for monitoring multivariate process variability when the measurement error is not negligible. Also note that this unexpected phenomenon of the sample generalized variance chart in the presence of measurement error does not only occur for uncorrelated processes, as shown in the following example.

Example 1. Suppose we have (1) with $n = 10, p = 2$,

$$\Sigma_0 = \begin{pmatrix} \frac{5}{4} & -\frac{3}{4} \\ -\frac{3}{4} & \frac{5}{4} \end{pmatrix}, \quad \Sigma_1 = \begin{pmatrix} \frac{\lambda_1+\lambda_2}{2} & \frac{\lambda_2-\lambda_1}{2} \\ \frac{\lambda_2-\lambda_1}{2} & \frac{\lambda_1+\lambda_2}{2} \end{pmatrix} \quad \text{and} \quad \Sigma_\varepsilon = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix},$$

where λ_1 and λ_2 are two positive constants satisfying $\lambda_1\lambda_2 = 3$. Also suppose that we choose the false alarm rate to be 0.05, yielding an in-control average run length of 20. Figure 1 represents curves of the power of the sample generalized variance chart (2) with and without measurement error for this case versus the value of λ_1. Note that the power of the sample generalized variance chart (2) without measurement error is 0.409 because it only depends on the ratio

$$r_0 = \frac{|\Sigma_1|}{|\Sigma_0|} = \frac{|A|\left|\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\right||A'|}{|A|\left|\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\right||A'|} = \lambda_1\lambda_2 = 3,$$

which is a fixed value in λ. Here

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

In contrast, the power of the sample generalized variance chart (2) with measurement error only depends on the ratio

$$\begin{aligned} r_\varepsilon = \frac{|\Sigma_1 + \Sigma_\varepsilon|}{|\Sigma_0 + \Sigma_\varepsilon|} &= \frac{|A|\left|\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\right||A'|}{|A|\left|\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\right||A'|} \\ &= \frac{(2+\lambda_1)(1+\lambda_2)}{6} = \frac{(2+\lambda_1)(1+\frac{3}{\lambda_1})}{6} = \frac{\lambda_1}{6} + \frac{1}{\lambda_1} + \frac{5}{6}. \end{aligned}$$

This first decreases and then increases to ∞ in λ_1 . As a result, the associated power first decreases, then increases to 1.

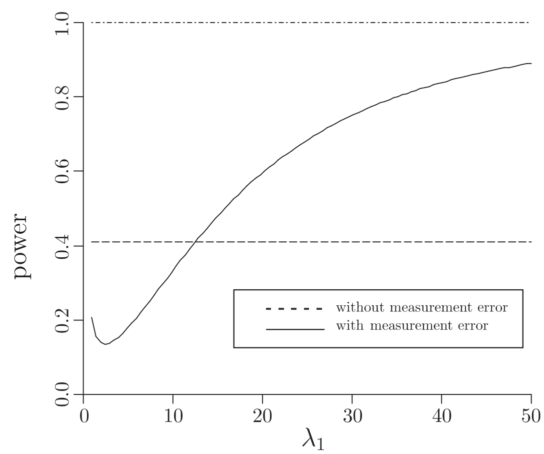

Figure 1. Plot of the power of the control chart (2) with and without measurement error for Example 1 versus the value of λ_1 for the case when $p = 2$.

3. The Likelihood Ratio Test Chart

When no measurement error exists, the control chart based on the likelihood ratio test for monitoring multivariate process variability is equivalent to repeated tests of significance of the form $H_0 : \Sigma_x = \Sigma_0$ vs. $H_1 : \Sigma_x \neq \Sigma_0$. The likelihood ratio statistic derived from a random sample $X_1, \ldots, X_n$ is

$$\Lambda(X_1, \ldots, X_n) = \left(\frac{e}{n}\right)^{\frac{np}{2}} \left(\frac{|A|}{|\Sigma_0|}\right)^{\frac{n}{2}} e^{-\frac{1}{2}tr(\Sigma_0^{-1}A)},$$

where $A = (n-1)S_x$, $S_x = \sum_{i=1}^n (X_i - \bar{X})(X_i - \bar{X})'/(n-1)$, and tr is the trace operator. It is known that the likelihood ratio test based on the statistic $\Lambda(X_1, \ldots, X_n)$ is not an unbiased test. Sugiura and Nagao (1968) showed that the modified likelihood ratio test based on the statistic $\Lambda^*(X_1, \ldots, X_n)$, obtained by substituting $n-1$ for the sample size n in $\Lambda(X_1, \ldots, X_n)$, is an unbiased test. Subsequently, the critical region for the unbiased likelihood ratio test is

$$\{(X_1, \ldots, X_n) : \Lambda^*(X_1, \ldots, X_n) < c_\alpha\}, \tag{7}$$

where the critical value c_α is determined by the significance level α. It is well known that the power of the test (7) at $\Sigma_x = \Sigma_1$ only depends on the eigenvalues $\lambda_1, \cdots, \lambda_p$ of the matrix $\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2}$. Nagao (1967) also showed that the power of the test is a nondecreasing function in $|\lambda_i - 1|, i = 1, \ldots, p$. Anderson (1984) showed that $-2\log\Lambda^*(X_1, \ldots, X_n)$ is asymptotically distributed as $\chi^2_{p(p+1)/2}$. Consequently, the control limit for the $-2\log\Lambda^*(X_1, \ldots, X_n)$ chart can be taken as UCL=$\chi^2_{p(p+1)/2,\alpha}$ (there is only one control limit). However, it is worth noting that there is an appreciable discrepancy between the real in-control average run length and the value of $1/\alpha$ for the $-2\log\Lambda^*(X_1, \ldots, X_n)$ chart when the sample size n is not large enough. For any finite sample n, an easy way to accurately calculate the in-control and out-of-control average run lengths of (7) is to use Monte-Carlo simulation.

Under (1), the control chart derived from the unbiased likelihood ratio test to detect changes in Σ_x is equivalent to repeated tests of significance of the form $H_0 : \Sigma_y = \Sigma_0 + \Sigma_\varepsilon$ vs. $H_1 : \Sigma_y \neq \Sigma_0 + \Sigma_\varepsilon$. Subsequently, the critical region for the unbiased likelihood ratio test is given by

$$\{(Y_1, \ldots, Y_n) : \Lambda_e(Y_1, \ldots, Y_n) < f_\alpha\}, \tag{8}$$

where

$$\Lambda_e(Y_1, \ldots, Y_n) = \Big(\frac{e}{n-1}\Big)^{\frac{(n-1)p}{2}} \Big(\frac{|B|}{|\Sigma_0 + \Sigma_\varepsilon|}\Big)^{\frac{n-1}{2}} e^{-\frac{1}{2}tr[(\Sigma_0+\Sigma_\varepsilon)^{-1}B]},$$

$B = (n-1)S_y$, and f_α is determined by the significance level α. Strictly speaking, the test based on the statistic $\Lambda_e(Y_1, \ldots, Y_n)$ for testing $H_0 : \Sigma_y = \Sigma_0 + \Sigma_\varepsilon$ vs. $H_1 : \Sigma_y \neq \Sigma_0 + \Sigma_\varepsilon$ is not a likelihood ratio test under the condition that Σ_ε is known because the maximum likelihood estimator of Σ_x under the unrestricted parameter space equals $S_y - \Sigma_\varepsilon$ only when $S_y - \Sigma_\varepsilon$ is positive definite with probability 1 (this is not always true). To avoid complexity and apply the previous (no measurement error) results to the current situation, in the test statistic $\Lambda_e(Y_1, \ldots, Y_n)$ we always employ the estimator $S_y - \Sigma_\varepsilon$ to estimate Σ_x. If the covariance matrix of X_i changes from $\Sigma_x = \Sigma_0$ to $\Sigma_x = \Sigma_1$, transforming to

$\Sigma_0^{-1/2}Y$ has the matrices Σ_0, Σ_1, and Σ_ε as $I, \Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2}$, and $\Sigma_0^{-1/2}\Sigma_\varepsilon\Sigma_0^{-1/2}$, respectively. Since $\Sigma_0^{-1/2}\Sigma_\varepsilon\Sigma_0^{-1/2}$ is symmetric and positive semidefinite, there exists an orthogonal matrix P such that $P'\Sigma_0^{-1/2}\Sigma_\varepsilon\Sigma_0^{-1/2}P = D$, where $D = \mathrm{diag}(d_1, \ldots, d_p)$ is a diagonal matrix with $d_1 \geq 0, \ldots, d_p \geq 0$ the eigenvalues of $\Sigma_0^{-1/2}\Sigma_\varepsilon\Sigma_0^{-1/2}$. Thus, after a further transformation of Y $(P'\Sigma_0^{-1/2}Y)$, the matrices $I, \Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2}$, and $\Sigma_0^{-1/2}\Sigma_\varepsilon\Sigma_0^{-1/2}$ become $I, P'\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2}P$, and D, respectively. Hence we use $I, P'\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2}P$, and D to represent the in-control, out-of-control covariance matrices of X_i, and the measurement error covariance matrix, respectively.

Now, is it possible that at the out-of-control matrix $\Sigma_x = \Sigma_1$ the power of the test (8) with measurement error be larger than the power of the test (7) without measurement error, as could happen with the sample generalized variance chart discussed in Section 2. It turns out that this question is too difficult to answer. However, under certain assumptions, the power of the test (8) with measurement error is always less than or equal to the power of the test (7) without measurement error.

Based on earlier remarks, the power of the test (8) with measurement error only depends on the eigenvalues of $(I+D)^{-1/2}(P'\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2}P+D)(I+D)^{-1/2}$ after the transformation of Y or, equivalently, $(\Sigma_0+\Sigma_\varepsilon)^{-1/2}(\Sigma_1+\Sigma_\varepsilon)(\Sigma_0+\Sigma_\varepsilon)^{-1/2}$ before the transformation of Y. Under the assumption that $P'\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2}P$ is a diagonal matrix, since the diagonal elements are the eigenvalues of $P'\Sigma_0^{-1/2}\Sigma_1 \Sigma_0^{-1/2}P$ and the matrices $P'\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2}P$ and $\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2}$ have the same eigenvalues, we conlude that $P'\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2}P = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$, where $\lambda_1, \ldots, \lambda_p$ are the eigenvalues of $\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2}$. Subsequently, the matrix

$$\begin{aligned}
&(I + D)^{-\frac{1}{2}}(P'\Sigma_0^{-\frac{1}{2}}\Sigma_1\Sigma_0^{-\frac{1}{2}}P + D)(I + D)^{-\frac{1}{2}} \\
&\quad = (I + D)^{-\frac{1}{2}}[\mathrm{diag}(\lambda_1, \ldots, \lambda_p) + D](I + D)^{-\frac{1}{2}} \\
&\quad = \mathrm{diag}\Big(\frac{\lambda_1 + d_1}{1 + d_1}, \ldots, \frac{\lambda_p + d_p}{1 + d_p}\Big)
\end{aligned}$$

has the eigenvalues $(\lambda_i + d_i)/(1 + d_i), i = 1, \ldots, p$. Since $d_i \geq 0, i = 1, \ldots, p$, we have

$$\Big|\frac{\lambda_i + d_i}{1 + d_i} - 1\Big| = \Big|\frac{\lambda_i - 1}{1 + d_i}\Big| \leq |\lambda_i - 1|.$$

Therefore, based on the result of Nagao (1967), the power of the test (7) at $\Sigma_x = \Sigma_1$ without measurement error is always greater than or equal to the power of the test (8) at $\Sigma_x = \Sigma_1$ with measurement error. Note that $P'\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2}P$ is a diagonal matrix if $\Sigma_1 = c\Sigma_0$, where $c > 0$ is a constant. Namely, the out-of-control covariance matrix Σ_1 is proportional to its in-control covariance matrix Σ_0.

As the power of the test (7) at $\Sigma_x = \Sigma_1$ without measurement error only depends on the eigenvalues of $\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2}$, it remains unchanged for any other form of the out-of-control Σ_1 as long as the matrix $\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2}$ gives the same set of eigenvalues. On the other hand, the matrix $(I+D)^{-1/2}(P'\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2}P+D)(I+D)^{-1/2}$ may have a different set of eigenvalues for different forms of the out-of-control Σ_1 although the matrix $\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2}$ gives the same eigenvalues $\lambda_1, \ldots, \lambda_p$. Therefore, the test (8) with measurement error may give different power at different forms of the out-of-control Σ_1 although the test (7) without measurement error gives the same power as long as the out-of-control Σ_1 has $\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2}$ with the same eigenvalues. Now, if the assumption $\Sigma_\varepsilon = k\Sigma_0$ holds, where $k \geq 0$ is a constant, then the matrix $D = P'\Sigma_0^{-1/2}\Sigma_\varepsilon\Sigma_0^{-1/2}P = kI_p$ for any orthogonal matrix P. Hence, we can choose an orthogonal matrix P such that $P'\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2}P = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$, where $\lambda_1, \ldots, \lambda_p$ are the eigenvalues of $\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2}$ since the diagonal elements and eigenvalues of $P'\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2}P$ are the same and $P'\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2}P$ and $\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2}$ have the same eigenvalues. As a result, the matrix $(I+D)^{-1/2}(P'\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2}P + D)(I+D)^{-1/2} = (I+D)^{-1/2}[\mathrm{diag}(\lambda_1, \ldots, \lambda_p) + D](I+D)^{-1/2}$ has the eigenvalues $(\lambda_1 + k)/(1+k), \ldots, (\lambda_p + k)/(1+k)$. This shows that at any out-of-control Σ_1, as long as the matrix $\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2}$ gives the same eigenvalues, both the powers of the test (7) without measurement error and the test (8) with measurement error are constant under the assumption $\Sigma_\varepsilon = k\Sigma_0$. Further, due to $|(\lambda_i + k)/(1+k) - 1| = |(\lambda_i - 1)/(1+k)| \leq |\lambda_i - 1|$, the power of the test (7) without measurement error is always greater than or equal to the power of the test (8) with measurement error under this assumption, based on the result of Nagao (1967).

Example 2. Suppose at (1) that $\Sigma_0 = I_2, \Sigma_1 = P\,\mathrm{diag}(\lambda_1, \lambda_2)P'$, and $\Sigma_\varepsilon = \mathrm{diag}(\nu_1, \nu_2)$, where P is a 2×2 orthogonal matrix and $\lambda_1, \lambda_2, \nu_1$, and ν_2 are positive constants. Then the power of the test (7) at Σ_1 without measurement error depends on the values of λ_1 and λ_2, which are the two eigenvalues of $\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2}$ for any orthogonal matrix P. It is well known that the 2×2 nontrivial orthogonal matrices (up to transposition) are

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \text{(rotation)} \qquad \text{and} \qquad \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \text{(reflection)}.$$

Substitute either of them for P in Σ_1 to get

$$\Sigma_1 = \begin{bmatrix} \lambda_1\cos^2\theta + \lambda_2\sin^2\theta & (\lambda_1 - \lambda_2)\sin\theta\cos\theta \\ (\lambda_1 - \lambda_2)\sin\theta\cos\theta & \lambda_1\sin^2\theta + \lambda_2\cos^2\theta \end{bmatrix}.$$

Subsequently,

$$
\begin{aligned}
&(\Sigma_0+\Sigma_\varepsilon)^{-\frac{1}{2}}(\Sigma_1+\Sigma_\varepsilon)(\Sigma_0+\Sigma_\varepsilon)^{-\frac{1}{2}} \\
&= \begin{bmatrix} 1+\nu_1 & 0 \\ 0 & 1+\nu_2 \end{bmatrix}^{-\frac{1}{2}} \begin{bmatrix} \lambda_1\cos^2\theta+\lambda_2\sin^2\theta+\nu_1 & (\lambda_1-\lambda_2)\sin\theta\cos\theta \\ (\lambda_1-\lambda_2)\sin\theta\cos\theta & \lambda_1\sin^2\theta+\lambda_2\cos^2\theta+\nu_2 \end{bmatrix} \\
&\quad\times \begin{bmatrix} 1+\nu_1 & 0 \\ 0 & 1+\nu_2 \end{bmatrix}^{-\frac{1}{2}} \\
&= \begin{bmatrix} \frac{\lambda_1\cos^2\theta+\lambda_2\sin^2\theta+\nu_1}{1+\nu_1} & \frac{(\lambda_1-\lambda_2)\sin\theta\cos\theta}{\sqrt{(1+\nu_1)(1+\nu_2)}} \\ \frac{(\lambda_1-\lambda_2)\sin\theta\cos\theta}{\sqrt{(1+\nu_1)(1+\nu_2)}} & \frac{\lambda_1\sin^2\theta+\lambda_2\cos^2\theta+\nu_2}{1+\nu_2} \end{bmatrix}.
\end{aligned}
$$

Hence, the power of the test (8) at $\Sigma_x=\Sigma_1$ with measurement error depends on the values of λ_1^* and λ_2^*, which are the eigenvalues of $(\Sigma_0+\Sigma_\varepsilon)^{-1/2}(\Sigma_1+\Sigma_\varepsilon)(\Sigma_0+\Sigma_\varepsilon)^{-1/2}$. Assuming $\lambda_1=5, \lambda_2=4, \nu_1=2$, and $\nu_2=1.5$, in Figure 2 we draw the curves of $|\lambda_1^*-1|$ and $|\lambda_2^*-1|$ versus the value of θ. From the figure, the values of $|\lambda_1^*-1|$ and $|\lambda_2^*-1|$ do not increase or decrease simultaneously in θ and this causes difficulty in evaluating power. However, the values of $|\lambda_1^*-1|$ and $|\lambda_2^*-1|$ with measurement error are respectively less than the counterparts $|\lambda_1-1|=4$ and $|\lambda_2-1|=3$ when no measurement error exists. As a result, at $\Sigma_x=\Sigma_1$, the power of the test (8) with measurement error is less than the power of the test (7) without measurement error, based on the result of Nagao (1967).

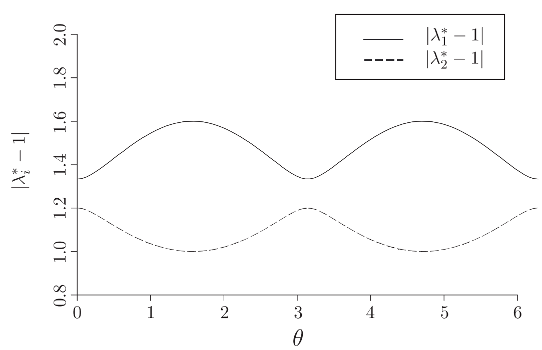

Figure 2. Plot of the distance of the two eigenvalues λ_1^* and λ_2^* of $(\Sigma_0+\Sigma_\varepsilon)^{-\frac{1}{2}}(\Sigma_1+\Sigma_\varepsilon)(\Sigma_0+\Sigma_\varepsilon)^{-\frac{1}{2}}$ to 1 for Example 2 versus the value of θ for the case when $p=2$.

Remark. In this paper we have not considered the case where the measurement error covariance matrix also changes during the monitoring period. Such changes clearly add complexity to the issue of control chart performance. Additionally, we have assumed that the in-control covariance matrix of the true quality characteristics is known. In practice, this must be estimated from some preliminary data. The effect of the estimation of the covariance matrix of the true quality characteristics will affect the performance of the control chart schemes considered here.

Appendix

Lemma. *Assume that* (1) *holds. Then, for a given value of* $r_0 = |\Sigma_1|/|\Sigma_0|$, *the value of* $r_\varepsilon = |\Sigma_1 + \Sigma_\varepsilon|/|\Sigma_0 + \Sigma_\varepsilon|$ *is bounded below, but not above, for any positive definite* Σ_ε.

Proof. After an appropriate transformation of Y, we can take $\Sigma_0 = I$ and suppose both Σ_1 and Σ_ε are positive definite. Hence, the ratio $r_0 = |\Sigma_1|/|\Sigma_0| = |\Sigma_1|$. Since Σ_1 is positive definite, there exists a nonsingular matrix B such that $B'\Sigma_1 B = I$. Further, there exists an orthogonal matrix T such that $T'(B'\Sigma_\varepsilon B)T = A$, where $A = \mathrm{diag}(a_1, \ldots, a_p)$ is a diagonal matrix with diagonal elements $a_1 > 0, \ldots, a_p > 0$ as the eigenvalues of $B'\Sigma_\varepsilon B$. Note that $|A| = \prod_{i=1}^p a_i = |\Sigma_\varepsilon||B'||B| = |\Sigma_\varepsilon|/|\Sigma_1|$ is a positive constant for any positive definite Σ_ε. We have both

$$\begin{aligned}
|\Sigma_1 + \Sigma_\varepsilon| &= |B'^{-1}||B'\Sigma_1 B + B'\Sigma_\varepsilon B||B^{-1}| = |I + B'\Sigma_\varepsilon B||\Sigma_1| \\
&= |I + T'B'\Sigma_\varepsilon BT||\Sigma_1| = |I + A||\Sigma_1|, \\
|I + A| &= (1 + a_1)(1 + a_2)\cdots(1 + a_p) \\
&= a_1 \cdots a_p + (a_1 \cdots a_{p-1} + \cdots + a_2 \cdots a_p) + \cdots + (a_1 + a_2 + \cdots + a_p) + 1.
\end{aligned}$$

Since for a given value of $r_0 = |\Sigma_1|$, and hence a given positive value of $|A| = \prod_{i=1}^p a_i$ for positive definite Σ_ε, we can choose some $a_i > 0, 1 \le i \le p$, large and some $a_j > 0, 1 \le j \ne i \le p$, small, the determinant $|I + A|$ and hence $|\Sigma_1 + \Sigma_\varepsilon|$ is not bounded above. Subsequently, for a given value of r_0, the ratio r_ε is not bounded above for any positive definite Σ_ε. On the other hand, for a given value of $r_0 = |\Sigma_1|$, it is obvious that the minimum value of r_ε is at $a_1 = \cdots = a_p$.

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