Statistica Sinica **9**(1999), 1119-1130

ON THE STATIONARITY AND THE EXISTENCE OF MOMENTS OF CONDITIONAL HETEROSKEDASTIC ARMA MODELS

Shiqing Ling

University of Western Australia

Abstract: Sufficient and necessary conditions for the existence of a unique second order stationary solution of conditional heteroskedastic autoregressive moving-average (CHARMA) models proposed by Tsay (1987) are derived. The solution is strictly stationary and ergodic, and has a causal representation. When the CHARMA model reduces to some special cases, it is shown that the conditions are equivalent to those already known in the literature. Based on Tweedie's (1988) result, sufficient conditions for the existence of finite-order moments of CHARMA models are also derived.

Key words and phrases: ARCH model, CHARMA model, ergodicity, existence of finite-order moments, Markov chain, stationarity.

1. Introduction

Tsay (1987) proposed the CHARMA (conditional heteroskedastic autoregressive moving-average) model,

$$y_t - \mu = \sum_{i=1}^{p} \psi_i (y_{t-i} - \mu) + \sum_{i=1}^{q} \theta_i a_{t-i} + a_t, \qquad \text{(Observation Equation)} \qquad (1.1)$$

$$a_t = \sum_{i=1}^{r} \delta_{it} a_{t-i} + \sum_{i=1}^{s} w_{it} (y_{t-i} - \mu) + w_{0t} (\hat{y}_{t-1}(1) - \mu) + e_t, \qquad \text{(Innovation Equation)} \qquad (1.2)$$

where the orders, p, q, r and s, are finite and non-negative integers, μ, ψ_i's and θ_i's are constants, δ_{it}'s, w_{it}'s and e_t's are random variables, and $\hat{y}_{t-1}(1) = E(y_t|F_{t-1})$, where F_{t-1} is the $\sigma-$field generated by $\{e_{t-i}, w_{t-i}, \delta_{t-i}|i = 1, 2, \ldots\}$, $w_t = (w_{0t}, \ldots, w_{st})^T$, $\delta_t = (\delta_{1t}, \ldots, \delta_{rt})^T$ and A^T denotes the transpose of a vector or matrix A. We consider the following.

Assumption 1. $\{e_t\}$ is a sequence of independent and identically distributed random variables with mean zero and finite positive variance σ^2;

Assumption 2. $\{w_t\}$ and $\{\delta_t\}$ are two sequences of i.i.d. random vectors with mean zero and non-negative definite constant covariance matrices $E(w_t w_t^T) = \Omega = (\Omega_{ij})_{(s+1)\times(s+1)}$ and $E(\delta_t \delta_t^T) = \Delta = (\Delta_{ij})_{r\times r}$, respectively;

Assumption 3. $\{e_t\}$, $\{w_t\}$ and $\{\delta_t\}$ are mutually independent;

Assumption 4. y_t and a_t are F_t-measurable;

Assumption 5. The equations $z^p - \psi_1 z^{p-1} - \cdots - \psi_p = 0$ and $z^q + \theta_1 z^{q-1} + \cdots + \theta_q = 0$ have no common root.

The CHARMA model (1.1)-(1.2) better summarizes the available information at time $t-1$ and is also a natural model for conditional heteroscedastic time series, both statistically and intuitively. It shares many characteristics of linear models and is more flexible than Weiss' (1984) ARMA-ARCH model in applications. Another important advantage is that the CHARMA model is easily extended to the multivariate case, as compared with Engle's (1982) ARCH model and Bollerslev's (1986) GARCH model. See the work by Ling and Deng (1993) and Wong and Li (1997).

Tsay (1987) investigated invertibility and parameter estimation for the CHARMA model (1.1)-(1.2) under the assumption $\mathrm{tr}\{\Delta + \Omega\,\mathrm{Cov}\,(Z_{s,t})[\mathrm{Var}\,(y_t)/\mathrm{Var}\,(a_t)]\} - \Omega_{11} < 1$, where $\mathrm{tr}(A)$ denotes the trace of the matrix A, $\mathrm{Var}(X)$ denotes the variance of the random variable X and $\mathrm{Cov}\,(Z_{s,t})$ is the correlation matrix of the process $Z_{s,t} = (y_t, \ldots, y_{t-s})^T$. It is obvious that the assumption depends on the existence of $\mathrm{Var}\,(y_t)$ and $\mathrm{Var}\,(a_t)$. However conditions under which y_t and a_t have finite variances are yet to be investigated. Moreover for some asymptotic properties of the estimated parameters, the strict stationarity, ergodicity and existence of the fourth and eighth moments of the CHARMA model (1.1)-(1.2) are required. The conditions under which these requirements are satisfied have not been found. In this paper, we derive the sufficient and necessary conditions for the existence of a unique second order stationary solution of model (1.1)-(1.2). It is shown that the solution is strictly stationary and ergodic and has a causal representation. In some special cases, these conditions reduce to those already known in the literature. Based on Tweedie's (1988) result, sufficient conditions for the existence of finite-order moments of the CHARMA model (1.1)-(1.2) are also derived.

The paper is organized as follows. Section 2 investigates the stationarity condition and Section 3 investigates the existence of finite-order moments.

Throughout the paper, the following notation will be used: $\rho(A)$ is the maximum eigenvalue of the matrix A in absolute value, $\mathrm{vec}(A)$ is the usual column-stacking vector of the matrix A, and $\otimes$ denotes the Kronecker product of matrices. The properties of matrix operations, $\mathrm{vec}(ABC) = (C^T \otimes A)\mathrm{vec}(B)$ and $(A \otimes B)^T = A^T \otimes B^T$, will be also used.

2. Conditions for Stationarity

Following Tsay (1987), with $P = \max(p, s)$ and $Q = \max(q, r)$, we have

$$\hat{y}_{t-1}(1) - \mu = E[(y_t - \mu)|F_{t-1}] = \sum_{i=1}^{P} \psi_i(y_{t-i} - \mu) + \sum_{i=1}^{Q} \theta_i a_{t-i}, \tag{2.1}$$

where $\psi_i = 0$ for $i > p$ and $\theta_i = 0$ for $i > q$.

From (1.1) and (2.1),

$$a_t = \sum_{i=1}^{P} (\psi_i w_{0t} + w_{it})(y_{t-i} - \mu) + \sum_{i=1}^{Q} (\delta_{it} + w_{0t}\theta_i) a_{t-i} + e_t, \tag{2.2}$$

where $w_{it} = 0$ for $i > s$ and $\delta_{it} = 0$ for $i > r$. Further, from (1.1) and (2.2),

$$\begin{aligned} y_t - \mu = &\sum_{i=1}^{P} (\psi_i + w_{0t}\psi_i + w_{it})(y_{t-i} - \mu) \\ &+ \sum_{i=1}^{Q} (\theta_i + w_{0t}\theta_i + \delta_{it}) a_{t-i} + e_t. \end{aligned} \tag{2.3}$$

Let

$$U_t = (y_t - \mu, \ldots, y_{t-P+1} - \mu, a_t, \ldots, a_{t-Q+1})^T_{(P+Q)\times 1},$$

$$\Phi = \left(\begin{array}{ccc|ccc} \psi_1 & \cdots & \psi_P & \theta_1 & \cdots & \theta_Q \\ \multicolumn{2}{c}{I_{(P-1)\times(P-1)}} & \mathbf{0}_{(P-1)\times 1} & \multicolumn{3}{c}{\mathbf{0}_{(P-1)\times Q}} \\ \hline & & & 0 & \cdots & 0 \\ \multicolumn{3}{c|}{\mathbf{0}_{Q\times P}} & \multicolumn{2}{c}{I_{(Q-1)\times(Q-1)}} & \mathbf{0}_{(Q-1)\times 1} \end{array} \right),$$

$$\begin{aligned} \Gamma_t = &\left(\begin{array}{ccc|ccc} w_{0t}\psi_1 + w_{1t} & \cdots & w_{0t}\psi_P + w_{Pt} & w_{0t}\theta_1 + \delta_{1t} & \cdots & w_{0t}\theta_Q + \delta_{Qt} \\ \multicolumn{3}{c|}{\mathbf{0}_{(P-1)\times P}} & \multicolumn{3}{c}{\mathbf{0}_{(P-1)\times Q}} \\ \hline w_{0t}\psi_1 + w_{1t} & \cdots & w_{0t}\psi_P + w_{Pt} & w_{0t}\theta_1 + \delta_{1t} & \cdots & w_{0t}\theta_Q + \delta_{Qt} \\ \multicolumn{3}{c|}{\mathbf{0}_{(Q-1)\times P}} & \multicolumn{3}{c}{\mathbf{0}_{(Q-1)\times Q}} \end{array} \right) \\ &= \Gamma B_t, \end{aligned}$$

where $I_{r\times r}$ is the $r \times r$ identity matrix, $\Gamma = (1, 0, \ldots, 0, 1, 0, \ldots, 0)^T_{(P+Q)\times 1}$ whose first and $(P+1)-st$ components are 1, and $B_t = (w_{0t}\psi_1 + w_{1t}, \ldots, w_{0t}\psi_P + w_{pt}, w_{0t}\theta_1 + \delta_{1t}, \ldots, w_{0t}\theta_Q + \delta_{Qt})$.

We can write (2.2) and (2.3) equivalently in the vector form,

$$U_t = (\Phi + \Gamma B_t)U_{t-1} + e_t\Gamma. \tag{2.4}$$

It is easy to show that equations (1.1)-(1.2) are equivalent to (2.4). The following theorem gives a simple necessary and sufficient condition for the existence of a F_t- measurable second-order stationary solution. Another necessary and sufficient condition can be found in the Appendix.

Theorem 2.1. *Under Assumptions* 1-3 *and* 5*, a necessary and sufficient condition for model* (1.1)-(1.2) *to have a unique* F_t- *measurable second-order stationary solution* $\{(y_t, a_t)\}$ *is that*

$$\rho(\Phi \otimes \Phi + C) < 1, \tag{2.5}$$

where $C = (\Gamma \otimes \Gamma)E(B_t \otimes B_t)$*. Furthermore,* $\{y_t\}$ *and* $\{a_t\}$ *are strictly stationary and ergodic.*

Proof. If $\rho(\Phi \otimes \Phi + C) < 1$, it is easy to show that $\sum_{k=1}^{m}(\Phi \otimes \Phi + C)^{k-1}\text{vec}(\Gamma\Gamma^T)$ converges as $m \to \infty$, Corollary 5.6.14 in Horn and Johnson (1990, p.299). The sufficiency of (2.5) follows from Theorem A.1 in the Appendix.

Suppose now that $\rho(\Phi \otimes \Phi + C) \geq 1$. Writing $\Phi \otimes \Phi + C$ in Jordan form, $\Phi \otimes \Phi + C = P\Lambda P^{-1}$, where except for the main diagonal elements and several ones on the first upper diagonal, other elements of Λ are zero. Then there exists at least one element $\lambda \geq 1$ on the main diagonal of Λ. However by Theorem A.1 in the Appendix, we know that $\sum_{k=1}^{m}(\Phi \otimes \Phi + C)^{k-1}\text{vec}(\Gamma\Gamma^T) = \sum_{k=1}^{m} P\Lambda^{k-1}P^{-1}\text{vec}(\Gamma\Gamma^T)$ converges as $m \to \infty$. Therefore, $\text{vec}(\Gamma\Gamma^T)$ must be orthogonal to the eigenvector $z = (z_1, \ldots, z_{P+Q})^T$ corresponding to λ. Since $z^T\text{vec}(\Gamma\Gamma^T) = 0$, we obtain that $z_1 + z_{P+1} + z_{P(P+Q)+1} + z_{P(P+Q)+P+1} = 0$ and hence

$$\begin{aligned}
\lambda z^T &= z^T(\Phi \otimes \Phi + C) = z^T(\Phi \otimes \Phi) + z^T(\Gamma \otimes \Gamma)E(B_t \otimes B_t) \\
&= z^T(\Phi \otimes \Phi) + (z_1 + z_{P+1} + z_{P(P+Q)+1} + z_{P(P+Q)+P+1})E(B_t \otimes B_t) \\
&= z^T(\Phi \otimes \Phi).
\end{aligned} \tag{2.6}$$

From (2.6), we know that λ is also an eigenvalue of $\Phi \otimes \Phi$.

Since $\sum_{k=1}^{m}(\Phi \otimes \Phi + C)^{k-1}\text{vec}(\Gamma\Gamma^T)$ converges, from Lemma A.1 in the Appendix, we know that $\rho(\Phi) < 1$. Furthermore $\rho(\Phi \otimes \Phi) < 1$ (see Problem 1.8 in Srivastava and Khatri (1979), p.33) and hence $|\lambda| < 1$. This contradicts $|\lambda| \geq 1$ and the necessity of (2.5) is proved.

Next we show uniqueness. We only need to prove that the F_t-measurable second-order stationary solution of (2.4) is unique. Suppose that there are two such solutions, denoted S_t and $S_t^{'}$. Then $V_t = S_t - S_t^{'}$ is F_t-measurable and satisfies

$$V_t = (\Phi + \Gamma B_t)V_{t-1}.$$

It is easy to see that $E(V_tV_t^T) \leq 2[E(S_tS_t^T) + E(S_t^{'}S_t^{'T})]$, where the right side is a constant matrix. Since $\rho(\Phi \otimes \Phi + C) < 1$, we have $\text{vec}(V_tV_t^T) = (\Phi \otimes \Phi + C)\text{vec}E(V_{t-1}V_{t-1}) = \lim_{m\to\infty}(\Phi \otimes \Phi + C)^m\text{vec}E(V_{t-m}V_{t-m}^T) = \mathbf{0}$. Hence

$V_t = 0$ a.s, i.e., $S_t = S_t^{'}$ a.s. Finally the strict stationarity and ergodicity follow directly from Theorem A.1 in the Appendix. This completes the proof.

Remark.

(i) The sufficient condition can be extended directly to the multivariate case. However, from the above proof, we see that in general condition (2.5) is not necessary for such an extension.

(ii) When $\delta_{it} = w_{jt} = 0$, $i = 1, \ldots, r$, $j = 0, 1, \ldots, s$, model (1.1)-(1.2) reduces to the usual ARMA model with a constant conditional variance, and condition (2.5) is equivalent to the well-known necessary and sufficient stationary condition that all roots of the equation $z^p - \psi_1 z^{p-1} - \cdots - \psi_p = 0$ lie inside the unit circle.

(iii) When $\mu = 0$, $w_{0t} = 0$ and $\delta_{it} = \theta_i = 0$ for $i = 1, \ldots, Q$, model (1.1)-(1.2) can be rewritten as a random coefficient autoregressive model, $y_t = \sum_{i=1}^{P}(\psi_i + w_{it})y_{t-i} + e_t$, and condition (2.5) is equivalent to the condition given by Nicholl and Quinn (1982).

Corollary 2.1. *For model* (1.1)-(1.2) *with* $\mu = 0$, $w_{it} = \theta_j = 0$, $i = 0, 1, \ldots, P$, $j = 1, \ldots, Q$, *i.e.,* $y_t = \sum_{i=1}^{P} \psi_i y_{t-i} + a_t$ *and* $a_t = \sum_{i=1}^{Q} \delta_{it} a_{t-i} + e_t$, *condition* (2.5) *means that all roots of the equation*

$$z^P - \psi_1 z^{P-1} - \cdots - \psi_P = 0 \tag{2.7}$$

lie inside the unit circle and $\rho[E(G_t \otimes G_t)] < 1$, *where*

$$G_t = \begin{pmatrix} \delta_{1t} & \cdots & \delta_{Qt} \\ & I_{(Q-1)\times(Q-1)} & O_{(Q-1)\times 1} \end{pmatrix}.$$

This is the condition in Ling and Deng (1993).

Proof. In this case,

$$\begin{aligned} \Phi \otimes \Phi + C &= E[(\Phi + \Gamma B_t) \otimes (\Phi + \Gamma B_t)] \\ &= E[\begin{pmatrix} M & \bar{\boldsymbol{\delta}}_{\mathbf{t}} \\ \mathbf{0} & G_t \end{pmatrix} \otimes \begin{pmatrix} M & \bar{\boldsymbol{\delta}}_{\mathbf{t}} \\ \mathbf{0} & G_t \end{pmatrix}] \\ &= \begin{pmatrix} E[M \otimes \begin{pmatrix} M & \bar{\boldsymbol{\delta}}_{\mathbf{t}} \\ \mathbf{0} & G_t \end{pmatrix}] & * \\ \mathbf{0} & E[G_t \otimes \begin{pmatrix} M & \bar{\boldsymbol{\delta}}_{\mathbf{t}} \\ \mathbf{0} & G_t \end{pmatrix}] \end{pmatrix}, \end{aligned}$$

where $M = \begin{pmatrix} \psi_1 & \cdots & \psi_P \\ & I_{(P-1)\times(P-1)} & O_{(P-1)\times 1} \end{pmatrix}$, $\bar{\delta}_t = \begin{pmatrix} \delta_{1t} & \cdots & \delta_{Qt} \\ & O_{(P-1)\times(Q)} & \end{pmatrix}$ and $*$ consists of some suitable elements. Note that

$$E\left[M\otimes\begin{pmatrix} M & \bar{\delta}_t \\ \mathbf{0} & G_t\end{pmatrix}\right] = M\otimes\left(\begin{array}{c|c} M & \mathbf{0} \\ \hline 0\cdots 0 & 0 \quad\cdots\quad 0 \\ \mathbf{0} & I_{(Q-1)\times(Q-1)}\ \mathbf{0}\end{array}\right).$$

By a direct calculation, we have

$$\mathrm{Det}(\lambda I - E\left[M\otimes\begin{pmatrix} M & \bar{\delta}_t \\ \mathbf{0} & G_t\end{pmatrix}\right]) = \lambda^{QP}\ \mathrm{Det}(\lambda I - M\otimes M).$$

Similarly we can show that

$$\mathrm{Det}(\lambda I - E\left[G_t\otimes\begin{pmatrix} M & \bar{\delta}_t \\ \mathbf{0} & G_t\end{pmatrix}\right]) = \lambda^{QP}\ \mathrm{Det}[\lambda I - E(G_t\otimes G_t)].$$

Thus $\rho(\Phi\otimes\Phi + C) < 1$ if and only if $\rho(M\otimes M) < 1$ and $\rho[E(G_t\otimes G_t)] < 1$. Note that $\rho(M\otimes M) < 1$ if and only if $\rho(M) < 1$. After some algebra, we know that $\rho(M) < 1$ is equivalent to condition (2.7). This completes the proof.

Example. Consider the model $y_t = \psi_1 y_{t-1} + a_t$ with $a_t = w_{1t}y_{t-1} + \delta_{1t}a_{t-1} + e_t$, a special case of model (1.1)-(1.2). Here

$$\Phi\otimes\Phi + C = \begin{pmatrix}\psi_1 & 0 \\ 0 & 0\end{pmatrix}\otimes\begin{pmatrix}\psi_1 & 0 \\ 0 & 0\end{pmatrix} + E\{\begin{pmatrix} w_{1t} & \delta_{1t} \\ w_{1t} & \delta_{1t}\end{pmatrix}\otimes\begin{pmatrix} w_{1t} & \delta_{1t} \\ w_{1t} & \delta_{1t}\end{pmatrix}\}$$

$$= \begin{pmatrix} \psi_1^2+\sigma_w^2 & 0 & 0 & \sigma_\delta^2 \\ \sigma_w^2 & 0 & 0 & \sigma_\delta^2 \\ \sigma_w^2 & 0 & 0 & \sigma_\delta^2 \\ \sigma_w^2 & 0 & 0 & \sigma_\delta^2 \end{pmatrix},$$

where $\sigma_w^2 = Ew_{1t}^2$ and $\sigma_\delta^2 = E\delta_{1t}^2$. After some algebra,

$$\mathrm{Det}(\lambda I - \Phi\otimes\Phi - C) = \lambda^2[\lambda^2 - \lambda(\psi_1^2+\sigma_w^2+\sigma_\delta^2) - \psi_1^2\sigma_\delta^2].$$

Thus the eigenvalues of $\Phi\otimes\Phi + C$ are $\lambda_{1,2} = 0$ and $\lambda_{3,4} = \{\psi_1^2+\sigma_w^2+\sigma_\delta^2 \pm [(\psi_1^2+\sigma_w^2+\sigma_\delta^2)^2 - 4\psi_1^2\sigma_\delta^2]^{1/2}\}/2$. By some algebraic calculations, we know that $|\lambda_{3,4}| < 1$ if and only if $\sigma_\delta^2 < 1$ and $|\psi_1| < [(1-\sigma_w^2-\sigma_\delta^2)/(1-\sigma_\delta^2)]^{1/2}$. It is obvious that if $\sigma_w^2 \neq 0$, then the admissible region of ψ_1 is smaller than that of the corresponding coefficient of an AR(1) model with a constant conditional variance.

3. Conditions for the Existence of Moments

Sufficient conditions for the existence of finite-order moments are given by the following theorem.

Theorem 3.1. *Suppose that Assumptions* 1-5 *hold and that* $\rho(E(\Phi+\Gamma B_t)^{\otimes 2m}) < 1$ *and* $Ee_t^{2m} < \infty$, *where* $m > 1$. *Then* (1.1)-(1.2) *admits a unique stationary solution* $\{(y_t, a_t)\}$ *with* $Ey_t^{2m} < \infty$ *and* $Ea_t^{2m} < \infty$, *where* $A^{\otimes n}$ *denotes* $\underbrace{A \otimes A \otimes \cdots \otimes A}_{n\ factors}$.

Before proving Theorem 3.1, we illustrate a concept and Tweedie's (1988) criteria. Let $\{X_t; t = 1, 2, \ldots\}$ be a temporally homogeneous Markov chain with a locally compact complete separable metric state space $(S, \mathcal{B})$. The transition probability is $P(X, A) = Pr(X_n \in A | X_{n-1} = x)$, where $x \in S$ and $A \in \mathcal{B}$.

Lemma 3.1. (Tweedie (1988, Theorem 2)) *Suppose that* $\{X_t\}$ *is a Feller chain, i.e.,* $\{X_t\}$ *is a Markov chain and for each bounded continuous function* g *on* S, *the function* $E(g(X_t)|X_{t-1} = x)$ *is also continuous in* x.

(i) *If there exists, for some compact set* $A \in \mathcal{B}$, *a non-negative function* g *and* $\varepsilon > 0$ *satisfying*

$$\int_{A^c} P(x, dy) g(y) \le g(x) - \varepsilon, \qquad x \in A^c, \tag{3.1}$$

then there exists a σ*-finite invariant measure* μ *for* P *with* $0 < \mu(A) < \infty$;

(ii) *if*

$$\int_A \mu(dx) [\int_{A^c} P(x, dy) g(y)] < \infty, \tag{3.2}$$

then μ *is finite and hence* $\pi = \mu/\mu(S)$ *is an invariant probability measure;*

(iii) *if*

$$\int_{A^c} P(x, dy) g(y) \le g(x) - f(x), \qquad x \in A^c,$$

then μ *admits a finite* f*-moment, i.e.,* $\int_S \mu(dy) f(y) < \infty$.

Remark. For the Markov chain generated by a time series model, Lemma 3.1 tells us that the model admits a strictly stationary solution with stationary distribution π if conditions (3.1) and (3.2) are satisfied.

Proof of Theorem 3.1. The process $\{U_t\}$ defined by (2.4) is a Markov chain with state space R^{P+Q}, and is also a Feller chain.

Define $g(U) = 1 + (U^{\otimes m})^T W U^{\otimes m}$, where $U \in R^{P+Q}$, W is defined by $\mathrm{vec}(W) = (I - D)^{-1}\mathrm{vec}H$, $D = E(\Phi + \Gamma B_t)^{\otimes 2m}$, and H is a positive definite $(P+Q)^m \times (P+Q)^m$ matrix. Similar to the proof of Theorem 5 in Feigin and Tweedie (1985), we can show that under Assumptions 1-5, if $\rho(D) < 1$ then W

is positive definite and there is a compact and bounded set $A \subset R^{P+Q}$ and a constant $\delta > 0$ such that, when $U \in A^c$, the function $g(U)$ satisfies

$$E\{g(U_t)|U_{t-1} = U\} \leq (1-\delta)g(U), \tag{3.3}$$

where U_t is defined by (2.4). By (3.3), it is not difficult to show that the function $g(U)$ satisfies conditions (3.1)-(3.2) in Lemma 3.1. By Lemma 3.1, there is an invariant probability measure π for the process $\{U_t\}$.

Let $f(U) = \delta g(U)$. From (3.3) and Lemma 3.1 (iii), we know that $E_\pi f(U_t)$ is finite. Thus $E_{\pi_1} y_t^{2m} < \infty$ and $E_{\pi_2} a_t^{2m} < \infty$, where π_1 and π_2 are the marginal distributions specified by π, corresponding to y_t and a_t, respectively.

Since the function $f(x) = x^{1/m}$ is concave $(x > 0)$, by Jensen's inequality, $E_{\pi_1} y_t^2 \leq (E_{\pi_1} y_t^{2m})^{1/m} < \infty$. Similarly $E_{\pi_2} a_t^2 < \infty$. Thus π_1 and π_2 are the second-order stationary distributions of y_t and a_t, respectively, i.e., $\{(y_t, a_t)\}$ is the second-order stationary solution of model (1.1)-(1.2). From Theorem 2.1, the solution is unique . This completes the proof.

Example. Consider the model $y_t = a_t$ with $a_t = \delta_{1t} a_{t-1} + e_t$. In this case, $(\Phi + \Gamma B_t)^{\otimes 2m} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}^{\otimes 2m} \delta_{1t}^{2m}$. The condition given in Theorem 3.1 reduces to $E\delta_{1t}^{2m} < 1$. Furthermore if $\delta_{1t} \sim N(0, \sigma^2)$, the condition reduces to $\sigma^{2m} \prod_{i=1}^{m-1}(2i-1) < 1$, which is the same as Engle's (1982) condition for the finiteness of the $2m$th moments of the first order ARCH model.

Remark.

(i) From the definition of $e_t\Gamma$ in (2.4), we cannot prove that the Markov chain $\{U_t\}$ is $\phi-$irreducible, and hence its geometrical ergodicity cannot be established by Theorem 1 in Feigin and Tweedie (1985). Similarly, since we cannot show that the compact set A in Lemma 3.1 (i) is *small* , Theorem 2 in Feigin and Tweedie (1985) cannot be used for the finiteness of finite-order moments.

(ii) For various special CHARMA models, we believe that the conditions in Theorem 3.1 can be simplified, but we shall not pursue the details here. The conditions for the existence of finite-order moments usually require us to check the eigenvalues of a higher order matrix. Numerically, the verification can be easily done.

Acknowledgement

I thank my supervisor Professor W. K. Li for his encouragement and discussion, and thank the editor and two referees for their comments that greatly improved the paper. The research was supported by the Studentship in Department of Statistics at The University of Hong Kong, and the Australia Research Council Large Grant in Department of Economics at University of Western Australia.

Appendix

Theorem A.1. *Under Assumptions* 1-3 *and* 5, *in order that there exists a* F_t-*measurable second-order stationary solution* $\{(y_t, a_t)\}$ *satisfying* (1.1)-(1.2), *it is necessary and sufficient that*

$$\sum_{k=1}^{m}(\Phi \otimes \Phi + C)^k \ vec(\Gamma\Gamma^T) \tag{A.1}$$

converges as $m \to \infty$, *where* Φ *and* Γ *are as defined in* (2.4) *and* C *is as defined in Theorem* 2.1. *Moreover the* $\{y_t\}$ *and* $\{a_t\}$ *are strictly stationary, ergodic, and have the following causal representations,*

$$y_t = \Gamma_1^T \sum_{k=1}^{\infty}[\prod_{i=0}^{k-1}(\Phi + \Gamma B_{t-i})]\Gamma e_{t-k} + e_t \tag{A.2}$$

and

$$a_t = \Gamma_2^T \sum_{k=1}^{\infty}[\prod_{i=0}^{k-1}(\Phi + \Gamma B_{t-i})]\Gamma e_{t-k} + e_t, \tag{A.3}$$

where $\Gamma_1 = (1, 0, \ldots, 0)^T_{(P+Q)\times 1}$, $\Gamma_2 = (0, \ldots, 0, 1, 0, \ldots, 0)^T_{(P+Q)\times 1}$ *with* $(P+1)-$*st component* 1, *and the right sides of* (A.2) *and* (A.3) *converge in mean square.*

Proof. We first show the sufficiency. Iterating equation (2.4), we have

$$\begin{aligned} U_t &= \Gamma e_t + \sum_{k=1}^{m}[\prod_{i=0}^{k-1}(\Phi + \Gamma B_{t-i})]\Gamma e_{t-k} + \prod_{i=0}^{m}(\Phi + \Gamma B_{t-i})U_{t-m-1} \\ &\triangleq S_{t,m} + R_{t,m}, \end{aligned}$$

where $R_{t,m} = \prod_{i=0}^{m}(\Phi + \Gamma B_{t-i})U_{t-m-1}$.

Note that $E(B_t \otimes B_t) = E(B_{t-j} \otimes B_{t-j})$ and $Ee^2_{t-j} = \sigma^2$ for any j, and $E(e_{t-i}e_{t-j}) = 0$ if $i \neq j$,

$$\begin{aligned} &\text{vec}E(S_{t,m}S^T_{t,m}) \\ =&\text{vec}E\{(\sum_{k=1}^{m}[\prod_{i=0}^{k-1}(\Phi+\Gamma B_{t-i})]\Gamma e_{t-k})(\sum_{k=1}^{m}[\prod_{i=0}^{k-1}(\Phi+\Gamma B_{t-i})]\Gamma e_{t-k})^T\} + \ \text{vec}(\Gamma\Gamma^T)\sigma^2 \end{aligned}$$

$$
\begin{aligned}
=&\sum_{k=1}^{m} E\ \mathrm{vec}\{[\prod_{i=0}^{k-1}[(\Phi+\Gamma B_{t-i})]\Gamma\Gamma^T[\prod_{i=0}^{k-1}(\Phi+\Gamma B_{t-i})]^T\}\sigma^2+\ \mathrm{vec}(\Gamma\Gamma^T)\sigma^2\\
=&\sum_{k=1}^{m} E\{\prod_{i=0}^{k-1}[(\Phi+\Gamma B_{t-i})\otimes(\Phi+\Gamma B_{t-i})]\}\mathrm{vec}(\Gamma\Gamma^T)\sigma^2+\mathrm{vec}(\Gamma\Gamma^T)\sigma^2\\
=&\sum_{k=1}^{m}(E[(\Phi+\Gamma B_{t-i})\otimes(\Phi+\Gamma B_{t-i})])^k\mathrm{vec}(\Gamma\Gamma^T)\sigma^2+\mathrm{vec}(\Gamma\Gamma^T)\sigma^2\\
=&\sum_{k=1}^{m}[\Phi\otimes\Phi+(\Gamma\otimes\Gamma)E(B_{t-i}\otimes B_{t-i})]^k\mathrm{vec}(\Gamma\Gamma^T)\sigma^2+\mathrm{vec}(\Gamma\Gamma^T)\sigma^2\\
=&\sum_{k=1}^{m}(\Phi\otimes\Phi+C)^k\mathrm{vec}(\Gamma\Gamma^T)\sigma^2+\mathrm{vec}(\Gamma\Gamma^T)\sigma^2, \qquad \text{(A.4)}
\end{aligned}
$$

where the identity $(\prod A_i)\otimes(\prod B_i)=\prod(A_i\otimes B_i)$ is used. By (A.1), $S_{t,m}$ converges in mean square as $m\to\infty$. Denote this limit by S_t. Then

$$S_t=\sum_{k=1}^{\infty}[\prod_{i=0}^{k-1}(\Phi+\Gamma B_{t-i})]\Gamma e_{t-k}+\Gamma e_t.$$

It is easy to verify that $\{S_t\}$ satisfies (2.4) and is F_t-measurable. By (A.4), $\mathrm{vec}E(S_tS_t^T)=\lim_{m\to\infty}\mathrm{vec}E(S_{t,m}S_{t,m}^T)$ is finite. Now let $y_t=\Gamma_1^TS_t$ and $a_t=\Gamma_2^TS_t$. Then $\{y_t\}$ and $\{a_t\}$ is an F_t-measurable second-order stationary solution of model (1.1)-(1.2) and have the causal representations (A.2) and (A.3), respectively. Since y_t and a_t are measurable functions of i.i.d. random variables $\{e_i,w_i,\delta_i,i=t,t-1,\ldots\}$, $\{y_t\}$ and $\{a_t\}$ are strictly stationary and ergodic.

The proof of the necessity part is omitted since it is quite similar to the proof of Theorem 2.2 of Nicholl and Quinn (1982). This completes the proof.

Lemma A.1. *If* $\sum_{k=1}^m(\Phi\otimes\Phi+C)^k vec(\Gamma\Gamma^T)$ *converges as* $m\to\infty$*, then* $\rho(\Phi)<1$.

Proof. Denote the matrix $\Omega=E(S_tS_t^T)/\sigma^2$. By stationarity,

$$\Omega=\Phi\Omega\Phi^T+E(\Gamma B_t\Omega B_t^T\Gamma^T)+\Gamma\Gamma^T.$$

Suppose that Φ has an eigenvalue λ with corresponding left eigenvector $z=(z_1,\ldots,z_{P+Q})^T$. Then

$$
\begin{aligned}
z^T\Omega\bar z&=z^T\Phi\Omega(\bar z^T\Phi)^T+z^TE(\Gamma B_t\Omega B_t^T\Gamma^T)\bar z+z^T\Gamma\Gamma^T\bar z\\
&=|\lambda|^2z^T\Omega\bar z^T+z^TE(\Gamma B_t\Omega B_t^T\Gamma^T)\bar z+|z_1+z_{P+1}|^2,
\end{aligned}
$$

where $\bar z$ is the complex conjugate of z.

We now show that $z_1 + z_{P+1} \neq 0$. Since z is a left eigenvector of Φ, we have

$$\begin{aligned}
&(z_1, \ldots, z_{P+Q})\Phi = (z_1, \ldots, z_{P+Q}) \\
&\cdot \left(\begin{array}{ccc|ccc}
\psi_1 & \cdots & \psi_P & \theta_1 & \cdots & \theta_Q \\
& I_{(P-1)\times(P-1)} & O_{(P-1)\times 1} & & O_{(P-1)\times Q} & \\
\hline
& & & 0 & \cdots & 0 \\
& O_{Q\times P} & & & I_{(Q-1)\times(Q-1)} & O_{(Q-1)\times 1}
\end{array}\right) \\
&= \lambda(z_1, \ldots, z_{P+Q}).
\end{aligned} \tag{A.5}$$

From (A.5), it is easy to obtain that

$$z_1\psi_i + z_{i+1} = \lambda z_i, \quad i = 1, \ldots, P-1, \quad z_1\psi_P = \lambda z_P \tag{A.6}$$

and

$$z_1\theta_i + z_{P+i+1} = \lambda z_{P+i}, \quad i = 1, \ldots, Q-1, \quad z_1\theta_Q = \lambda z_{P+Q}. \tag{A.7}$$

From (A.6), we get

$$z_1(\lambda^P - \psi_1\lambda^{P-1} - \cdots - \psi_P) = 0. \tag{A.8}$$

If $z_1 + z_{P+1} = 0$, then from (A.7),

$$z_1(\lambda^Q + \theta_1\lambda^{Q-1} + \cdots + \theta_Q) = 0. \tag{A.9}$$

From (A.8)-(A.9) and Assumption 5, we have that $z_1 = 0$ and hence $z_{P+1} = 0$. From (A.6) and (A.7), we obtain $z_1 = z_2 = \cdots = z_{P+Q} = 0$, i.e., $z = 0$. This contradicts the fact that z is a left eigenvector of Φ. Thus $z_1 + z_{P+1} \neq 0$, which implies that $|\lambda| < 1$. This completes the proof.

References

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *J. Econometrics* **31**, 307-327.

Engle, R. F. (1982). Autoregressive conditional heteroskedasticity with estimates of variance of U. K. inflation. *Econometrica* **50**, 987-1008.

Feigin, P. D. and Tweedie, R. L. (1985). Random coefficient autoregressive processes. *J. Time Ser. Anal.* **6**, 1-14.

Horn, R. A. and Johnson, C. R. (1990). *Matrix Analysis*. Cambridge University Press, Cambrioue.

Ling, Shiqing and Deng, W. C. (1993). Parametric estimate of multivariate autoregressive models with conditional heterocovariance matrix errors. *Acta Math. Appl. Sinica* **16**, 517-533

Nicholls, D. F. and Quinn, B. G. (1982). *Random Coefficient Autoregressive Model: An Introduction*. Lecture Notes in Statistics, Vol 11. Springer-Verlag, New York.

Srivastava, M. S. and Khatri, C. G. (1979). *An Introduction to Multivariate Statistics.* North Holland, New York.

Tsay, R. S. (1987). Conditional heteroscedastic time series models. *J. Amer. Statist. Assoc.* **7**, 590-604.

Tweedie, R. L. (1988). Invariant measure for Markov chains with no irreducibility assumptions. *J. Appl. Probab.* **25A**, 275-285.

Weiss, A. A. (1984). ARMA models with ARCH errors. *J. Time Ser. Anal.* **5**, 129-143.

Wong, H. and Li, W. K. (1997). On a multivariate conditional heteroscedastic model. *Biometrika* **84**, 111-123.

Department of Economics, University of Western Australia, Australia.

E-mail: sling@ecel.uwa.edu.au

(Received July 1996; accepted March 1999)