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AN ALTERNATE VARIABLES CONTROL CHART: THE UNIVARIATE AND MULTIVARIATE CASE

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Abstract: A variables control chart that can be used to monitor location and scale on a single chart is proposed. The procedure provides an alternative to the boxplot style of simultaneous control charts with the added advantage of performing equally well for both large and small subgroup sizes. The resulting control chart provides information regarding the process's proximity to the target value and its variability. Control limits analogous to the Shewhart limits are developed and several properties are investigated. The procedure is extended to the multivariate case and examples illustrating both the univariate and multivariate cases included.

Key words and phrases: Multivariate control chart, process control, simultaneous control chart.

1. Introduction

Traditional North American quality assurance techniques are being challenged by innovative philosophies imported from Japan. One particular philosophy discussed by many authors stresses the need for considering a nominal (target) value in assessing the quality levels of a process. In the past, attention has focused on bringing a process within specification while placing little emphasis on attaining target values. Sullivan (1984), among others, provides examples that stress the importance of controlling to a target value as well as the variability. Juran (1951), as well as others, briefly discusses integrating targets into the traditional Shewhart charts but fails to discuss details of those cases where the target differs from the mean. The proposed procedure incorporates the target into a simultaneous control charting procedure resulting in a chart that provides information regarding (i) proximity to the target and (ii) process variability.

Simultaneous control charts refer to plots that provide graphical expressions for a potpourri of statistics used in describing a process. Examples of this type of control chart include those proposed by White and Schroeder (1987) and Iglewicz and Hoaglin (1987) which use boxplots to express the information gathered at the subgroup level. These additions to the literature are representative of an ongoing series of improvements (e.g., Hackl and Ledolter (1991)) made to the original control charts developed by Shewhart (1931).

Boxplots are very informative but can be overpowering. Unless users are familiar with the plots, the vast amount of information contained in a single plot can be confusing. Proposed additions to the charts regarding robust and resistant measures only serve to further clutter the inferences drawn. In the hands of an experienced data analyst boxplots and the proposed resistant and robust measures can be enlightening. However they may only serve to detract from the general inferences required on the manufacturing floor. Due to their complexity, boxplots are rarely performed by hand. In those cases where a computer is available or where the data acquisition is automated, boxplots may be quite convenient; however, it is unlikely that the boxplot style of simultaneous control chart will see application on the manufacturing floor. Other drawbacks which limit the use of boxplot style simultaneous charts include their inability to deal with (i) small sample sizes, (ii) control limits for any measure of dispersion and (iii) multi-characteristic processes.

There is an abundance of other newly developed control charts and procedures designed to enhance the analysis of a process; most however do not reflect the current changes in quality assurance as pointed out in Woodall (1985). No longer is "conforming to specifications", without a nominal value, sufficient. Both proximity to target and process variability are of concern in today's quality. The proposed procedures attempt to adopt this philosophy by monitoring both proximity to target and process variability in a simultaneous control chart while avoiding some of the drawbacks associated with boxplot styled charts. The techniques used are similar to the traditional control charts outlined by Shewhart with the added advantage of appearing on a single plot. Control chart constants for the proposed measures are easily determined for sample sizes of 2 and larger, providing the practitioner with small sample simultaneous control chart procedures not available with boxplot techniques. The proposed control chart also includes discernible control limits for a measure of dispersion at the subgroup level and is easily extended to those cases where more than one variable will be used in monitoring the process. The procedure is straightforward and can be carried out by employees whose major task is to monitor and adjust the process. The plots can be produced by hand requiring only the aid of a hand-held calculator.

2. The Procedure

Letting $x_1, \ldots, x_n$ represent a subgroup of n observations, traditional Shewhart control charts monitor process location and process variability by tracking $\bar{x}$ and the range (R) (or standard deviation (S)) at each subgroup. The control limits for the subgroup means (i.e., $\bar{x}$'s) are of the form $\bar{\bar{X}} \pm A_2\bar{R}$ or $\bar{\bar{X}} \pm A_3\bar{S}$ where A_2, A_3 are statistically determined constants, $\bar{\bar{X}} = k^{-1}\sum_{j=1}^{k}\bar{x}_j$ is the mean of the k subgroup means, $\bar{R} = k^{-1}\sum_{j=1}^{k} R_j$ and $\bar{S} = k^{-1}\sum_{j=1}^{k} S_j$ are the means of the k subgroup ranges and standard deviations respectively. The control limits associated with the subgroup range and the subgroup standard deviation are of the form $D_3\bar{R}$ and $D_4\bar{R}$ or $B_3\bar{S}$ and $B_4\bar{S}$ respectively where D_3, D_4, B_3 and B_4 are also statistically determined constants.

For each subgroup of n observations, the proposed measures of process performance are derived from the mean square error (MSE) around the target value. The measures monitored will be $(\bar{x} - T)^2$ and $MSE = (n-1)^{-1}\sum_{i=1}^{n}(x_i - T)^2$ (the sample variance $S^2 = (n-1)^{-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$ will receive occasional attention) where T denotes the target value of the process. Assuming the process measurements are $N(\mu, \sigma^2)$, the control limits associated with the measures will be (i) $A\sigma^2$ for $(\bar{x} - T)^2$, (ii) $C\sigma^2$ for MSE and (iii) $B\sigma^2$ for S^2 where A, B and C are constants derived from the statistical distributions associated with each of the measures. Only upper control limits are discussed for each of the suggested process summaries as small values of these measures are considered desirable.

Assuming $X \sim N(\mu, \sigma^2)$ it follows that $(\bar{x} - T)^2 \sim n^{-1}\sigma^2\chi^2_{1,\lambda}$ (Johnson and Kotz (1970)), where $\chi^2_{1,\lambda}$ denotes the non-central chi-square distribution with one degree of freedom (df) and non-centrality parameter $\lambda = n[(\mu - T)/\sigma]^2$. Defining the UCL to be the $(1-\alpha)100\%$ percentile of the distribution function associated with $(\bar{x} - T)^2$ results in $UCL_{(\bar{x}-T)^2} = n^{-1}\sigma^2\chi^2_{1,\lambda,(1-\alpha)} = A\sigma^2$. In order to determine A for a particular process, $[(\mu - T)/\sigma]^2$ must be known. In practice, $\bar{\bar{X}} = k^{-1}\sum_{j=1}^{k}\bar{x}_j$ and $\bar{S}^2 = k^{-1}\sum_{j=1}^{k} S_j^2$, where k denotes the number of subgroups and $\bar{x}_j, S_j^2$ the sample mean and variance of the jth subgroup, are substituted for μ and σ^2 respectively (Shewhart (1931)).

The MSE will follow a $(n-1)^{-1}\sigma^2\chi^2_{n,\lambda}$ distribution with n df and $\lambda = n[\mu - T)/\sigma]^2$. Analogous to the UCL for $(\bar{x} - T)^2$, $UCL_{MSE} = (n-1)^{-1}\sigma^2\chi^2_{n,\lambda,(1-\alpha)} = C\sigma^2$. $\bar{\bar{X}}$ and $\bar{S}^2$ will again be substituted for μ and σ^2 respectively resulting in UCL for MSE of the form $UCL_{MSE} = C\bar{S}^2$. The SAS (1994) code for calculating C is $CINV((1-\alpha), n, \lambda)/(n-1)$, which for $(1-\alpha) = 0.9973$ provides values of C for various λ and n.

S^2 will be distributed as $\sigma^2\chi^2_{n-1,0}$, where $\chi^2_{n-1,0}$ denotes the central Chi-square (i.e., $\lambda = 0$) distribution resulting in $UCL_{S^2} = (n-1)^{-1}\sigma^2\chi^2_{n-1,0,(1-\alpha)} = B\sigma^2$. The UCL_{S^2} depends upon the value of the population parameter σ^2 which will be estimated by $\bar{S}^2$, resulting in $UCL_{S^2} = B\bar{S}^2$. B can be calculated using $CINV((1-\alpha), (n-1), 0)/(n-1)$ (SAS (1994)) for values of n. The value of B associated with $(1-\alpha) = 0.9973$ results in a more stringent UCL than those associated with the traditional S charts as only the lower control limit for S^2 is used. In doing so we effectively make the control chart a single-tailed problem. Identification and investigation of "sharp" declines in subgroup variation is encourage, but not viewed as an "error".

The procedure for creating the proposed plots is similar to that of traditional $\bar{X}$ and S charts. The subgroup means and variances, $\bar{x}$ and S^2, as well as $\bar{\bar{X}}$ and $\bar{S}^2$ must be determined. In addition the practitioner must determine $(\bar{x}-T)^2$ for each subgroup, noting the cases where $\bar{x} < T$ and calculating the MSE for each subgroup using $(S^2 + (n/(n-1))(\bar{x}-T)^2)$. Finally the UCL for each of the measures must be found using the appropriate algorithms presented above.

The suggested plotting strategy is to first plot the values of $(\bar{x}-T)^2$ for each subgroup along with the $UCL_{(\bar{x}-T)^2}$. The plotting characters "$-$" for $\bar{x} < T$ and "+" for $\bar{x} \geq T$ are suggested when plotting the subgroup values of $(\bar{x}-T)^2$ as they will be of assistance in identifying any trends which may be obscured through examining squared differences form the target. The plotting characters will also be used in conjunction with the detection rules discussed in Section 7 to assist the practitioner in identifying systematic runs and trends.

The practitioner then plots the MSE and the UCL for both S^2 and MSE. The plotting characters used here are of little consequence as any trends that may occur should be obvious regardless of the plotting characters used. Plotting the MSE rather than S^2 is suggested, as the MSE must always be greater than or equal to $(\bar{x}-T)^2$ for each subgroup. This is not neccesarily the case for S^2. In the case of successive subgroups (i.e., ith and ith+1) where $(\bar{x}-T)^2_i$ is greater than S^2_i and $(\bar{x}-T)^2_{i+1}$ less than S^2_{i+1} the plots will overlap and may clutter the graphical inference.

Similar to the standard control charts, those subgroups with either (or both) of the proposed measures above the appropriate UCL should be highlighted and investigated. As well, if any sharp declines, unusual trends or actions appear they too should be investigated. Any decisions and actions made at this stage will be similar to those taken for the traditional control charts. If for example the subgroup means appear to be drifting away from the target (i.e., $(\bar{x}-T)^2$ is moving away from 0) or if the overall variability is changing in a systematic fashion, the process should be investigated. Trend and run rules that will assist in identifying systematic shifts in the process are discussed in Section 7.

In some subgroups it may occur that both MSE and $(\bar{x}-T)^2$ are within their respective control limits but S^2 exceeds its UCL (this can only occur when the $(\bar{x}-T)^2$ is small and S^2 large). In cases such as this, the difference between plotted points $(\bar{x}-T)^2$ and MSE will be large indicating that S^2 is large. To prevent practitioners from missing significant increases in the variability of a process (i.e., significant increases in S^2), it is suggested that the UCL_{S^2} be used as a "warning limit" for the MSE. That is, for all subgroups where the MSE exceeds the UCL_{S^2}, but is less than the UCL_{MSE}, S^2 should be included in the plot. For clarity, a different plotting symbol for S^2 is suggested for this situation. In those cases where the $MSE > UCL_{MSE}$ practitioners should plot S^2 in order

UCL_{MSE}

UCL_{S^2}

$UCL_{(\bar{x}-T)^2}$

Figure 1. Alternate variables control chart for Braverman example.

From Figure 1 we see that none of the subgroups exceed the UCL for $(\bar{x}-T)^2$ with most appearing to be close to 0 indicating reasonable proximity to the target. Subgroups 7, 12 and 26 have the largest departures from the target but are within the control limits. Nineteen of the subgroups have sampling means below the target as denoted by the negative signs. The longest sequence of similar signs is five (subgroups 1 to 5), indicating that the longest run of subgroup means either above or below (in this case below) the target is five. There does not appear to be any significant drifting from the target nor does there appear to be any cyclical relationships occurring. However before any formal inferences can be drawn the MSE for each subgroup must be investigated.

None of the subgroups exceed the UCL for MSE, but subgroup 15 has a MSE that exceeds the UCL for S^2. Closer investigation finds $(\bar{x} - T)^2$ to be small for subgroup 15 resulting in a large S^2 (i.e., $S_{15}^2 = 252.250$) which exceeds the $UCL_{S^2} = 228.5$ suggesting that the process variance has increased and the process should be examined for assignable cause. Analysis of the rest of the plot follows in a predictable fashion. Subgroups 14, 24 and 28 require further investigation as they appear to have small variability as well as being quite close to the target. Additional information such as individual deviations from the target for subgroup 15, for example, may prove interesting and can easily be added to the plot.

4. The Multivariate Case

The methods and sampling schemes used to gather information in the multivariate case are assumed identical to the univariate case where subgroups consisting of n samples are drawn periodically from the process. In the general multivariate case, a set of p characteristics is measured on each sampling unit and recorded in a p-dimensional vector $\underset{\sim}{X}' = [x_1, \ldots, x_p]$ with each x_i representing the numerical value associated with one of the p quality characteristics. The subgroup mean in the multivariate case ($\bar{\underset{\sim}{X}}$) will be a vector of means of the form

$$\bar{\underset{\sim}{X}}' = \Big[n^{-1} \sum_{i=1}^{n} \underset{\sim}{X}_i\Big]' = \Big[n^{-1} \sum_{i=1}^{n} x_{1i}, \ldots, n^{-1} \sum_{i=1}^{n} x_{pi}\Big].$$

Assuming k subgroups, the grand mean ($\bar{\bar{\underset{\sim}{X}}}$) is also a vector which for the p-variable case is

$$\bar{\bar{\underset{\sim}{X}}}' = \Big[k^{-1} \sum_{j=1}^{k} \bar{\underset{\sim}{X}}_j\Big]' = \Big[(nk)^{-1} \sum_{j=1}^{k} \sum_{i=1}^{n} x_{1ij}, \ldots, (nk)^{-1} \sum_{j=1}^{k} \sum_{i=1}^{n} x_{pij}\Big].$$

The target will be a vector $\underset{\sim}{T}' = [T_1, \ldots, T_p]$, with T_i the target value for the ith variable.

The measure $\tilde{T}_p = (\bar{\underset{\sim}{X}} - \underset{\sim}{T})' \Sigma^{-1} (\bar{\underset{\sim}{X}} - \underset{\sim}{T})$, with Σ denoting the covariance matrix, is used to assess proximity to the target value. For each of the k subgroups, $(\bar{\underset{\sim}{X}} - \underset{\sim}{T})' \Sigma^{-1} (\bar{\underset{\sim}{X}} - \underset{\sim}{T})$ provides a scalar measure of proximity to the target. As the center of mass ($\bar{\underset{\sim}{X}}$) drifts from the target ($\underset{\sim}{T}$), the generalized distance $\tilde{T}_p$ will reflect these departures by increasing in magnitude. Increasing values of $\tilde{T}_p$ reflect greater departures from the target while small values (minimum of zero) reflect close proximity to the target. $\tilde{T}_p$ can be determined for each subgroup and will be used to monitor the process's proximity to the target. This multivariate analogue of $(\bar{x} - T)^2/\sigma^2$ uses generalized Mahalanobis (1936) distances in assessing the process's ability to produce at the target. In the univariate case $(\bar{x}-T)^2$ is the measure of interest and the process variability (i.e., σ^2) is reflected in the control limits. In the multivariate case the dispersion is incorporated in the actual measurement rather than being reflected in the control limits.

The proposed multivariate measure of variability within a subgroup is $MSE_p = (n-1)^{-1}\sum_{i=1}^{n}(\underset{\sim}{X}_i - \underset{\sim}{T})'\Sigma^{-1}(\underset{\sim}{X}_i - \underset{\sim}{T})$ which assesses the amount of dispersion around the target within each subgroup. MSE_p confounds proximity to the target and the inherent variability within a subgroup in a single measure but can be partitioned into $(n-1)^{-1}\sum_{i=1}^{n}(\underset{\sim}{X}_i-\bar{\underset{\sim}{X}})'\Sigma^{-1}(\underset{\sim}{X}_i-\bar{\underset{\sim}{X}})$ and $(\bar{\underset{\sim}{X}}-\underset{\sim}{T})'\Sigma^{-1}(\bar{\underset{\sim}{X}}-\underset{\sim}{T})$, with the first component being a measure of dispersion around the subgroup mean and the second the proposed measure of proximity to the target within a subgroup ($\tilde{T}_p$). The measures used to monitor the process are (i) $\tilde{T}_p = (\bar{\underset{\sim}{X}}-\underset{\sim}{T})'\Sigma^{-1}(\bar{\underset{\sim}{X}}-\underset{\sim}{T})$,

$$\text{(ii) } MSE_p = (n-1)^{-1}\sum_{i=1}^{n}(\underset{\sim}{X}_i - \underset{\sim}{T})'\Sigma^{-1}(\underset{\sim}{X}_i - \underset{\sim}{T}) \quad \text{and}$$

$$\text{(iii) } S_p^2 = (n-1)^{-1}\sum_{i=1}^{n}(\underset{\sim}{X}_i - \bar{\underset{\sim}{X}})'\Sigma^{-1}(\underset{\sim}{X}_i - \bar{\underset{\sim}{X}}).$$

The upper control limits for the proposed multivariate measurements are based on the assumption that the process measurements (i.e., p-dimensional vectors) are multivariate normal variates with mean vector $\underset{\sim}{\mu}$ and dispersion matrix Σ (i.e., $\text{MVN}_p(\underset{\sim}{\mu}, \Sigma)$). The UCL will be of the form $UCL_{\tilde{T}_p} = n^{-1}\chi^2_{p,\lambda,(1-\alpha)} = D$. The SAS (1994) code for calculating D is $CINV((1-\alpha),\ p,\ \lambda)/n$ and will produce D for various λ, p and n when $(1-\alpha) = 0.9973$. The UCL for MSE_p will be $UCL_{MSE_p} = (n-1)^{-1}\chi^2_{np,\lambda,(1-\alpha)} = E$. The SAS (1994) code for calculating E is $CINV((1-\alpha),\ np,\ \lambda)/(n-1)$ and will provide E for values of λ, p and n when $(1-\alpha) = 0.9973$. In both cases the non-centrality parameter is defined to be $\lambda = n(\underset{\sim}{\mu} - \underset{\sim}{T})'\Sigma^{-1}(\underset{\sim}{\mu} - \underset{\sim}{T})$. Analogous to the univariate case and consistent with techniques used in Shewhart charts, $\bar{\bar{\underset{\sim}{X}}}$ and $\bar{\underset{\sim}{S}}$ will be substiuted for $\underset{\sim}{\mu}$ and Σ where $\bar{\underset{\sim}{S}} = k^{-1}\sum_{j=1}^{k}\underset{\sim}{S}_j$, and $\underset{\sim}{S}_j = n^{-1}\sum_{j=1}^{n}(\underset{\sim}{X}_i - \bar{\underset{\sim}{X}})(\underset{\sim}{X}_i - \bar{\underset{\sim}{X}})'$ for $j = 1, \ldots, n$. The UCL for S_p^2 will be $UCL_{S_p^2} = (n-1)^{-1}\chi^2_{(n-1)p,(1-\alpha)} = F$ and can be calculated using $CINV((1-\alpha),(n-1)p,0)/(n-1)$ for values of p, $(1-\alpha)$ and n. Note that when $p = 1$, the multivariate measures and their associated control limits are analogous to their univariate counterparts outlined in Section 2, differing only by where the dispersion is incorporated in the calculations.

The proposed plotting strategy is to determine and then sketch the control limits $UCL_{\tilde{T}_p}$, $UCL_{s_p^2}$ and UCL_{MSE_p} on the chart. The limits are a function of the subgroup sample size (n), number of quality characteristics considered (p), the degree of off-targetness ($(\underset{\sim}{\mu}-\underset{\sim}{T})'\Sigma^{-1}(\underset{\sim}{\mu}-\underset{\sim}{T})$), and the level of significance (α) desired. Unlike the univariate and Shewhart procedures, the limits are completely specified once n, α, and $(\underset{\sim}{\mu}-\underset{\sim}{T})'\Sigma^{-1}(\underset{\sim}{\mu}-\underset{\sim}{T})$ are known. In essence the limits in the univariate procedures are also known once the sample size and level of α are fixed, however traditional procedures incorporate the overall measure of dispersion in the control limits rather than the measurement, making plotting and calculating somewhat simpler.

The plotting characters used in the multivariate case are of little consequence and any useful character can be used. The MSE_p rather than S_p^2 is used to monitor dispersion within a subgroup, as the MSE_p must always be greater than or equal to $\tilde{T}_p$ resulting in no crossovers in the plot. As a result it is suggested that only $\tilde{T}_p$ and MSE_p be plotted for each subgroup and S_p^2 added in those cases where MSE_p exceeds either the $UCL_{S_p^2}$ or UCL_{MSE_p}. Inferences drawn will be similar to those inferences drawn from the multivariate control charts based on Hotelling's T^2 (Hotelling (1947)) as discussed by Alt (1986).

5. Multivariate Example

Sultan (1986) discusses a problem from a steel manufacturing process that measured the Brinnel hardness (x) and tensile strength (y) for 30 samples. To illustrate the multivariate control chart procedure the 30 samples of the form $(x, y)'$ were aligned in subgroups of five using the sequential sample numbers.

Subgroups

	1		2		3		4		5		6
1	143, 34.2	6	178, 51.5	11	175, 57.3	16	182, 57.2	21	160, 45.5	26	195, 58.0
2	200, 57.0	7	162, 45.9	12	187, 58.5	17	177, 50.6	22	183, 53.9	27	134, 45.7
3	160, 47.5	8	215, 59.1	13	187, 58.2	18	204, 55.1	23	179, 51.2	28	187, 42.0
4	181, 53.4	9	161, 48.4	14	186, 57.0	19	178, 50.9	24	194, 57.7	29	135, 40.5
5	148, 47.8	10	141, 47.3	15	172, 49.4	20	196, 57.9	25	181, 55.6	30	159, 58.0

From the subgroup information, $\underset{\sim}{\bar{\bar{X}}} = \begin{bmatrix} 174.67 \\ 51.80 \end{bmatrix}$ and $\underset{\sim}{\bar{S}} = \begin{bmatrix} 332.13 & 67.48 \\ 67.48 & 29.62 \end{bmatrix}$ which when substituted for $\underset{\sim}{\mu}$ and Σ results in $n^{-1}\lambda = (\underset{\sim}{\bar{\bar{X}}} - \underset{\sim}{T})'\underset{\sim}{\bar{S}}^{-1}(\underset{\sim}{\bar{\bar{X}}} - \underset{\sim}{T}) = 0.6174$

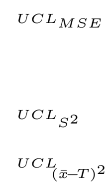

Figure 2. Simultaneous multivariate control chart.

6. Properties

The run length of a control procedure is defined to be the number of sampling periods observed before an out-of-control signal arises, with the average run length (ARL) used to describe the performance of the control procedure. The ARLs for $(\bar{x}-T)^2$ and MSE can be determined assuming the process measurements are $N(\mu_0, \sigma_0^2)$ for (i) subgroups of size n, (ii) perturbations from the target of $(\mu - T)^2$ and (iii) variance inflaters β (i.e., $\beta\sigma_0^2$) for various "steady states" (i.e., $\lambda_0 = n(\mu_0 - T)^2/\sigma_0^2$) using

$$\text{ARL}_{MSE} = 1/\{1 - \Pr(MSE > UCL_{MSE} | MSE \sim \beta\sigma_0^2(n-1)^{-1}\chi^2_{n,n(\mu-T)^2/(\beta\sigma_0^2)}, \\ UCL_{MSE} = \sigma_0^2(n-1)^{-1}\chi^2_{n,\lambda_0}(0.9973))\},$$

where $\chi^2_{n,\lambda_0}(0.9973)$ and $\chi^2_{1,\lambda_0}(0.9973)$ represent the value of χ^2 variates associated with $(1-\alpha) = 0.9973$. The resulting ARLs provide insights into how quickly we expect the charting procedure to identify shifts in the proximity to the target and/or changes in the MSE.

The ARLs for various values of β (representing a change in the variability) and $(\mu - T)^2$ (representing a shift in location) provide performance assessment of the Alternate Variables Control chart that can be investigated and compared with other charting procedures. The ARLs can be determined using $(1 - PROBCHI(CINV((1-\alpha), df, n^{*}mtstead)/b, df, (n^{*}mtpert)/b))^{**}(-1)$ for subgroups of size n, variance inflaters b, "steady state" deviations from the target $mtstead$ (i.e., $(\mu_0 - T)^2$ in units of σ_0^2), perturbations from the target $mtpert$ (i.e., $(\mu - T)^2$ in units of σ_0^2) and $df = 1$ for the $(\bar{x} - T)^2$ chart and $df = n$ for the MSE chart.

The ARLs provide insights into the strengths and weaknesses of the control chart procedures. In the case of the Alternate Variables Control chart the ARLs suggest that the procedure's ability to identify changes in the process are comparable to the abilities of the Shewhart procedures. For example, the ARL associated with the $(\bar{x} - T)^2$ chart when the process is in-control and centered at the target (i.e., $\beta = 1$, $(\mu - T)^2 = 0$ and $\lambda_0 = 0$) is 370.4 for all subgroup sizes, which is identical to the ARL for Shewhart's $\bar{x}$ chart when the process is in-control. On the other hand, a process whose "steady state" is such that it is centered $\sqrt{0.5}$ standard deviations from the target (i.e., $(\mu_0 - T)^2/\sigma_0^2 = 0.5$), has an ARL of 71.3 associated with a shift in location of 1 standard deviation (i.e., $(\mu - T)^2/\sigma_0^2 = 1$) when using subgroups of size 4 . The equivalent ARL for the Shewhart Chart is between 120 and 200 (Champ and Woodall (1987)).

The ability of the Alternate Variables Control chart to detect changes in the process varies by situation. For example, in the "steady state" case where the process is centered 1 standard deviation from the target (i.e., $(\mu_0 - T)^2/\sigma_0^2 = 1$) and where the process remains centered 1 standard deviation (i.e., $(\mu - T)^2/\sigma_0^2 = 1$) from the target, as indicated earlier the ARL for subgroup size 3 is 370.4. This result suggests that the average number of subgroups of size 3 encountered prior to an out-of-control signal (where the process remains centered 1 standard deviation from the target) is 370.4. The ARL associated with the Alternate Variables Control Chart in the case where the same process experiences a shift such that it is now centered at the target (i.e., $(\mu - T)^2/\sigma_0^2 = 0$) is 2640.9 suggesting that the chart's ability to detect a process's shift closer to the target is poor. This particular process shift may be more quickly identified through the use of supplemental detection rules (see Section 7). The ARLs associated with the MSE portion of the chart provide similar insights as to how quickly shifts in the variability and/or proximity to the target are picked up by the MSE portion of the chart.

7. Detection Rules

Several of the common detection rules used to enhance the ARLs of the Shewhart procedures can be adapted for use with the Alternate Variables Control chart procedures. The adapted Western Electric (1956) Handbook's Test for (i) two out of three consecutive points between $\mu + 2\sigma$ and $\mu + 3\sigma$ (or $\mu - 2\sigma$ and $\mu - 3\sigma$) becomes two out of three consecutive points between the 95.44% and 99.73% percentiles; (ii) four out of five consecutive points between $\mu + 1\sigma$ and $\mu+3\sigma$ (or $\mu-1\sigma$ and $\mu-3\sigma$) becomes four out of five consecutive point between the 68.26% and 99.73% percentiles (iii) six consecutive points increasing or decreasing becomes six consecutive points with the same sign and either steadily increasing or decreasing; and (iv) fifteen points in the interval $(\mu \pm 1\sigma)$ becomes fifteen consecutive points inside the 68.26% percentile.

The multipliers used in determining the 68.26% and 95.44% for the $(\bar{x} - T)^2$ chart (A6826, A9544) and MSE chart (C6826, C9544) can be calculated using $CINV((1-\alpha),\, 1,\, \lambda)/n$ in the case of A and $CINV((1-\alpha),\, n,\, \lambda)/(n-1)$ in the case of C using $(1-\alpha) = 0.6826$ and $(1-\alpha) = 0.9544$ respectively (SAS (1994)). The 68.26% and 95.44% for the $(\bar{x} - T)^2$ chart become $A6826\sigma^2$ and $A9544\sigma^2$ and can be calculated using $A6826\bar{S}^2$ and $A9544\bar{S}^2$. Similarly the 68.26% and 95.44% for the MSE are $C6826\sigma^2$ and $C9544\sigma^2$ and calculated using $C6826\bar{S}^2$ and $C9544\bar{S}^2$.

The detection rules outlined have similar probabilistic interpretations of the Western Electric rules. Rules (i), (ii) and (iv) have identical probabilistic interpretations of the equivalent Western Electric rules, while rule (iii) has similar but not exact probability interpretations. If $\mu \neq T$ then the probability of occurrence for Rule (iii) will be slightly larger or smaller than the Western Electric rules depending on the magnitude and sign of $(\mu - T)$.

8. Multivariate ARLs, Masking and Identifying Assignable Causes

The proposed multivariate version of the Alternate Variables Control chart summarizes the behaviour of a process with respect to multidimensional targets and dispersion. Analogous to the univariate case the ARLs associated with the multivariate procedures provide insights into the ability of the charting procedures to detect shifts in the process. Assuming the process measurements are $\text{MVN}_p(\underset{\sim}{\mu}_0, \Sigma_0)$ the ARLs associated with $\tilde{T}_p$ and MSE_p for (i) subgroup sizes n, (ii) process perturbations λ_{pert} and "steady states" λ_{ucl} can be defined as

$$ARL_{\tilde{T}_p} = 1/\Big\{1 - \Pr(\tilde{T}_p > UCL_{\tilde{T}_p} | \tilde{T}_p \sim n^{-1}\chi^2_{p,n\lambda_{pert}},$$
$$UCL_{\tilde{T}_p} = n^{-1}\chi^2_{p,n\lambda_{ucl}}(0.9973))\Big\},$$

$$\mathrm{ARL}_{MSE_p} = 1/\{1 - \Pr(MSE_p > UCL_{MSE_p} | MSE_p \sim (n-1)^{-1}\chi^2_{np,n\lambda_{pert}},$$
$$UCL_{MSE_p} = (n-1)^{-1}\chi^2_{np,n\lambda_{ucl}}(0.9973))\},$$

where $\lambda_{pert} = (\underset{\sim}{\mu} - \underset{\sim}{T})'\Sigma^{-1}(\underset{\sim}{\mu} - \underset{\sim}{T})$ represents perturbations resulting from shifts in proximity to the target and/or a changes in the dispersion matrix experienced by the process and $\lambda_{ucl} = (\underset{\sim}{\mu}_0 - \underset{\sim}{T})'\Sigma_0^{-1}(\underset{\sim}{\mu}_0 - \underset{\sim}{T})$ represents the degree of off-targetness associated with the process when in-control. The SAS (1994) code $(1 - PROBCHI(CINV((1-\alpha), df, n^*lucl), df, (n^*lpert)))^{**}(-1)$ produces the ARLs associated with subgroups of size n, dimension p, "steady states" of *lucl*, perturbations of *lpert* and $df = p$ for the $\tilde{T}_p$ chart and $df = np$ for the MSE_p chart.

Although the proposed multivariate procedures allow users to detect simultaneous movements that may not be uncovered by univariate charts, situations may arise where the procedures mask univariate movements. For example, it may occur that one process variable moves closer to its target while simultaneously a second variable moves further from its target, but the values of $\tilde{T}_p$ and MSE_p are unchanged. If practitioners suspect such a situation they may wish to examine the univariate Alternate Variables Control charts for the individual variables. Alternatively practitioners may wish to examine subsets of the p quality variables simultaneously or attach more importance to particular variables and less to others within the set of p quality variables. The contributions of individual quality variables, subsets of the p quality variables or weighted subsets of the quality variables can be examined using the proposed multivariate procedures in conjunction with a weight function.

The weight function is a $p \times p$ diagonal matrix where w_i represents the weight assigned to the ith quality variable such that $0 < w_i < p$ for all i and $\sum_{i=1}^{p} w_i = p$. The resulting measures $\tilde{T}_{p|w}$, $MSE_{p|w}$ and $S^2_{p|w}$ become $(\bar{\underset{\sim}{X}} - \underset{\sim}{T})'\underset{\sim}{W}'\Sigma^{-1}\underset{\sim}{W}(\bar{\underset{\sim}{X}} - \underset{\sim}{T})$, $(n-1)^{-1}\sum_{i=1}^{n}(\underset{\sim}{X}_i - \underset{\sim}{T})'\underset{\sim}{W}'\Sigma^{-1}\underset{\sim}{W}(\underset{\sim}{X}_i - \underset{\sim}{T})$, and $(n-1)^{-1}\sum_{i=1}^{n}(\underset{\sim}{X}_i - \bar{\underset{\sim}{X}})'\underset{\sim}{W}'\Sigma^{-1}\underset{\sim}{W}(\underset{\sim}{X}_i - \bar{\underset{\sim}{X}})$ respectively and provide practitioners with the ability to attach more or less importance to various quality characteristics in the multivariate setting.

Several authors have examined methods for examining and decomposing a variable's (or group of variables) contribution in a multivariate setting (e.g., Hawkins (1993)). The proposed weighting method allows practitioners to attach more or less importance to the individual quality characteristic by decreasing or increasing the individual weights assigned to each. The weight function effectively alters the regions that will produce values of $\tilde{T}_{p|w}$, $MSE_{p|w}$ and $S^2_{p|w}$ that fall below the respective UCLs previously established. These regions will in general be p-dimensional ellipsoids whose axes and shape are altered by changes

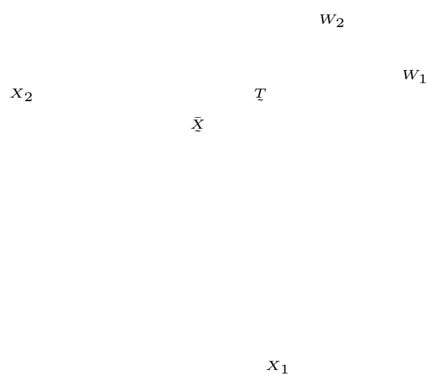

Figure 3. Bivariate bounds for $\bar{X}_1$ for $UCL_{\tilde{T}_{p|w}} = 4.435$ using $W_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $W_2 = \begin{pmatrix} 1.5 & 0 \\ 0 & 0.5 \end{pmatrix}$

When used in this fashion, the weight function effectively alters the covariance structure and the resulting measurement region responsible for values of

$\tilde{T}_{p|w}$ below the $UCL_{\tilde{T}_{p|w}}$. A weight function with $w_1 = 2.0$, $w_2 = 0$ in the bivariate case applies all the influence or weight to x_2 while the one with $w_1 = 0$, $w_2 = 2.0$ applies the entire influence to x_1. Increasing the differences among the w_i's allows the practitioner to shift the influence of the individual characteristics, in turn allowing practitioners to decompose and examine the process measurements in various ways.

9. Comments

A simultaneous control chart has been presented that mirrors recent changes in quality assurance. The proposed chart is analogous to the traditional $\bar{X}$ and S chart providing much of the information available from the traditional control charts while incorporating additional pertinent information. The additional information has been incorporated in a single plot. The calculations required for the univariate case are straightforward requiring only a hand-held calculator. The procedure is easily adapted to computer analysis and graphics.

Many of the properties derived and examined for the traditional control charts apply to the new procedure as the measures used are similar in nature. The $(\bar{x} - T)^2$ portion of the chart allows the practitioner to assess the stability of the location of the process while also providing an assessment of proximity to the target. The MSE portion provides an assessment of the stability of the variability of the process. Replacing the target value in the new procedure with $\bar{\bar{X}}$ results in a plot that provides all the information included in the traditional control chart. However in current philosophy emphasis is placed on proximity to the target value when monitoring a process.

The simultaneous nature of the proposed procedure has been achieved without substituting clarity. The resultant control chart purveys more information than is available in traditional charts while not being dramatically different from the traditional charts in their motivation or inference. Although the new procedure does not provide all the information that may be contained in a boxplot style chart, it does have features such as (i) explicit boundary values for S^2 and MSE and (ii) small sample results that are unavailable in boxplot procedures.

The proposed multivariate procedure is similar to the traditional Hotelling T^2 style of chart but results in a plot that allows investigation of both proximity to the target and overall variability where as Hotelling's procedure confounds these measures. The motivation for the multivariate chart is identical to that for the univariate chart as is also the case for the inferences. The performance of the multivariate chart provides users with serious alternatives to current multivariate charts as their ARLs are similar to those of the univariate Shewhart charts. Detection and weighting rules designed to enhance the performance of the multivariate charting procedures need to be developed.

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