Supplement to "Two-sample tests for relevant differences in the eigenfunctions of covariance operators"

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Abstract

This supplement contains the technical details required for the arugments given in Section 2.2 of the main paper.

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1. Technical details

We begin with the proof of Proposition 2.1.

1.1 Proof of Proposition 2.1

Below let $\int := \int_0^1$. According to the definitions of $\hat{\tau}_j^X(\lambda), \hat{v}_j^X(t,\lambda), \tau_j^X$, and v_j^X, a simple calculation shows that for almost all $t \in [0,1]$,

$$\int (C^X(t,s) + (\hat{C}_m^X(t,s,\lambda) - C^X(t,s)))(v_j^X(s) + (\hat{v}_j^X(s,\lambda) - v_j^X(s)))ds \qquad (6.1)$$
$$= (\tau_j^X + (\hat{\tau}_j^X(\lambda) - \tau_j^X))(v_j^X(t) + (\hat{v}_j^X(t,\lambda) - v_j^X(t))).$$

The sequence $\{v_j^X\}_{j\in\mathbb{N}}$ forms an orthonormal basis of $L^2([0,1])$, and hence for each natural number j there exist coefficients $\{\xi_{i,\lambda}\}_{i\in\mathbb{N}}$ such that

$$\hat{v}_j^X(t,\lambda) - v_j^X(t) = \sum_{i=1}^{\infty} \xi_{i,\lambda} v_i^X(t), \qquad (6.2)$$

for almost every t in $[0,1]$. By rearranging terms in (6.1), we see that

$$\int C^X(t,s)(\hat{v}_j^X(s,\lambda) - v_j^X(s))ds + \int \left(\hat{C}_m^X(t,s,\lambda) - C^X(t,s)\right) v_j^X(s)ds \qquad (6.3)$$

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$$= \tau_j^X (\hat{v}_j^X(t,\lambda) - v_j^X(t)) + \left(\hat{\tau}_j^X(\lambda) - \tau_j^X\right) v_j^X(t) + G_{j,m}(t,\lambda),$$

where

$$G_{j,m}(t,\lambda) = \int [C^X(t,s) - \hat{C}_m^X(t,s,\lambda)][\hat{v}_j^X(s,\lambda) - v_j^X(s)]ds + [\hat{\tau}_j^X(\lambda) - \tau_j^X][\hat{v}_j^X(t,\lambda) - v_j^X(t)].$$

Taking the inner product on the left and right hand sides of (6.3) with v_k, for $k \neq i$, and employing (6.2) yields

$$\tau_k^X \xi_{k,\lambda} + \iint \left(\hat{C}_m^X(t,s,\lambda) - C^X(t,s)\right) v_j^X(s) v_k^X(t) ds dt = \tau_j^X \xi_{k,\lambda} + \langle G_{j,m}(\cdot,\lambda), v_k^X \rangle,$$

which implies that

$$\xi_{k,\lambda} = \frac{\langle \hat{C}_m^X(\cdot,\cdot,\lambda) - C^X, v_j^X \otimes v_k^X \rangle}{\tau_j^X - \tau_k^X} - \frac{\langle G_{j,m}(\cdot,\lambda), v_k^X \rangle}{\tau_j^X - \tau_k^X}, \tag{6.4}$$

for all $\lambda \in [0,1]$ and $k \neq i$. Furthermore, by the parallelogram law,

$$\xi_{i,\lambda} = \langle v_j^X, \hat{v}_j^X(\cdot,\lambda) - v_j^X \rangle = -\frac{1}{2} \|\hat{v}_j^X(\cdot,\lambda) - v_j^X\|^2. \tag{6.5}$$

Let $S_{j,X} = \min\{\tau_{j-1}^X - \tau_j^X, \tau_j^X - \tau_{j+1}^X\}$ for $j \geq 2$ and $S_{1,X} = \tau_1^X - \tau_2^X$. By Assumption 2.3 and the fact that $j \leq d$ we have $S_{j,X} > 0$. Hence, Lemma 2.2 in Horváth and Kokoszka (2012) (see also Section 6.1 of Gohberg et al. (1990)) implies for all $\lambda \in [0,1]$,

$$\sqrt{\lambda} \|\hat{v}_j^X(\cdot,\lambda) - v_j^X\| \leq \frac{1}{S_{j,X}} \left\| \sqrt{\lambda} [\hat{C}_m^X(\cdot,\cdot,\lambda) - C^X] \right\|. \tag{6.6}$$

Further,

$$\begin{aligned} \sqrt{\lambda}[\hat{C}_m^X(t,s,\lambda) - C^X(t,s)] &= \frac{\sqrt{\lambda}}{\lfloor m\lambda \rfloor} \sum_{i=1}^{\lfloor m\lambda \rfloor} (X_i(t)X_i(s) - C^X(t,s)) \\ &= \frac{1}{\sqrt{m}} \frac{\sqrt{m\lambda}}{\sqrt{\lfloor m\lambda \rfloor}} \frac{1}{\sqrt{\lfloor m\lambda \rfloor}} \sum_{i=1}^{\lfloor m\lambda \rfloor} (X_i(t)X_i(s) - C^X(t,s)). \end{aligned}$$

It is easy to show using the Cauchy–Schwarz inequality that the sequence $X_i(\cdot)X_i(\cdot) - C^X(\cdot,\cdot) \in L^2([0,1])^2$ is $L^{2+\kappa}$-m-approximable for some $\kappa > 0$ if X_i is L^p-m-approximable for some $p > 4$. Lemma B.1 from the Supplementary Material of Aue et al. (2018) can be generalized to $L^{2+\kappa}$-m-approximable random variables taking values in $L^2([0,1]^2)$, from which it follows that

$$\sup_{\lambda \in [0,1]} \frac{1}{\sqrt{\lfloor m\lambda \rfloor}} \left\| \sum_{i=1}^{\lfloor m\lambda \rfloor} (X_i(\cdot)X_i(\cdot) - C^X(\cdot,\cdot)) \right\| = O_{\mathbb{P}}(\log^{(1/\kappa)}(m)).$$

Using this and combining with (6.6), we obtain the bound

$$\sup_{\lambda \in [0,1]} \left\| \sqrt{\lambda}[\hat{C}_m^X(\cdot,\cdot,\lambda) - C^X] \right\| = O_{\mathbb{P}}\left(\log^{(1/\kappa)}(m)\sqrt{m}\right), \tag{6.7}$$

and the estimate (2.28). Furthermore, using the bound that

$$|\hat{\tau}_j^X(\lambda) - \tau_j^X| \leq \big\|\hat{C}_m^X(\cdot,\cdot,\lambda) - C^X\big\|,$$

we obtain by similar arguments that

$$\sup_{\lambda\in[0,1]} \sqrt{\lambda}|\hat{\tau}_j^X(\lambda) - \tau_j^X| = O_{\mathbb{P}}\Big(\frac{\log^{(1/\kappa)}(m)}{\sqrt{m}}\Big). \tag{6.8}$$

Using the triangle inequality, Cauchy–Schwarz inequality, and combining (6.7) and (6.8), it follows

$$\begin{aligned}\sup_{\lambda\in[0,1]} \lambda\|G_{j,m}(\cdot,\lambda)\| \leq \sup_{\lambda\in[0,1]} \sqrt{\lambda}\Big\|[\hat{C}_m^X(\cdot,\cdot,\lambda) - C^X]\Big\| \sup_{\lambda\in[0,1]} \sqrt{\lambda}\|\hat{v}(\cdot,\lambda) - v_j^X\| \\ + \sup_{\lambda\in[0,1]} \sqrt{\lambda}|\hat{\tau}_j^X(\lambda) - \tau_j^X| \sup_{\lambda\in[0,1]} \sqrt{\lambda}\|\hat{v}(\cdot,\lambda) - v_j^X\| = O_{\mathbb{P}}\Big(\frac{\log^{(2/\kappa)}(m)}{m}\Big).\end{aligned} \tag{6.9}$$

Let

$$R_{j,m}(t,\lambda) = \frac{1}{\sqrt{m}}\sum_{k\neq j}\frac{v_k^X(t)}{\tau_j^X - \tau_k^X}\int_0^1\int_0^1 \hat{Z}_m^X(s_1,s_2,\lambda)v_k^X(s_2)v_j^X(s_1)ds_1ds_2.$$

Combining (6.2), (6.4) and (6.5), we see that for almost all $t \in [0,1]$ and for all $\lambda \in [0,1]$,

$$\lambda[\hat{v}_j^X(\cdot,\lambda) - v_j^X(t)] = \frac{m\lambda}{\lfloor m\lambda\rfloor}R_{j,m}(t,\lambda) - \sum_{k\neq j}\frac{\langle\lambda G_{j,m}(\cdot,\lambda), v_k^X\rangle}{\tau_j^X - \tau_k^X}v_k^X(t) - \frac{1}{2}\|\hat{v}_j^X(\cdot,\lambda) - v_j^X\|^2 v_j^X(t),$$

with the convention that $(m\lambda/\lfloor m\lambda\rfloor)R_{j,m}(t,\lambda) = 0$ for $\lambda < 1/m$. Using this identity and the triangle inequality, we obtain

$$\begin{aligned}\sup_{\lambda\in[0,1]}\Big\|\lambda[\hat{v}_j^X(\cdot,\lambda) - v_j^X(t)] - \frac{m\lambda}{\lfloor m\lambda\rfloor}R_{j,m}(t,\lambda)\Big\| \\ \leq \frac{1}{2}\sup_{\lambda\in[0,1]}\lambda\|\hat{v}_j^X(\cdot,\lambda) - v_j^X\|^2 + \sup_{\lambda\in[0,1]}\Big\|\sum_{k\neq j}\frac{\langle\lambda G_{j,m}(\cdot,\lambda), v_k^X\rangle}{\tau_j^X - \tau_k^X}v_k^X(t)\Big\|.\end{aligned} \tag{6.10}$$

The first term on the right-hand side of (6.10) can be bounded by bound (2.28). In order to bound the second term we have, using the orthonormality of the v_k^X (Parseval's identity) and the fact that $1/(\tau_j^X - \tau_k^X)^2 \leq 1/S_{j,X}^2$ for all $k \neq i$, that

$$\begin{aligned}\Big\|\sum_{k\neq j}\frac{\langle\lambda G_{j,m}(\cdot,\lambda), v_k^X\rangle}{\tau_j^X - \tau_k^X}v_k^X(\cdot)\Big\| &= \Big(\sum_{k\neq j}\frac{\langle\lambda G_{j,m}(\cdot,\lambda), v_k^X\rangle^2}{(\tau_j^X - \tau_k^X)^2}\Big)^{1/2} \\ &\leq \frac{1}{S_{j,X}}\Big(\sum_{k\neq j}\langle\lambda G_{j,m}(\cdot,\lambda), v_k^X\rangle^2\Big)^{1/2} \leq \frac{1}{S_{j,X}}\|\lambda G_{j,m}(\cdot,\lambda)\|.\end{aligned}$$

Therefore

$$\sup_{\lambda\in[0,1]}\Big\|\sum_{k\neq j}\frac{\langle\lambda G_{j,m}(\cdot,\lambda), v_k^X\rangle}{\tau_j^X - \tau_k^X}v_k^X(\cdot)\Big\| \leq \sup_{\lambda\in[0,1]}\frac{1}{S_{j,X}}\|\lambda G_{j,m}(\cdot,\lambda)\| = O_{\mathbb{P}}\Big(\frac{\log^{(2/\kappa)}(m)}{m}\Big),$$

where the last estimate follows from (6.9). Using these bounds in (6.10), we obtain that

$$\sup_{\lambda\in[0,1]}\Big\|\lambda[\hat{v}_j^X(\cdot,\lambda)-v_j^X(t)]-\frac{m\lambda}{\lfloor m\lambda\rfloor}R_{j,m}(t,\lambda)\Big\|=O_{\mathbb{P}}\Big(\frac{\log^{(2/\kappa)}(m)}{m}\Big).$$

Given the convention that $(m\lambda/\lfloor m\lambda\rfloor)R_{j,m}(t,\lambda)=0$ for $0\le\lambda<1/m$, the result follows then by showing that

$$\sup_{\lambda\in[1/m,1]}\Big|\frac{m\lambda}{\lfloor m\lambda\rfloor}-1\Big|\Big\|R_{j,m}(t,\lambda)\Big\|=O_{\mathbb{P}}\Big(\frac{\log^{(2/\kappa)}(m)}{m}\Big).$$

This result is a consequence of $\sup_{\lambda\in[1/m,1]}\left|\frac{m\lambda}{\lfloor m\lambda\rfloor}-1\right|\le 1/m$, and $\sup_{\lambda\in[1/m,1]}\|R_{j,m}(t,\lambda)\|=O_{\mathbb{P}}(1)$.

1.2 Proof of Proposition 2.3

Before proceeding with this proof, we develop some notation as well as a rigorous definition of the constant ζ_j. Recall the notations (2.31), (2.26) and (2.27) and define the random variables

$$\tilde{X}_i(s_1,s_2)=X_i(s_1)X_i(s_2)-C^X(s_1,s_2);\quad \tilde{Y}_i(s_1,s_2)=Y_i(s_1)Y_i(s_2)-C^Y(s_1,s_2). \tag{6.11}$$

Further let the random variables $\overline{X}_i^{(j)}$ and $\overline{Y}_i^{(j)}$ be defined by

$$\begin{aligned}\overline{X}_i^{(j)}&=\textstyle\int_0^1\int_0^1\tilde{X}_i(s_1,s_2)f_j^X(s_1,s_2)ds_1ds_2\ ,\\ \overline{Y}_i^{(j)}&=\textstyle\int_0^1\int_0^1\tilde{Y}_i(s_1,s_2)f_j^Y(s_1,s_2)ds_1ds_2,\end{aligned} \tag{6.12}$$

with the functions f_j^X, f_j^Y given by

$$f_j^X(s_1,s_2)=-v_j^X(s_1)\sum_{k\neq j}\frac{v_k^X(s_2)}{\tau_j^X-\tau_k^X}\int_0^1 v_k^X(t)v_j^Y(t)dt, \tag{6.13}$$

$$f_j^Y(s_1,s_2)=-v_j^Y(s_1)\sum_{k\neq j}\frac{v_k^Y(s_2)}{\tau_j^Y-\tau_k^Y}\int_0^1 v_k^Y(t)v_j^X(t)dt. \tag{6.14}$$

Firstly, we note that by using the orthonormality of the eigenfunctions v_j^X and v_j^Y, and Assumption 2.3, we get that

$$\|f_j^X\|^2=\iint(f_j^X(s_1,s_2))^2ds_1ds_2=\|v_j^X\|^2\sum_{k\neq j}\frac{\Big(\int_0^1 v_k^X(t)v_j^Y(t)dt\Big)^2}{(\tau_j^X-\tau_k^X)^2}\le 1/S_{j,X}^2<\infty.$$

Let

$$\sigma_{X,j}^2=\sum_{\ell=-\infty}^{\infty}\mathrm{cov}(\overline{X}_0^{(j)},\overline{X}_\ell^{(j)}),\quad\text{and}\quad \sigma_{Y,j}^2=\sum_{\ell=-\infty}^{\infty}\mathrm{cov}(\overline{Y}_0^{(j)},\overline{Y}_\ell^{(j)}).$$

Based on these quantities, ζ_j is defined as

$$\zeta_j=2\sqrt{\frac{\sigma_{X,j}^2}{\theta}+\frac{\sigma_{Y,j}^2}{1-\theta}}. \tag{6.15}$$

Proof of Proposition 2.3. We can write

$$\begin{aligned} \hat{Z}^{(j)}_{m,n}(\lambda) &= \sqrt{m+n}\int_0^1 (\hat{D}^{(j)}_{m,n}(t,\lambda))^2 - \lambda^2 D_j^2(t))dt \\ &= \sqrt{m+n}\Big\{\int_0^1 (\hat{D}^{(j)}_{m,n}(t,\lambda) - \lambda D_j(t))^2 \\ &\qquad +2\lambda D_j(t)(\hat{D}^{(j)}_{m,n}(t,\lambda) - \lambda D_j(t))^2 dt \\ &= \sqrt{m+n}\int_0^1 (\tilde{D}^{(j)}_{m,n}(t,\lambda))^2 dt \\ &\qquad +2\lambda\sqrt{m+n}\int_0^1 D_j(t)\tilde{D}^{(j)}_{m,n}(t,\lambda)dt + o_{\mathbb{P}}(1) \end{aligned} \tag{6.16}$$

uniformly with respect to $\lambda \in [0,1]$, where the process $\tilde{D}^{(j)}_{m,n}(t,\lambda)$ is defined in (2.31) and Proposition 2.2 was used in the last equation. Observing (2.32) gives

$$\hat{Z}^{(j)}_{m,n}(\lambda) = \tilde{Z}^{(j)}_{m,n}(\lambda) + o_{\mathbb{P}}(1) \tag{6.17}$$

uniformly with respect to $\lambda \in [0,1]$, where the process $\tilde{Z}^{(j)}_{m,n}$ is given by

$$\tilde{Z}^{(j)}_{m,n}(\lambda) = 2\lambda\sqrt{m+n}\int_0^1 D_j(t)\tilde{D}^{(j)}_{m,n}(t,\lambda)dt. \tag{6.18}$$

Consequently the assertion of Proposition 2.3 follows from the weak convergence

$$\{\tilde{Z}^{(j)}_{m,n}(\lambda)\}_{\lambda\in[0,1]} \rightsquigarrow \{\lambda\zeta_j\mathbb{B}(\lambda)\}_{\lambda\in[0,1]}.$$

We obtain, using the orthogonality of the eigenfunctions and the notation (2.6), that

$$\begin{aligned} \tilde{Z}^{(j)}_{m,n}(\lambda) &= 2\lambda\sqrt{m+n}\Big\{\frac{1}{\sqrt{m}}\int_0^1 \hat{Z}^X_m(s_1,s_2,\lambda)\int_0^1\int_0^1 D_j(t)\sum_{k\neq j}\frac{v_k^X(t)}{\tau_j^X - \tau_k^X}dt v_j^X(s_1)v_k^X(s_2)ds_1ds_2 \\ &\qquad -\frac{1}{\sqrt{n}}\int_0^1 Z^Y_n(s_1,s_2,\lambda)\int_0^1\int_0^1 D_j(t)\sum_{k\neq j}\frac{v_k^Y(t)}{\tau_j^Y - \tau_k^Y}dt v_j^Y(s_1)v_k^Y(s_2)ds_1ds_2\Big\} \\ &= 2\lambda\sqrt{m+n}\Big\{\frac{1}{m}\sum_{i=1}^{\lfloor m\lambda\rfloor}\overline{X}_i^{(j)} + \frac{1}{n}\sum_{i=1}^{\lfloor n\lambda\rfloor}Y_i^{(j)}\Big\}, \end{aligned} \tag{6.19}$$

where the random variables $\overline{X}_i^{(j)}$ and $\bar{Y}_i^{(j)}$ are defined above. We now aim to establish that

$$\Big\{\frac{1}{\sqrt{m}}\sum_{i=1}^{\lfloor m\lambda\rfloor}\overline{X}_i^{(j)}\Big\}_{\lambda\in[0,1]} \rightsquigarrow \sigma_{X,j}\{\mathbb{B}^X(\lambda)\}_{\lambda\in[0,1]}, \tag{6.20}$$

where $\mathbb{B}^X$ is a standard Brownian motion on the interval $[0,1]$. In the following we use the symbol $\|\cdot\|$ simultaneously for L^2-norm on the space $L^2([0,1])$ and $L^2([0,1]^2)$ as the particular meaning is always clear from the context. Firstly, we note that by using the orthonormality of the eigenfunctions v_j^X and v_j^Y, and Assumption 2.3, we get that

$$\|f_j^X\|^2 = \iint (f_j^X(s_1,s_2))^2 ds_1 ds_2 = \|v_j^X\|^2 \sum_{k\neq j} \frac{\left(\int_0^1 v_k^X(t) v_j^Y(t) dt\right)^2}{(\tau_j^X - \tau_k^X)^2} \leq 1/S_{j,X}^2 < \infty.$$

The following calculation is similar to Lemma A.3 in Aue et al. (2020). Let

$$\tilde{X}_i^{(m)}(t,s) = X_{i,m}(t)X_{i,m}(s) - \mathbb{E}X_0(t)X_0(s),$$

where $\{X_{i,m}\}_{i\in\mathbb{Z}}$ is the mean zero m-dependent sequence used in definition of m-approximability (see Assumption 2.2). Moreover, if $q = p/2$ with p given in Assumption 2.2, then we have by the triangle inequality and Minkowski's inequality that

$$\begin{aligned}\left\{\mathbb{E}\|\tilde{X}_i - \tilde{X}_i^{(m)}\|^q\right\}^{1/q} &\leq \left\{\mathbb{E}(\|X_i(\cdot)(X_i(\cdot) - X_{i,m}(\cdot))\| + \|X_{i,m}(\cdot)(X_i(\cdot) - X_{i,m}(\cdot))\|)^q\right\}^{1/q} \\ &\leq \left\{\mathbb{E}(\|X_i(\cdot)(X_i(\cdot) - X_{i,m}(\cdot))\|^q\right\}^{1/q} + \left\{\mathbb{E}\|X_{i,m}(\cdot)(X_i(\cdot) - X_{i,m}(\cdot))\|^q\right\}^{1/q}.\end{aligned} \tag{6.21}$$

Using the definition of the norm in $L^2([0,1])$, it is clear that

$$\|X_i(\cdot)(X_i(\cdot) - X_{i,m}(\cdot))\| = \|X_i\|\|X_i - X_{i,m}\|,$$

and hence we obtain from the Cauchy–Schwarz inequality applied to the expectation on the concluding line of (6.21) and stationarity that

$$\begin{aligned}(\mathbb{E}(\|X_i(\cdot)(X_i(\cdot) - X_{i,m}(\cdot))\|^q)^{1/q} + (\mathbb{E}\|X_{i,m}(\cdot)(X_i(\cdot) - X_{i,m}(\cdot))\|^q)^{1/q} \\ \leq (\mathbb{E}\|X_0\|^{2q})^{1/2q}(\mathbb{E}\|X_0 - X_{0,m}\|^{2q})^{1/2q}.\end{aligned}$$

It follows from this and (6.21) that

$$\sum_{m=1}^{\infty} (\mathbb{E}\|\tilde{X}_i - \tilde{X}_i^{(m)}\|^q)^{1/q} \leq (\mathbb{E}\|X_0\|^p)^{1/p} \sum_{m=1}^{\infty} (\mathbb{E}\|X_0 - X_{0,m}\|^p)^{1/p} < \infty. \tag{6.22}$$

Now let $\overline{X}_{i,m}^{(j)}$ be defined as $\overline{X}_i^{(j)}$ in (6.12) with X_i replaced by $X_{i,m}$. We obtain using the Cauchy–Schwarz inequality that

$$(\mathbb{E}[\overline{X}_i^{(j)} - \overline{X}_{i,m}^{(j)}]^q)^{1/q} \leq \|f_j^X\|(\mathbb{E}\|\tilde{X}_i - \tilde{X}_i^{(m)}\|^q)^{1/q}.$$

By (6.22) it follows that

$$\sum_{m=1}^{\infty} (\mathbb{E}[\overline{X}_i^{(j)} - \overline{X}_{i,m}^{(j)}]^q)^{1/q} < \infty$$

and therefore the sequence $\overline{X}_i^{(j)}$ satisfies the assumptions of Theorem 3 in Wu (2005). By this result the weak convergence in (6.20) follows. By the same arguments it follows that

$$\left\{\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor n\lambda \rfloor} \overline{Y}_i^{(j)}\right\}_{\lambda\in[0,1]} \rightsquigarrow \sigma_{Y,j}\{E\mathbb{B}^Y(\lambda)\}_{\lambda\in[0,1]}, \tag{6.23}$$

where $\mathbb{B}^Y$ is a standard Brownian motion on the interval $[0,1]$ and

$$\sigma^2_{Y,j} = \sum_{\ell=-\infty}^{\infty} \mathrm{cov}(\overline{Y}_0^{(j)}, \overline{Y}_\ell^{(j)}).$$

Since the sequences $\{X_i\}_{i\in\mathbb{R}}$ and $\{Y_i\}_{i\in\mathbb{R}}$ are independent, we have that (6.20) and (6.23) may be taken to hold jointly where the Brownian motions $\mathbb{B}^X$ and $\mathbb{B}^Y$ are independent. It finally follows from this and (6.19) that

$$\{\tilde{Z}^{(j)}_{m,n}(\lambda)\}_{\lambda\in[0,1]} \rightsquigarrow \Big\{2\lambda\Big(\frac{\sigma_{X,j}}{\sqrt{\theta}}\mathbb{B}^X(\lambda) + \frac{\sigma_{Y,j}}{\sqrt{1-\theta}}\mathbb{B}^Y(\lambda)\Big)\Big\}_{\lambda\in[0,1]} \overset{\mathcal{D}}{=} \big\{\lambda\zeta_j\mathbb{B}(\lambda)\big\}_{\lambda\in[0,1]} ,$$

which completes the proof of Proposition 2.3. □

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