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IMPROVING ON THE MLE OF A POSITIVE NORMAL MEAN

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Abstract. We study the problem of estimating the mean of a univariate normal population when the mean is known to be non-negative and loss is squared error. The MLE is superior to the UMVUE, but it is known that the MLE is itself inadmissible. A long unsolved problem is to find an explicit estimator which dominates the MLE. This paper is devoted to producing a class of such estimators. We believe, but have not proved, that some members of our class are admissible.

Key words and phrases: Minimaxity, squared error loss, location parameter.

1. Introduction

This paper gives explicit improvements to the MLE of a normal mean θ with $\theta \in [0, \infty)$.

Let $X \sim N(\theta, 1)$, $\theta \in [0, \infty)$ and $\delta(X)$ be an estimator of the mean θ with loss

$$L(\theta, \delta) = (\delta - \theta)^2. \tag{1.1}$$

The estimator X is the UMVUE, X_+ is the MLE, where

$$a_+ = \begin{cases} a, & \text{if } a \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\theta \geq 0$, it is clear that X_+ dominates X. It has long been known that the positive-part estimator X_+ is itself inadmissible (see for example, Brown (1971)).

Rukhin (1990) described a fascinating connection whereby asymptotically, the problem of variance estimation can be reduced to the estimation of a positive normal mean.

Katz (1961) showed that X_+ is a minimax estimator of θ. He also showed that the generalized Bayes estimator with respect to the uniform prior on $[0, \infty)$ is an admissible and minimax estimator of θ, but that it does not dominate X_+ uniformly.

Shao and Strawderman (1993) found explicit improvements over the positive-part James-Stein estimator in the problem of estimating the mean vector of a multivariate normal distribution. The present paper grew out of that study and the methods of proof are quite similar in the two papers. The problem of improving the UMVUE of a noncentrality parameter is also closely related and is the subject of a forthcoming paper.

The similarities have to do with the fact that the improving estimator in each of these problems must "wiggle" sufficiently about the estimator to be improved.

Our results fall into two broad classes. The first set of results describe explicit estimators which change X_+ on the set $(-\infty,0]\cup[1,\sqrt{3}]$. These results depend on some properties of central chi-square distributions. These results and other preliminary results are given in Section 2. In Section 3, we study alternative estimators of the form

$$\delta(X)=X_+-ag(X^2)I_{\{1\le X\le\sqrt{3}\}}-akh(X^2)I_{\{X\le 0\}}, \tag{1.2}$$

where $g(\cdot)$ is an even symmetric piecewise linear function about $X^2=2$ with $g(1)=g(3)=0$, $g'(1)<0$ and $|g'(\cdot)|\equiv 1$ a.s. on $[1,3]$. Hence, the simplest function $g(\cdot)$ is "W" shaped on $[1,3]$. To fully specify $g(\cdot)$, then, it suffices to specify the value c^* in (2,3) such that $g(c^*)$ attains its minimum. Values of a and c^* and conditions on k and $h(\cdot)$ are given such that $\delta(X)$ dominates X_+. In Section 4, we give some numerical results and some comments. It is easy to see that estimators of the form (1.2) cannot themselves be admissible. Therefore we investigate a more general class of estimators in Section 5.

This second class of estimators is of the form

$$\delta(X)=X_+-ag(X^2)I_{\{X\ge x_0\}}-akh(X^2)I_{\{X<x_0\}}, \tag{1.3}$$

where x_0 is a given positive number. We give conditions on a, k, $g(\cdot)$ and $h(\cdot)$ such that (1.3) dominates X_+. In this case, $g(\cdot)$ will be "W" shaped on $[x_0,\infty)$ and $h(\cdot)$ will be bounded continuous and nonpositive. We believe, but have not proved, that admissible improvements can be found in this class.

Some proofs of technical results are given in the Appendix.

2. Preliminaries

Our first result in this section is a lemma analogous to Stein's (1981) lemma for the evaluation of expectation of cross products appearing in risk functions. Its proof is straightforward and is omitted.

Lemma 2.1. *Let $X\sim N(\theta,1)$, $H(\cdot)$ be a continuous function on $[a^2,b^2]$, and $H'(\cdot)$ have at most a finite number of discontinuities* $0\le a^2=a_0^2<a_1^2<\cdots<$

$a_k^2 < a_{k+1}^2 = b^2$. *If, for* $i = 0, \ldots, k+1$, *both* $H'((a_i^+)^2)$ *and* $H'((a_i^-)^2)$ *are finite, then*

$$\begin{aligned} &E(X-\theta)XH(X^2)I_{\{a\le X\le b\}} \\ &= E[H(X^2)+2X^2H'(X^2)]I_{\{a\le X\le b\}} + aH(a^2)e^{-(a-\theta)^2/2} - bH(b^2)e^{-(b-\theta)^2/2}. \end{aligned} \tag{2.1}$$

Note that if $H(a^2) = H(b^2) = 0$, then the lemma essentially reduces to Stein's lemma in one dimension.

If $X \sim N(\theta, 1)$, the density of X can be rewritten as

$$\frac{1}{\sqrt{2\pi}} e^{-\theta^2/2} \sum_{j=0}^{\infty} \frac{\theta^j x^j}{j!}\, e^{-x^2/2}.$$

The next series of lemmas have to do with properties of functions $G_n(u)$ defined in Lemma 2.2. These properties play a crucial role in the development of Section 3. The function $G_n(u)$ is essentially (modulo constants) $q_n(2+u) - q_n(2-u)$ restricted to $u \in [0,1]$ where $q_n(u)/[2^{n/2+1}\Gamma(n/2+1)]$ is the density of a central chi-square with $n+2$ degrees of freedom,

$$q_n(u) = e^{-u/2}u^{n/2}.$$

Lemma 2.2. *Let*

$$G_n(u) = e^{-(2+u)/2}(2+u)^{n/2} - e^{-(2-u)/2}(2-u)^{n/2} \tag{2.2}$$

and the domain of $G_n(u)$ *be* $[0,1]$. *The following properties hold*:

1. $[G_2(u)/G_0(u)]' + u \ge 0,$ (2.3)
2. $[G_3(u)/G_1(u)]' + 2u \ge 0,$ (2.4)
3. $G_0(u)$ *and* $G_1(u)$ *are monotone decreasing, and* $G_n(u)$ *is monotone increasing for all* $n \ge 2$. (2.5)

Proof. See Appendix.

Lemma 2.3. *If* $0 < d < 0.5$, *and* $n \ge 4$, *then* $e((1+d)/(3-d))^{n/2}$ *is monotone decreasing in* n, *and*

$$e\Big(\frac{1+d}{3-d}\Big)^n < 1. \tag{2.6}$$

Proof. Since $0 < (1+d)/(3-d) < 1$, (2.6) is monotone decreasing in n, and

$$e\Big(\frac{1+d}{3-d}\Big)^{n/2} < e\Big(\frac{1+d}{3-d}\Big)^2 \le e\Big(\frac{3}{5}\Big)^2 < 1.$$

Lemma 2.4. *Let* $F_n(u) = e^{-(2+u)/2}(2+u)^{1/2+n} - e^{(2-u)/2}(2-u)^{1/2+n}$. *Then for* $n \geq 2$

$$F_{n+1}(u) = 4F_n(u) - (4-u^2)F_{n-1}(u), \tag{2.7}$$

$$\begin{aligned} F_{n+1}(u) - (2+u)F_n(u) &= (2-u)[F_n(u) - (2+u)F_{n-1}(u)] \\ &= (2-u)^n[F_1(u) - (2+u)F_0(u)], \end{aligned} \tag{2.8}$$

$$F_{n+1}(u) = \frac{(2+u)^{n+1} - (2-u)^{n+1}}{2u}(F_1(u) - (2+u)F_0(u)) + (2+u)^{n+1}F_0(u) \tag{2.9}$$

and

$$G_{2n+1}(u) = F_n(u). \tag{2.10}$$

Proof. See Shao and Strawderman (1993), Lemma 2.5 with $p = 3$.

A key result of this section is the following:

Theorem 2.1. *Let* $u \in [0,1]$. *For all* $j \geq 3$, $G_{2j+2}(u)/G_{2j}(u)$ *and* $G_{2j+3}(u)/G_{2j+1}(u)$ *are both positive monotone increasing functions of* u.

Proof. See Appendix.

The next lemmas use these properties of $G_n(u)$'s to establish inequalities used in the remainder of the paper.

Lemma 2.5. *Let* $c_j \in (0,1)$ *be such that* $\int_0^{c_j} G_j(u)du = \int_{c_j}^1 G_j(u)du$; *then for all* $j \geq 6$, $c_j < c_{j+2}$.

Proof. See Appendix.

By calculating we have the following results:

Table 2.1. The table of c_j

j	0	1	2	3	4	5
c_j	0.710751	0.692336	0.841921	0.712458	0.702803	0.699699

j	6	7	8	9	10	11
c_j	0.699742	0.702042	0.706146	0.711710	0.718439	0.726062

If we take c to be any number between (c_0, c_9), and let

$$I(j) = -\int_0^c G_j(u)du + \int_c^1 G_j(u)du, \tag{2.11}$$

then $I(j) > 0$ for $j = 0, 1, 2, 3$ and all $j \geq 9$.

3. A Class of Improved Estimators

In this section we use the results of Section 2, particularly Theorem 2.1, to find a class of estimators dominating X_+ for the loss (1.1). We consider estimators of the form

$$\delta(a, g, h, X) = X_+ - ag(X^2)I_{\{1 \le X \le \sqrt{3}\}} - akh(X^2)I_{\{X<0\}}, \tag{3.1}$$

where $g(t)$ is an even symmetric linear function around the point $t = 2$ such that $g(1) = g(3) = 0$.

Theorem 3.1. *Let $X \sim N(\theta, 1)$ with $\theta \in [0, \infty)$. The loss function is given by $(d - \theta)^2$. Let $c \in (c_0, c_9)$ (see Lemma 2.5) and d satisfy*

$$\int_0^c G_9(u)du \le \int_c^{1-d} G_9(u)du \tag{3.2}$$

and

$$\int_0^c G_{10}(u)du \le \int_c^{1-d} G_{10}(u)du.$$

Define

$$g(t) = \begin{cases} 1 + 2c - t, & \text{if } 2 \le t < 2 + c, \\ t - 3, & \text{if } 2 + c \le t < 3, \end{cases} \tag{3.3}$$

and extend the definition of $g(t)$ to $[1, 2)$ so that $g(t)$ is symmetric about $t = 2$, and

$$0 \ge h(t) \ge -1. \tag{3.4}$$

Let $0 < \underline{\theta} < \overline{\theta}$ be two constants such that if $\theta \ge \overline{\theta}$, then

$$\frac{\theta^9}{9!}\int_0^1 g'(2+u)G_9(u)du + 2\sum_{j=4}^{8}\frac{\theta^j}{j!}\int_0^{1-d} g'(2+u)G_j(u)du \ge 0, \tag{3.5}$$

and if $0 \le \theta \le \underline{\theta}$, then

$$\int_0^1 g'(2+u)G_0(u)du + 2\sum_{j=4}^{8}\frac{\theta^j}{j!}\int_0^{1-d} g'(2+u)G_j(u)du \ge 0. \tag{3.6}$$

If $k \ge K$ and $0 < a \le \min(A, A_1, B(k), D(k))$ where

$$A = \frac{d}{2}\Big[1 - e\Big(\frac{1+d}{3-d}\Big)^2\Big], \tag{3.7}$$

$$A_1 = \min\Big\{\frac{4\int_1^3 g'(t)q_j(t)dt}{\int_1^3 g^2(t)q_{j-1}(t)dt} \ |j = 0, 1, 2, 3\Big\}, \tag{3.8}$$

$$K = \frac{2\sum_{j=4}^{8} \frac{\overline{\theta}^j}{j!} |\int_0^{1-d} g'(2+u)G_j(u)du|}{\underline{\theta}\int_0^\infty |h(x^2)|e^{-x^2/2-x\overline{\theta}}dx}, \tag{3.9}$$

$$B(k) = \frac{\int_1^3 g'(t)e^{-t/2}dt}{\int_1^{\sqrt{3}} g^2(x^2)e^{-x^2/2}dx + k^2\int_0^\infty h^2(x^2)e^{-x^2/2}dx}, \tag{3.10}$$

and

$$D(k) = \frac{\underline{\theta}}{k}, \tag{3.11}$$

then

$$\delta(X) = X_+ - ag(X^2)I_{\{1\le X\le\sqrt{3}\}} - akh(X^2)I_{\{-\infty<X\le 0\}}$$

dominates X_+.

Proof. The difference in risk between $\delta(X)$ and X_+ is given by

$$\begin{aligned}\Delta R(\theta) &= R(\delta(X),\theta) - R(X_+,\theta)\\ &= E[a^2g^2(X^2) - 4axg'(X^2)]I_{\{1\le X\le\sqrt{3}\}}\\ &\quad + E[(ak)^2h^2(X^2) + 2ak\theta h(X^2)]I_{\{X<0\}},\end{aligned}$$

so

$$\begin{aligned}\sqrt{2\pi}\Delta R(\theta) &= \int_1^{\sqrt{3}}[a^2g^2(x^2) - 4axg'(x^2)]e^{-(x-\theta)^2/2}dx\\ &\quad + \int_0^\infty [(ak)^2h^2(x^2) + 2ak\theta h(x^2)]e^{-(x+\theta)^2/2}dx\\ &= a^2\int_1^{\sqrt{3}} g^2(x^2)e^{-(x-\theta)^2/2}dx - 2ae^{-\theta^2/2}\sum_{j=0}^\infty \frac{\theta^j}{j!}\int_1^3 g'(t)e^{-t/2}t^{j/2}dt\\ &\quad + (ak)^2\int_0^\infty h^2(x^2)e^{-(x+\theta)^2/2}dx + 2ak\theta\int_0^\infty h(x^2)e^{-(x+\theta)^2/2}dx. \end{aligned} \tag{3.12}$$

Consider case I: $\theta \ge \overline{\theta}$:

Take $ak \le \underline{\theta}$, then the sum of the last two terms of (3.12) is non-positive. By Lemma 2.5 and (3.5),

$$\begin{aligned}2\sum_{j=4}^\infty \frac{\theta^j}{j!}\int_1^3 g'(t)q_j(t)dt &\ge 2\sum_{j=4}^8 \frac{\theta^j}{j!}\int_{1-d}^1 g'(2+u)G_j(u)du + \sum_{j=9}^\infty \frac{\theta^j}{j!}\int_1^3 g'(t)q_j(t)dt\\ &> \sum_{j=4}^\infty \frac{\theta^j}{j!}\int_{1-d}^1 G_j(u)du.\end{aligned}$$

When $j \geq 4$, then by Lemma 2.3 and since $q_j(t)$ is monotone increasing, we have

$$\begin{aligned}\int_{1-d}^{1} G_j(u)du &= \int_{1-d}^{1} e^{-(2+u)/2}(2+u)^{j/2}[1-e^u(\frac{2-u}{2+u})^{j/2}]du \\ &\geq [1-e(\frac{1+d}{3-d})^{j/2}]\int_{3-d}^{3} q_j(t)dt \\ &\geq \frac{d}{2}[1-e(\frac{1+d}{3-d})^2]\int_{1}^{3} q_j(t)dt \\ &= A\int_{1}^{3} q_j(t)dt.\end{aligned}$$

$$\begin{aligned}\sum_{j=4}^{\infty}\frac{\theta^j}{j!}\int_{1-d}^{1} g'(2+u)G_j(u)du &> A\sum_{j=4}^{\infty}\frac{\theta^j}{j!}\int_{1}^{3} g^2(t)q_j(t)dt \\ &> A\sum_{j=4}^{\infty}\frac{\theta^j}{j!}\int_{1}^{3} g^2(t)q_{j-1}(t)dt.\end{aligned}$$

By (3.8) and (3.11), (3.12) is non-positive.

Consider case II: $0 \leq \theta < \underline{\theta}$:

Because $h(x^2)$ is non-positive,

$$\int_0^{\infty} h(x^2)e^{-x^2/2}dx < 0.$$

And by (3.12), (3.4), (3.6) (3.10) and (3.8), we know that if $0 < a < \min\{A, A_1, D(k)\}$, then

$$\begin{aligned}\sqrt{2\pi}\Delta R(\theta) \leq\, & a^2 e^{-\theta^2/2}\int_1^{\sqrt{3}} g^2(x^2)e^{-x^2/2}dx \\ & + (ak)^2\int_0^{\infty} h^2(x^2)e^{-(x+\theta)^2/2}dx - ae^{-\theta^2/2}\int_1^3 g'(t)q_0(t)dt \\ & < e^{-\theta^2/2}a^2\{\int_1^{\sqrt{3}} g^2(x^2)e^{-x^2/2}dx + k^2\int_0^{\infty} h^2(x^2)e^{-x^2/2}dx\} \\ & - ae^{-\theta^2/2}\int_1^3 g'(t)q_0(t)dt \leq 0.\end{aligned}$$

Consider case III: $\underline{\theta} \leq \theta \leq \overline{\theta}$:

By (3.12) and (3.10),

$$\begin{aligned}\sqrt{2\pi}\Delta R(\theta) \leq & -2ae^{-\theta^2/2}\sum_{j=4}^{8}\frac{\theta^j}{j!}\int_0^{1-d} g'(2+u)G_j(u)du \\ & + a^2k^2\int_0^{\infty} h^2(x^2)e^{-(x+\theta)^2/2}dx + 2ak\theta\int_0^{\infty} h(x^2)e^{-(x+\theta)^2/2}dx \\ \leq\, & ak\theta\int_0^{\infty} h(x^2)e^{-(x+\theta)^2/2}dx - 2ae^{-\theta^2/2}\sum_{j=4}^{8}\frac{\theta^j}{j!}\int_0^{1-d} g'(2+u)G_j(u)du.\end{aligned}$$

This last expression is negative by (3.9). Hence $\Delta R(\theta) \le 0$. This completes the proof.

Next, we give some value of c, A, N, k and a for which the estimator given in the theorem improves X_+.

If $c = 0.711$, since $c_0 < c < c_9$, then, by the inequalities

$$\int_0^c G_9(u)du \le \int_c^{1-d} G_9(u)du$$

and

$$\int_0^c G_{10}(u)du \le \int_c^{1-d} G_{10}(u)du$$

we get $d > 0.000985$.

Let

$$g(t) = \begin{cases} 1-t, & \text{if } 1 \le t < 1.289, \\ t-1.578, & \text{if } 1.289 \le t < 2, \\ 2.422-t, & \text{if } 2 \le t < 2.711, \\ t-3, & \text{if } 2.711 \le t < 3. \end{cases}$$

It can be seen, from the following table,

j	0	1	2	3
$\int_1^3 g'(t)q_j(t)dt$	0.000133764	0.00633025	0.00768492	0.00111563
$\int_1^3 g^2(t)q_{j-1}(t)dt$	0.0231128	0.031081	0.0424611	0.0588857

that $A_1 \ge 0.0231498$.

Take $N \ge 12$. We can get $\overline{\theta} = 6$ and $\underline{\theta} = 1.5$.

j	4	5	6	7	8
$\int_0^{1-d} g'(2+u)G_j(u)du$	$-.01180561$	$-.0506126$	$-.0950321$	$-.13757$	$-.137936$

Let $h(\cdot) = -1$, so $K \ge 97.7063$. If we take $k = 98$, then

$$B(98) = 1.11129 \times 10^{-8}, \quad D(98) = 0.0153061.$$

So

$$\delta(X) = X_+ - 1.11129 \times 10^{-8} g(X^2) I_{\{1 \le X \le \sqrt{3}\}} + 0.00000108906 I_{\{-\infty < X \le 0\}}$$

dominates X_+.

4. Some Numerical Results

Theorem 3.1 gives conditions on a and k, such that $\delta_{a,k}(X) = X_+ - ag(X^2) I_{\{1<X<\sqrt{3}\}} + akI_{\{X<0\}}$ dominates X_+. The calculations at the end of Section 3 indicate that the constant a must be quite small and the constant k quite large in order to satisfy the conditions of the theorem.

The following tables suggest however that a could be substantially larger and k substantially smaller and still lead to an estimator dominating X_+. For example, if a changes from 10^{-8} to 10^{-4} and k from 100 to 1 simultaneously, it appears that the resulting estimator still dominates X_+.

Table 4.1. $\frac{\Delta R(\theta,a,k)}{a}$, for $a = 10^{-4}$, $k = 50, 10, 1$

θ	$k = 50$	$k = 10$	$k = 1$
0	−0.00000445821	−0.0000345377	−0.0000357785
0.4	−34.5548	−6.91548	−0.696626
0.8	−42.4934	−8.50671	−0.859693
1.2	−34.6221	−6.93217	−0.701931
2.0	−11.403	−2.27878	−0.225842
4.0	−0.0311755	−0.00577131	−0.0000553673
6.0	−0.00000514318	−0.00000395614	−0.00000368905
8.0	$-2.60539\ 10^{-10}$	$-2.59559\ 10^{-10}$	$-2.59328\ 10^{-10}$

Table 4.2. $\frac{\Delta R(\theta,a,k)}{a}$, for $a = 10^{-8}$, $a = 10^{-7}$, $a = 10^{-6}$, $a = 10^{-5}$, $a = 10^{-4}$, and $k = 0.5$

θ	$a = 10^{-8}$	$a = 10^{-7}$	$a = 10^{-6}$	$a = 10^{-5}$	$a = 10^{-4}$
0	−0.000267054	−0.000267296	−0.00026521	−0.000244354	−0.0000357879
0.4	−0.351529	−0.351529	−0.351525	−0.35149	−0.351134
0.8	−0.435434	−0.435434	−0.435429	−0.435377	−0.434859
1.2	−0.35652	−0.35652	−0.356513	−0.356449	−0.355807
2.0	−0.11247	−0.11247	−0.112464	−0.112402	−0.111789
4.0	0.00022486	0.000224893	0.000225229	0.000228589	0.000262185
6.0	−0.00000371233	−0.00000371229	−0.00000370852	−0.00000372713	−0.00000367421
8.0	$-2.60053\ 10^{-10}$	$-2.60034\ 10^{-10}$	$-2.60046\ 10^{-10}$	$-2.5998\ 10^{-10}$	$-2.59328\ 10^{-10}$

The tables also indicate that k cannot be too small. The numerical results show that if $k = 0.5$ for example, then for any $10^{-8} < a < 10^{-4}$, $\delta_{a,0.5}(X)$ cannot dominate X_+.

The tables also indicate that the maximal improvement is quite small. This is consistent also with the relatively small gains obtainable over the positive-part James-Stein estimator in the multivariate normal case (see Shao and Strawderman (1994)).

5. More General Classes

The results of the previous two sections presented estimates which change the values of X_+ only on the set $(-\infty,0)\cup[1,\sqrt{3}]$. While the estimators improve upon X_+, they cannot themselves be generalized Bayes and hence are not admissible.

In this section, we present a class of improved estimators which allow change for X in $(-\infty,\infty)$ and which we believe (but have not proved) contain admissible improvements.

Theorem 5.1. *Let $g(x^2)$ be a continuous and piecewise differentiable W-shape function defined on $[x_0,\infty)$ with $g(x_0^2)<0$ and $g(\infty)=0$, i.e. there exist $0<x_0<x_1<x_2<x_3<x_4<x_5<x_6<\infty$ such that*

$$g(x_2^2)=g(x_4^2)=0$$

and

$$g'(x^2)=\begin{cases}\geq 0, & \text{if } x_1<x<x_3 \text{ or } x_5<x<\infty,\\ \leq 0, & \text{if } x_0<x<x_1 \text{ or } x_3<x<x_5.\end{cases}$$

Let $h(x^2)$ be a non-negative continuous function defined on $[-\infty,0]$, with $h(-\infty)=0$. Let $G(x^2)$ be a continuous function defined on $[0,x_0]$ with $0\leq h(0)=G(0)\leq 1$ and $G(x_0^2)=g(x_0^2)$.

Assume

(1) $\int_{x_0}^{x_6} g'(x^2)e^{-x^2/2}x\,dx>0$.

(2) *There exists a positive integer J such that*

$$\int_{x_0}^{x_6} g'(x^2)e^{-x^2/2}x^J dx>0 \qquad \text{and} \qquad \int_{x_3}^{x_6} g'(x^2)e^{-x^2/2}x^J dx>0.$$

(3) *For all $x\geq x_6$, $g^2(x^2)\leq Dxg'(x^2)$ for some fixed $D>0$.*

(4) *Let $0<B_1<B_2$ be two constants, N a fixed integer ($N\geq J$) such that if $\theta\geq B_2$, then*

$$\frac{\theta^N}{N!}\int_{x_0}^{x_6} g'(x^2)e^{-x^2/2}x^{N+1}dx\geq\frac{4}{3}\sum_{j=1}^{N-1}\frac{\theta^j}{j!}\Big|\int_{x_0}^{x_6} g'(x^2)e^{-x^2/2}x^{j+1}dx\Big|,$$

and if $0 \le \theta \le B_1$, *then*

$$\int_{x_0}^{x_6} g'(x^2)e^{-x^2/2}xdx \ge 4\sum_{j=1}^{N-1}\frac{\theta^j}{j!}\Big|\int_{x_0}^{x_6} g'(x^2)e^{-x^2/2}x^{j+1}dx\Big|.$$

(5) *Let*

$$k \ge \frac{2\sum_{j=1}^{N-1}|\int_{x_0}^{x_6} g'(x^2)e^{-x^2/2}x^{j+1}dx|\frac{B_2^j}{j!}}{B_1\int_0^\infty h(x^2)e^{-x^2/2-B_2x}dx}.$$

(6) *Let* $G(\cdot)$ *satisfy the following two extra conditions*:

$$k^2\int_0^{x_0} G^2(x^2)e^{-x^2/2}dx \le \int_{x_0}^\infty g^2(x^2)e^{-x^2/2}dx$$

and

$$2k\int_0^{x_0} xG(x^2)e^{-x^2/2}dx \le \int_{x_6}^\infty g'(x^2)e^{-x^2/2}xdx.$$

(7)

$$A_1 = \frac{1.5\int_{x_6}^\infty g'(x^2)e^{-x^2/2}x^{J+1}dx}{\int_{x_0}^{x_6} g^2(x^2)e^{-x^2/2}dx + D\int_{x_6}^\infty g'(x^2)e^{-x^2/2}xdx}.$$

(8)

$$A_2 = \frac{3\int_{x_0}^\infty g'(x^2)e^{-x^2/2}xdx}{k^2\int_0^\infty h^2(x^2)e^{-x^2/2}dx + 2\int_{x_0}^\infty g^2(x^2)e^{-x^2/2}dx}.$$

(9) $0 < a < \min(A_1, A_2)$.
Then

$$\delta(X) = X_+ - ag(X^2)I_{\{x_0\le X<\infty\}} + akG(X^2)I_{\{0\le X<x_0\}} + akh(X^2)I_{\{-\infty<X<0\}}$$

dominates X_+.

Proof. The difference in risk between $\delta(X)$ and X_+ is given by

$$\begin{aligned}\Delta R(\theta) &= R(\theta,\delta(X)) - R(\theta,X_+)\\ &= \frac{1}{\sqrt{2\pi}}\int_{x_0}^\infty [a^2g^2(x^2) - 4axg'(x^2)]e^{-(x-\theta)^2/2}dx\\ &\quad + \frac{1}{\sqrt{2\pi}}\int_0^{x_0}[a^2k^2G^2(x^2) + 2(x-\theta)akG(x^2)]e^{-(x-\theta)^2/2}dx\\ &\quad + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^0[a^2k^2h^2(x^2) - 2ak\theta h(x^2)]e^{-(x-\theta)^2/2}dx.\end{aligned}$$

So

$$\sqrt{2\pi}e^{\theta^2/2}\Delta R(\theta) \le a^2k^2\int_0^\infty h^2(x^2)e^{-x^2/2}dx + a^2k^2\int_0^{x_0} G^2(x^2)e^{-x^2/2+\theta x}dx$$
$$+ a^2\int_{x_0}^\infty g^2(x^2)e^{-x^2/2+\theta x}dx + 2ak\int_0^{x_0} G(x^2)e^{-x^2/2+\theta x}xdx$$
$$- 4a\int_{x_0}^\infty g'(x^2)e^{-x^2/2+\theta x}xdx - 2ak\theta\int_0^\infty h(x^2)e^{-x^2/2-\theta x}dx.$$

By assumption (6),

$$\sqrt{2\pi}e^{\theta^2/2}\Delta R(\theta)$$
$$\le a^2k^2\int_0^\infty h^2(x^2)e^{-x^2/2}dx + 2a^2\sum_{j=0}^\infty\int_{x_0}^\infty g^2(x^2)e^{-x^2/2}\frac{\theta^j x^j}{j!}dx$$
$$- 3a\sum_{j=0}^\infty\int_{x_6}^\infty g'(x^2)e^{-x^2/2}\frac{\theta^j}{j!}x^{j+1}dx - 4a\sum_{j=0}^\infty\int_{x_0}^{x_6} g'(x^2)e^{-x^2/2}\frac{\theta^j}{j!}x^{j+1}dx$$
$$- 2ak\theta\int_0^\infty h(x^2)e^{-x^2/2-\theta x}dx.$$

By assumptions (2), (3), (7) and (9), for all $j \ge 1$,

$$2a\int_{x_0}^\infty g^2(x^2)e^{-x^2/2}x^j dx - 3\int_{x_6}^\infty g'(x^2)e^{-x^2/2}x^{j+1}dx$$
$$\le 2a\int_{x_0}^{x_6} g^2(x^2)e^{-x^2/2}x^j dx - (3-2aD)\int_{x_6}^\infty g'(x^2)e^{-x^2/2}x^{j+1}dx$$
$$\le x_6^j[2a\int_{x_0}^{x_6} g^2(x^2)e^{-x^2/2}dx - (3-2aD)\int_{x_6}^\infty g'(x^2)e^{-x^2/2}xdx]$$
$$\le 0,$$

so

$$\sqrt{2\pi}e^{\theta^2/2}\Delta R(\theta) \le a^2k^2\int_0^\infty h^2(x^2)e^{-x^2/2}dx + 2a^2\int_{x_0}^\infty g^2(x^2)e^{-x^2/2}dx$$
$$- 3a\int_{x_6}^\infty g'(x^2)e^{-x^2/2}xdx - 4a\int_{x_0}^{x_6} g'(x^2)e^{-x^2/2}xdx$$
$$- 3a\sum_{j=1}^\infty\int_{x_6}^\infty g'(x^2)e^{-x^2/2}\frac{\theta^j}{j!}x^{j+1}dx$$
$$- 4a\sum_{j=1}^\infty\int_{x_1}^{x_6} g'(x^2)e^{-x^2/2}\frac{\theta^j}{j!}x^{j+1}dx$$
$$- 2ak\theta\int_0^\infty h(x^2)e^{-x^2/2-\theta x}dx.$$

By assumptions (8) and (9), for all $j \geq N > J$, $\int_{x_0}^{x_6} g'(x^2)e^{-x^2/2}x^{j+1}dx > 0$. Therefore

$$\begin{aligned}\sqrt{2\pi}e^{\theta^2/2}\Delta R(\theta) \leq & - a\int_{x_0}^{x_6} g'(x^2)e^{-x^2/2}xdx - 3a\int_{x_6}^{\infty} g'(x^2)e^{-x^2/2}\frac{\theta^N}{N!}x^{N+1}dx \\ & - 2ak\theta\int_0^{\infty} h(x^2)e^{-x^2/2-\theta x}dx \\ & + 4a\sum_{j+1}^{N-1}\frac{\theta^j}{j!}|\int_{x_0}^{x_6} g'(x^2)x^{-x^2/2}x^{j+1}dx| \\ \leq & \ 0,\end{aligned}$$

by assumptions (4) and (5). This completes the proof.

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Appendix

We give the proofs of several results in the body of the paper.

Proof of Lemma 2.2. Since

$$\frac{G_3(u)}{G_1(u)} = \frac{(2+u)^{3/2} - (2-u)^{3/2}e^u}{(2+u)^{1/2} - (2-u)^{1/2}e^u},$$

then

$$\begin{aligned}&[G_3(u)]'G_1(u) - [G_1(u)]'G_3(u) + 2u[G_1(u)]^2 \\ &= e^{-2}\{[(2+u)(1+2u)e^{-u} - (2-u)(1-2u)e^u] - [\frac{4u}{(4-u^2)^{1/2}} + 2u(4-u^2)^{1/2}]\}.\end{aligned}$$

Since $(4-u^2) \in [3,4]$, and the function $2/v + v$ is monotone increasing on $[\sqrt{3}, 2]$, thus

$$\frac{4u}{(4-u^2)^{1/2}} + 2u(4-u^2)^{1/2} = 2u[\frac{2}{(4-u^2)^{1/2}} + (4-u^2)^{1/2}] \leq 6u \tag{A.1}$$

and

$$\begin{aligned}&(2+u)(1+2u)e^{-u} - (2-u)(1-2u)e^u - 6u \\ &= -4\{\frac{u}{1!} + \frac{u^3}{3!} + \frac{u^5}{5!} + \cdots\} + 10u\{1 + \frac{u^2}{2!} + \frac{u^4}{4!} + \cdots\} \\ &\quad - 4u^2\{u + \frac{u^3}{3!} + \frac{u^5}{5!} + \cdots\} - 6u\end{aligned}$$

$$\geq u^3\{\frac{10}{2!}+\frac{10}{4!}u^2+\frac{10}{6!}u^4\}-u^3\{\frac{4}{3!}+\frac{4}{5!}u^2+\frac{4}{7!}u^4+\frac{4}{7!}u^6\}$$
$$-u^3\{4+\frac{4}{3!}u^2+\frac{4}{5!}u^4+\frac{4}{7!}u^6+\frac{8}{9!}u^6\}$$
$$\geq 0.$$

Hence (2.4) is true. Also

$$[\frac{G_2(u)}{G_0(u)}]'+u=[\frac{(2+u)-(2-u)e^u}{1-e^u}]'+u \tag{A.2}$$
$$=(1-e^u)^{-2}\{(1-e^u+ue^u)(1-e^u)+e^u(2+u-2e^u+ue^u)+u(1-e^u)^2\}. \tag{A.3}$$

Denote the numerator of (A.3) as $z(u)$. Then

$$z'(u)=1-e^{2u}+2ue^{2u}, \tag{A.4}$$
$$z''(u)=4ue^{2u}\geq 0,$$
$$z(0)=z'(0)=0,$$

so (2.3) is true. (2.5) is trivial.

Proof of Lemma 2.5. Let

$$\frac{G_{j+2}(u)}{G_j(u)}=\beta_j(u). \tag{A.5}$$

Since $G_j(u)\geq 0$, $G_{j+2}(u)\geq 0$, and by Theorem 2.1, $\beta(u)$ is monotone increasing, we have

$$-\int_0^{c_j}G_{j+2}(u)du+\int_{c_j}^1 G_{j+2}(u)du=-\int_0^{c_j}G_j(u)\beta_j(u)du+\int_{c_j}^1 G_j(u)\beta_j(u)du$$
$$>\beta_j(c_j)[-\int_0^{c_j}G_j(u)du+\int_{c_j}^1 G_j(u)du]$$
$$=0.$$

Therefore $c_{j+2}>c_j$.

Proof of Theorem 2.1. The positive property is trivial. Let $t=2+u$, $s=2-u$ and $F_j(u)=e^{-t/2}t^{p/2+j-1}-e^{-s/2}s^{p/2+j-1}$ with $p=3$. So

$$G_{2j+1}(u)=F_j(u). \tag{A.6}$$

By Lemma 2.3.1 of Katz (1961)

$$\frac{G_{2j+3}(u)}{G_{2j+1}(u)}=\frac{F_{j+1}(u)}{F_j(u)}=4+(u^2-4)\frac{F_{j-1}(u)}{F_j(u)},$$

so

$$[\frac{G_{2j+3}(u)}{G_{2j+1}(u)}]' = 2u\frac{F_{j-1}(u)}{F_j(u)} + (u^2-4)[\frac{F_{j-1}(u)}{F_j(u)}]'$$
$$= 2u\frac{G_{2j-1}(u)}{G_{2j+1}(u)} + (u^2-4)[\frac{G_{2j-1}(u)}{G_{2j+1}(u)}]'. \quad (A.7)$$

If we can show $G_9(u)/G_7(u)$ to be monotone increasing, then for all $j \geq 3$, $G_{2j+3}(u)/G_{2j+1}(u)$ will also be increasing. Here

$$\frac{G_9(u)}{G_7(u)} = \frac{F_4(u)}{F_3(u)} = \frac{\frac{F_4(u)}{F_0(u)}}{\frac{F_3(u)}{F_0(u)}} = \frac{(t^4-s^4)\frac{F_1(u)}{F_0(u)} - ts(t^3-s^3)}{(t^3-s^3)\frac{F_1(u)}{F_0(u)} - ts(t^2-s^2)}. \quad (A.8)$$

Since

$$[\frac{F_4(u)}{F_0(u)}]'[\frac{F_3(u)}{F_0(u)}] - [\frac{F_4(u)}{F_0(u)}][\frac{F_3(u)}{F_0(u)}]'$$
$$= [\frac{F_1(u)}{F_0(u)}]^2(t^2-s^2)^3 - (ts)'\frac{F_1(u)}{F_0(u)}(ts)^2(t-s)^2 + ts[\frac{F_1(u)}{F_0(u)}]'(ts)^2(t-s)^2$$
$$- 2ts\frac{F_1(u)}{F_0(u)}(t-s)(t^2-s^2)^2 + (ts)^2(t^2-s^2)(t-s)^2 \quad (A.9)$$
$$= [\frac{F_1(u)}{F_0(u)}]^2(8u)^3 + 8[-\frac{F_1(u)}{F_0(u)}](4-u^2)u^3(28+u^2)$$
$$+ 4(4-u^2)^2u^2\{8u + (4-u^2)[\frac{F_1(u)}{F_0(u)}]'\},$$

then, by Lemma 2.2, the last term of (A.9) is non-negative. Hence $G_9(u)/G_7(u)$ is monotone increasing. Similarly

$$[\frac{G_8(u)}{G_6(u)}]' \geq 0.$$

A similar argument, shows that $G_{2j+2}(u)/G_{2j}(u)$ is monotone increasing for all $j \geq 3$.

References

Brown, L. D. (1971). Admissible estimators, recurrent diffusions, and insoluble boundary value problems. Ann. Math. Statist. **42**, 855-903.

Katz, M. W. (1961). Admissible and minimax estimates of parameters in truncated spaces. Ann. Math. Statist. **32**, 136-142.

Rukhin, A. L. (1990). Comment on 'Developments in decision-theoretic variance estimation', by Maatta, J. M. and Casella, G. Statist. Sci. **5**, 113-116.

Shao, Peter Yi-Shi and Strawderman, William E. (1994). Improving on the James-Stein positive part estimator. Ann. Statist. **22**, 1517-1538.

Shao, Peter Yi-Shi and Strawderman, William E. (1993). Improving on the positive part of the UMVUE of a noncentrality parameter. J. Multivariate Anal. To appear.

Stein, C. M. (1981). Estimation of the mean of a multivariate normal distribution. Ann. Statist. **9**, 1135-1151.

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