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PREDICTION INTERVALS FOR WEIBULL ORDER STATISTICS

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Abstract: Using a conditional method, explicit formulae for computing quantiles pertinent to prediction intervals for future Weibull order statistics are developed for two cases: when only previous independent failure data are available, and when both previous independent failure data and early-failure data in current experiment are available. The second case includes the case when only current early-failure data are available. Comparisons of interval widths are made for different estimators of parameters and different ways of forming prediction intervals.

Key words and phrases: Ancillaries, BLIE, conditional method, equivariant statistic, extreme value distribution, k-out-of-n:F system, MLE, Type II censored data.

1. Introduction

Suppose that n components of the same type are put on a life test simultaneously, and the experimenter wants to set a prediction interval for the kth failure time. Before the experiment starts, one would obtain the prediction interval based on previously gathered data. If the experiment has started and some failures have occurred, one would then use both previous and current early-failure data. The prediction intervals developed in this paper can also be used to predict the total duration time in a Type II censoring experiment, and to predict the lifetime of an k-out-of-n:F system, as the lifetime of such a system is same as the kth failure time of the n components composing the system.

Prediction intervals for exponential distributions have been discussed by Lawless (1971, 1972, 1977), and Hsieh and Wang (1992). Odeh (1990) considered a prediction problem for the normal distribution. For the Weibull distribution, Lawless (1973b) used a conditional method to obtain a prediction interval for the first order statistic of a set of future observations, based on previously collected data; Hsieh (1996) used the same technique to obtain prediction intervals for future observations, based on only early-failure data of an on-going experiment. Here, we extend the prediction problem to the case of using both previous independent data and early-failure data of the on-going experiment. Explicit formulae are developed for computing quantiles of pertinent pivotal quantities. The idea of using both previous data and early-failure data of an on-going life test to predict a future component failure time is a promising new strategy.

The distribution theory for estimators of unknown parameters in Weibull models is complicate and can not be described in explicit forms. Nevertheless, using a conditional method, many problems become analytically manageable. The conditional method used in this paper is the one conditioned on ancillary statistics, which was first suggested by Fisher (1934) and promoted further by a number of others (Cox (1958), Buehler (1959)). Lawless (1973a, 1973b, 1974, 1978) applied this conditional method to different problems relating to the Weibull and extreme value distributions. In the conditional method, quantiles for constructing prediction intervals depend on ancillary statistics of observed data. Yet, the overall probability of coverage for the interval remains equal to the preassigned confidence coefficient. Hall, Prairie and Motlagh (1975), on the other hand, took an unconditional non-parametric approach. Engelhardt and Bain (1979) discussed an approximate procedure for predicting the kth smallest Weibull observation, but the procedure is complicated and requires interpolation of quantiles of an F-distribution with non-integer degrees of freedom. The procedure is hard to be implemented into a computer program.

The prediction intervals considered in this paper involve parameter estimators. We are particularly interested in equivariant estimators such as the maximum likelihood estimators (MLE), and the best linear invariant estimators (BLIE). Explicit formulae for computing conditional quantiles are developed for two cases depending on available data: when only previous failure data are available, or when both previous and current early-failure data are available. Some comparisons are made for prediction intervals using two different estimators. For each estimator, I also compare interval widths of two different ways of forming prediction intervals: using simulated quantiles of a pivotal statistic, and using conditional quantiles. Coverage probabilities of these procedures under log-normal distributions are also investigated. Comparisons are also made for using previous data alone, using current failure data alone, and using both previous and current data.

In the following discussion, we use a summation with asterisk defined by

$$\sum_{q,s}^{*} x_i = \sum_{i=1}^{s} x_i + (q-s)x_s.$$

2. Using Only Previous Independent Data

2.1. The problem and notation

Let $\vec{U} = (U_1, \ldots, U_p)$ be a vector of ordered Type II censored data with sample size m and number of failures p, taken from a Weib(θ, δ), where Weib(θ, δ) represents a Weibull distribution with scale parameter θ and shape parameter δ.

Let $(V_1, \ldots, V_n)$ is a vector of future order statistics of size n also taken from the same distribution, and the vector be independent of $\vec{U}$. The goal is to predict the kth order statistic V_k $(1 \leq k \leq n)$ based on $\vec{U}$ alone.

For theoretical developments, it is more convenient to work with logarithms of the U_is, and V_is. Let $X_i = \log(U_i)$, $i = 1, \ldots, p$; $Y_i = \log(V_i)$, $i = 1, \ldots, n$; and $\vec{X} = (X_1, \ldots, X_p)$. It is known that if U has a Weib(θ,δ), then $\log(U)$ has the smallest extreme value distribution with location parameter u and scale parameter b, where $u = \log \theta$ and $b = 1/\delta$. Such an extreme value distribution is denoted as SEV(u,b), and its reliability function is: $R(y) = \exp[-\exp(\frac{y-u}{b})]$, $-\infty < y < \infty$. The log transformation converts the scale parameter θ and the shape parameter δ of the Weib(θ, δ) to a location parameter u and a scale parameter b in an SEV, respectively. In estimation theory, it is easier to handle location and scale parameters, especially, when data are Type II censored order statistics. Two commonly used estimators for (u, b), based on order statistics, are the MLE and BLIE; they are well discussed in, e.g., Mann, Schafer and Singpurwalla (1974), Lawless (1982), and Bain and Engelhardt (1991).

If $(\tilde{u}, \tilde{b})$ denotes the estimators for (u, b), based on $\vec{X}$, then $(\tilde{u}, \tilde{b})$ is an equivariant estimator of (u, b): the distributions of $(\tilde{u}-u)/b$ and $\tilde{b}/b$ are parameter-free (See Lawless (1982), Theorem 4.1.1). Let $\tilde{\theta} = \exp(\tilde{u})$ and $\tilde{\delta} = 1/\tilde{b}$. If $(\tilde{u}, \tilde{b})$ is the MLE of (u, b) based on $\vec{U}$, then, by the invariance property of MLE, the associated $(\tilde{\theta}, \tilde{\delta})$ is the MLE of (θ, δ) based on $\vec{X}$.

Since we do not use any observation in the second sample, we choose $\tilde{u}$ as a 'middle' point for predicting Y_k. A γ-prediction interval for Y_k will be of the form $[\tilde{u} + t_1\tilde{b},\ \tilde{u} + t_2\tilde{b}]$, where t_1 and t_2 satisfy $\gamma = \Pr[\tilde{u} + t_1\tilde{b} < Y_k < \tilde{u} + t_2\tilde{b}]$; a γ-prediction interval for the future V_k is then given by $[\exp(\tilde{u}+t_1\tilde{b}),\ \exp(\tilde{u}+t_2\tilde{b})]$.

To choose t_1 and t_2 we consider the pivotal statistic $W_k^* = (Y_k - \tilde{u})/\tilde{b}$. It suffices to work out the cdf of W_k^*. Now the cdf of W_k^* can not be expressed in explicit form. Nonetheless, since W_k^* is pivotal, one can use simulation to obtain approximate quantiles, t_1 and t_2, if one wishes to do so. Here, however, we take an analytical approach. We use a conditional method to obtain an explicit formula for a conditional cdf of W_k^* which is used to construct the required prediction interval.

2.2. The conditional CDF of W_k^*

There are two parameters, u and b, to be estimated. We need at least two observations, i.e., $p \geq 2$. For $p = 2$, an explicit expression for the unconditional cdf of W_k^* can be found, which is given in the Appendix (as it is less important in practice).

Since the cdf of W_k^* for $p \geq 3$ is not available, data-independent quantiles for W_k^* can not be computed. Instead, we use a conditional method developed in

Lawless (1973a, 1973b, 1978) to obtain data-dependent analytical solutions which preserve the confidence coefficient. Define a vector of ancillaries $\vec{c} = (c_1, \ldots, c_p)$ with $c_i = (X_i - \tilde{u})/\tilde{b}$, $i = 1, \ldots, p$. Let $\nu = \sum_{i=1}^{p} c_i$, and $\varphi(z) = \sum_{m,p}^{*} \exp(c_i z)$. If for each $\vec{c}$, we choose $t \equiv t(\vec{c})$ such that $\Pr[W_k^* \le t(\vec{c}) \mid \vec{c}] = \gamma$, then

$$\Pr[W_k^* \le t(\vec{c})] = E(\Pr[W_k^* \le t(\vec{c}) \mid \vec{c}]) = \gamma. \tag{1}$$

So, it suffices to consider the conditional cdf of W_k^*. Using a conditional expectation theorem, and the fact that Y_k and $(\tilde{u}, \tilde{b})$ are independent, we get

$$\begin{aligned}
\Pr[W_k^* \le t \mid \vec{c}] &= \Pr[Y_k \le \tilde{u} + t\,\tilde{b} \mid \vec{c}] \\
&= E(\Pr[Y_k \le \tilde{u} + t\tilde{b} \mid \tilde{u}, \tilde{b}] \mid \vec{c}) \\
&= \sum_{i=k}^{n} \binom{n}{i} \sum_{j=0}^{i} \binom{i}{j} (-1)^j E([R(\tilde{u} + t\tilde{b})]^{n+j-i} \mid \vec{c}) \\
&= \sum_{i=k}^{n} \binom{n}{i} \sum_{j=0}^{i} \binom{i}{j} (-1)^j Q(t, n+j-i, \vec{c}) \\
&\text{(or)} \\
&= 1 - \sum_{i=0}^{k-1} \binom{n}{i} \sum_{j=0}^{i} \binom{i}{j} (-1)^j Q(t, n+j-i, \vec{c}),
\end{aligned} \tag{2}$$

where

$$\begin{aligned}
Q(t, l, \vec{c}) &\equiv E([R(\tilde{u} + t\tilde{b})]^l \mid \vec{c}) \\
&= E(\exp[-l \exp(Z_1 Z_2 + t Z_2)] \mid \vec{c}),
\end{aligned} \tag{3}$$

with $Z_1 = (\tilde{u} - u)/\tilde{b}$, and $Z_2 = \tilde{b}/b$. Using a general formula given in Lawless (1982), p. 148, the conditional pdf of (Z_1, Z_2) given $\vec{c}$, in our case, can be written as

$$f(z_1, z_2 \mid \vec{c}) = g(\vec{c}, p, m) z_2^{p-1} \exp\Big\{(\sum_{i=1}^{p} c_i) z_2 + p z_1 z_2 - [\sum_{m,p}^{*} \exp(c_i z_2)] e^{z_1 z_2}\Big\}$$

for $-\infty < z_1 < \infty$, $0 < z_2$, where $g(\vec{c}, p, m) = K_1/\Gamma(p)$ and

$$K_1 = \Big[\int_0^{\infty} z^{p-2} \exp(\nu z)[\varphi(z)]^{-p} dz\Big]^{-1}.$$

Hence, the conditional expectation of (3) is given by

$$Q(t, l, \vec{c}) = \int_0^{\infty} \int_{-\infty}^{\infty} \exp[-l e^{z_1 z_2 + t z_2}] f(z_1, z_2 | \vec{c}) dz_1 dz_2$$

$$= \frac{K_1}{\Gamma(p)} \int_o^{-\infty} \int_{-\infty}^{\infty} z_2^{p-1} \exp\Big\{ (\sum_{i=1}^{p} c_i) z_2 + p z_1 p z_2$$

$$- [l e^{t z_2} + \sum_{m,p}^{*} \exp(c_i z_2)] e^{z_1 z_2} \Big\} dz_1 dz_2$$

$$= K_1 \int_0^{\infty} \frac{z^{p-2} \exp[(\sum_{i=1}^{p} c_i) z]}{[l \exp(tz) + \sum_{m,p}^{*} \exp(c_i z)]^p} dz. \tag{4}$$

In particular, when $k = 1$, (2) reduces to formula (9) of Lawless (1973b).

Equation (1) implies that a $100(1-\alpha)\%$ prediction interval for V_k can be obtained by choosing t_1, and t_2 such that $\Pr[Y_k \le \tilde{u} + t_1 \tilde{b} \mid \vec{c}] = \alpha_1; \Pr[Y_k \le \tilde{u} + t_2 \tilde{b} \mid \vec{c}] = 1 - \alpha_2$, where $\alpha_1 + \alpha_2 = \alpha$, then forming an interval $[\tilde{u} + t_1 \tilde{b},\ \tilde{u} + t_2 \tilde{b}]$ for Y_k, and $[\exp(\tilde{u} + t_1 \tilde{b}), \exp(\tilde{u} + t_2 \tilde{b})]$ for V_k. Experience indicates that the conditional pdf of W_k^* for small k is skewed to right; consequently the prediction intervals are shorter by taking unequal tail-probabilities, say, $\alpha_1 < \alpha_2$, than by taking equal tail-probabilities. Since (2) is increasing in t, for a given probability γ, the corresponding quantile can be easily determined by a simple computation routine. To evaluate the integral in (4), we can adopt a method suggested in Lawless (1978) that one integrates the function over the ranges $z = 0$ to 1, $z = 1$ to 2, $z = 2$ to 3, and so on. After each integration the relative size of the addition to the total area thus far is computed is obseved, and when this becomes sufficiently small, say, less than 10^{-5}, the process stops. Usually, the process stops in only few steps and integration up to $z = 10$ is sufficient.

2.3. Comparison of different methods

To evaluate the performance of the conditional method, the proposed method is compared with the unconditional method by simulations. In each case, both MLE and BLIE procedures for estimating (u, b) are considered for $m = 10$, $p = 8$, $n = 4$, and $k = 3,\ 4$. The distribution parameters consedered are $u = 0$ and $b = 0.5,\ 1.0,\ 1.6$. These parameters are equivalent to $\theta = 1$ and $\delta = 2,\ 1,\ 0.625$, respectively, in a Weib(θ, δ) distribution. The confidence coefficient is 0.90 with $\alpha_1 = 0.01$ and $\alpha_2 = 0.09$.

The quantiles for the conditional method (CON) were computed through (2). The unconditional (UNC) quantiles for W_k^* were obtained through simulations. Since simulated quantiles vary from one simulation to another, I took the average of 100 simulated quantiles of the same assigned lower probability, each generated from a sample of size 1,000 from the standard SEV$(0, 1)$, as the distribution of W_k^* does not depend on parameters u or b. The number of repetitions for each parameter combination is 200. The results are presented in Table 1a. The table includes relative frequencies of covering future values V_k ($k = 3, 4$), average widths of prediction intervals, and standard deviations (SD) of the simulated interval widths. It indicates that all relative frequencies are close to the nominal 0.90. All methods show relatively close average widths and standard deviations. The conditional method with BLIE estimators has slightly shorter average widths for $b = 0.5,\ 1,\ 1.6$ when $k = 3$, and $b = 0.5,\ 1$ when $k = 4$. Yet, none of them is absolutely superior to the others. The large values of average widths and SDs for $k = 4$ and $b = 1.6$ in Table 1a are due to some large values of widths in the simulated samples. Alternative simulations would yield to different results. Yet, the relative frequencies of coverage are the focus of this table.

I also studied some coverage probabilities of the two procedures developed for the Weibull distribution when data were taken from the log-normal distribution, $\mathrm{LN}(u, b)$, such that the logarithm of a variate from that distribution has a normal distribution with mean u and standard deviation b. The procedures for constructing prediction intervals used in Table 1a were carried out under the situations that data were generated from $\mathrm{LN}(0, b)$ for $b = 0.5,\ 1,\ 1.6$. The standard normal deviates were generated according to Box and Muller (1958). Some simulation results are given in Table 1b. This table shows that when the data are from a log-normal distribution the estimated probabilities of covering the 3rd failure time (V_3) for the studied cases are close to the nominal 0.9. The procedures are less reliable for predicting the later failure time (V_4); the actual confidence levels for this case are around 0.8 for both unequal-tail and equal-tail intervals.

Table 1a. 90% Prediction intervals using different methods
$m = 10,\ p = 8,\ n = 4,\ u = 0,\ \alpha_1 = 0.01,\ \alpha_2 = 0.09$

			MLE			BLIE		
k	b	Method	Relative Frequency	Average Width	SD	Relative Frequency	Average Width	SD
3	0.5	UNC	0.895	1.280	0.427	0.895	1.281	0.432
		CON	0.900	1.280	0.428	0.895	1.274	0.440
	1.0	UNC	0.900	2.696	1.532	0.895	2.702	1.539
		CON	0.905	2.705	1.540	0.885	2.634	1.593
	1.6	UNC	0.885	6.267	5.200	0.885	6.305	5.382
		CON	0.885	6.322	5.249	0.875	6.291	5.279
4	0.5	UNC	0.930	2.126	1.017	0.930	2.129	1.030
		CON	0.930	2.111	1.008	0.930	2.111	1.008
	1.0	UNC	0.900	7.489	6.747	0.900	7.532	6.936
		CON	0.900	7.495	6.819	0.895	7.472	6.837
	1.6	UNC	0.910	33.075	51.987	0.910	33.023	51.985
		CON	0.905	33.350	52.853	0.905	33.351	52.887

Table 1b. Coverage probabilities of the 90% prediction intervals under log-normal distributions $m = 10,\ p = 8,\ n = 4,\ u = 0$

			Unequal-Tail ($\alpha_1 = 0.01,\ \alpha_2 = 0.09$)		Equal-Tail ($\alpha_1 = 0.05,\ \alpha_2 = 0.05$)	
k	b	Method	MLE	BLIE	MLE	BLIE
3	0.5	UNC	0.875	0.875	0.880	0.880
		CON	0.875	0.875	0.880	0.880
	1.0	UNC	0.890	0.890	0.890	0.890
		CON	0.895	0.885	0.890	0.890
	1.6	UNC	0.875	0.880	0.890	0.890
		CON	0.875	0.865	0.890	0.890
4	0.5	UNC	0.775	0.780	0.805	0.805
		CON	0.760	0.765	0.805	0.805
	1.0	UNC	0.770	0.775	0.800	0.800
		CON	0.770	0.770	0.800	0.800
	1.6	UNC	0.780	0.780	0.830	0.830
		CON	0.785	0.785	0.830	0.830

2.4. Numerical example

Mann and Fertig (1973) provided the following first 10 failure times (in hours) of 13 airplane components:

$$0.22,\ 0.50,\ 0.88,\ 1.00,\ 1.32,\ 1.33,\ 1.54,\ 1.76,\ 2.50,\ 3.00$$

(See also Lawless (1982), Example 4.1.1). These data fit a Weibull model well. Here, $m = 13$, $p = 10$, and both MLE and BLIE for (u, b) are considered. Suppose this kind of components are to be used in a 3-out-of-4:F system. The system has 4 components, and the system fails when 3 or more components fail to function properly. The failure time for this system is the third component failure time, i.e., V_3. We want to find a 90% prediction interval for V_3 in a sample of 4 components.

The prediction interval is computed using eight methods: combinations of unconditional/conditional, MLE/BLIE, and unequal-tail/equal-tail. The results are shown in Table 2. The first portion of the table is a sample of the first five simulations. It reveals that the simulated quantiles (t_1 and t_2) using the unconditional method vary from one simulation to another. Again, I used the average of 100 such quantiles to stabilize the values (those with asterisks). As expected, the intervals constructed using unequal tail-probabilities give shorter widths than those using equal tail-probabilities. Also, the conditional method gives slightly shorter width than that of the unconditional method.

Table 2. 90% Prediction intervals for V_3 using airplane data $m = 13,\ p = 10,\ n = 4,\ k = 3$

Method	$\tilde{u}$	$\tilde{b}$	t_1	t_2	P.I. for V_3	Width
Using Unequal Tail-Probabilities: $\alpha_1 = 0.01,\ \alpha_2 = 0.09$						
UNC/MLE	0.821	0.705	−2.410	1.003	[0.415, 4.613]	4.197
			−2.638	1.100	[0.354, 4.938]	4.584
			−2.471	1.087	[0.398, 4.893]	4.495
			−2.864	1.014	[0.301, 4.647]	4.345
			−2.651	1.039	[0.350, 4.732]	4.382
UNC/MLE*	0.821	0.705	−2.655	1.045	[0.349, 4.750]	4.400
UNC/BLIE*	0.873	0.715	−2.699	0.976	[0.347, 4.810]	4.463
CON/MLE	0.821	0.705	−2.578	1.040	[0.369, 4.735]	4.366
CON/BLIE	0.873	0.715	−2.613	0.953	[0.369, 4.733]	4.364
Using Equal Tail-Probabilities: $\alpha_1 = 0.05,\ \alpha_2 = 0.05$						
UNC/MLE*	0.821	0.705	−1.570	1.324	[0.751, 5.784]	5.033
UNC/BLIE*	0.873	0.715	−1.621	1.255	[0.751, 5.874]	5.123
CON/MLE	0.821	0.705	−1.566	1.318	[0.753, 5.762]	5.009
CON/BLIE	0.873	0.715	−1.617	1.229	[0.753, 5.764]	5.011

Note: * Quantiles t_1 and t_2 are based on the average of 100 quantiles generated from samples of size 1000.

2.5. Remarks on an approximate procedure

Engelhardt and Bain (1979) discussed an approximate 90% lower prediction limit (L_1) for the 5th failure time in a sample of 100 future items, based on the results of tests on the endurance of 23 deep-groove ball bearings provided by Lieblein and Zelen (1956). The data are in millions of revolutions before failure. Their approximations are $t_1 = -4.210$ and $L_1 = 10.59$. If we use (2) of this paper, then $t_1 = -4.344$ and $L_1 = 10.37$. A simulation study with 10,000 repetitions for quantiles of the pivotal statistic W_k^* gives $t_1 = -4.394$ and $L_1 = 10.124$. Here, the previous data is a complete sample ($m = p = 23$), and the second sample size ($n = 100$) is large. It is not clear how good the approximation will be, when the first sample is censored, or when the second sample is of small or medium size. Furthermore, the approximation procedure is complicated. Its importance becomes minor, especially, in the light of the exact procedure developed in this paper.

3. Using Both Previous and Current Data

In practice, it is sensible to use all available data. Suppose that in addition to the previously available data $\vec{X}$, we also have r $(1 \leq r \leq n)$ log failure times from the on-going life test, say, $Y_1, \ldots, Y_r$. Again, our goal is to construct a prediction interval for the kth failure time, V_k, of the second sample, where $r < k$ and $V_k \equiv \exp(Y_k)$. Here, we shall use both sets of available data.

Let x_i and y_i be realizations of X_i and Y_i, respectively. Then the MLE, $(\tilde{u}, \tilde{b})$, of (u, b) is obtained by solving for u and b in the following two equations:

$$e^u = \left[\frac{1}{r+p}\{\sum_{n,r}^{*} \exp(y_i/b) + \sum_{m,p}^{*} \exp(x_i/b)\}\right]^b,$$

and

$$b = \frac{\sum_{n,r}^{*} y_i \exp(y_i/b) + \sum_{m,p}^{*} x_i \exp(x_i/b)}{\sum_{n,r}^{*} \exp(y_i/b) + \sum_{m,p}^{*} \exp(x_i/b)} - \frac{1}{r+p}\{\sum_{n,r}^{*} y_i + \sum_{m,p}^{*} x_i\}.$$

For $k = 1$, one uses procedures discussed in Section 2. For $k \geq 2$, a different pivotal statistic will be used. We take $T_k^* = (Y_k - Y_r)/\tilde{b}$ as the pivot. The prediction interval for Y_k to be considered here is of the form $[Y_r + t_1\tilde{b},\ Y_r + t_2\tilde{b}]$. Using T_k^* guarantees that the values in the prediction interval will be at least as great as the last available observation Y_r. This makes sense as Y_k is always greater than Y_r. Should one use a pivotal similar to W_k^* of Section 2, a prediction interval for Y_k might include values less than Y_r, which is clearly not a desirable situation.

Again, we use the conditional method to determine t_1 and t_2. We first define ancillary statistics a_is and c_is for the current and previous data: $a_i = (Y_i - \tilde{u})/\tilde{b},\ i = 1, \ldots, r;\ c_i = (X_i - \tilde{u})/\tilde{b},\ i = 1, \ldots, p$. Let $\vec{a} = (a_1, \ldots, a_r)$ and $\vec{c} = (c_1, \ldots, c_p)$. We note that among the $r+p$ ancillaries, only $r+p-2$ of them are functionally independent. Let $\xi = \sum_{i=1}^{r} a_i + \sum_{j=1}^{p} c_j$.

To compute $\Pr[0 < T_k^* < t \mid \vec{a}, \vec{c}]$, we first find the joint pdf of $\{Y_1, \ldots, Y_r, Y_k, X_1, \ldots, X_p\}$, then find that of $\{W_k, Z_1, Z_2, a_1, \ldots, a_r, c_1, \ldots, c_{p-2}\}$, where $W_k = (Y_k - u)/b,\ Z_1 = (\tilde{u} - u)/\tilde{b}$, and $Z_2 = \tilde{b}/b$. Finally, the joint pdf of $\{W_k, Z_1, Z_2\}$ conditional on $\{\vec{a}, \vec{c}\}$ is found to be

$$f(w_k, z_1, z_2 \mid \vec{a}, \vec{c}) = \lambda \sum_{j=0}^{k-r-1} \binom{k-r-1}{j} (-1)^j z_2^{r+p-1}$$
$$\times \exp\{w_k + \xi z_2 + (r+p)z_1 z_2 - (n-k+j+1)e^{w_k} - B_j(z_2)e^{z_1 z_2}\}$$

for $a_r z_2 + z_1 z_2 < w_k < \infty$, $-\infty < z_1 < \infty$, $0 < z_2$, where

$$B_j(z) = \sum_{i=1}^{r} \exp(a_i z_2) + (k-r-1-j)\exp(a_r z_2) + \sum_{m,p}^{*} \exp(c_i z_2),$$

and λ is a normalization constant.

The cdf of T_k^* conditional on $\{\vec{a}, \vec{c}\}$ is

$$\begin{aligned}
&\Pr[0 < T_k^* < t \mid \vec{a}, \vec{c}] \\
&= \Pr[a_r Z_2 + Z_1 Z_2 < W_k < (a_r + t) Z_2 + Z_1 Z_2 \mid \vec{a}, \vec{c}] \\
&= \int_0^\infty \int_{-\infty}^\infty \int_{a_r z_2 + z_1 z_2}^{(a_r+t) z_2 + z_1 z_2} f(w_k, z_1, z_2 \mid \vec{a}, \vec{c}) dw_k dz_1 dz_2 \\
&= K_2 \sum_{j=0}^{k-r-1} \binom{k-r-1}{j} \left[\frac{(-1)^j}{(n-k+j+1)} \right] \\
&\quad \times \int_0^\infty z^{r+p-2} \exp(\xi z) \Big\{ \frac{1}{[A_{1j}(z)]^{r+p}} - \frac{1}{[A_{2j}(z)]^{r+p}} \Big\} dz, \qquad (5)
\end{aligned}$$

where

$$A_{1j}(z) = \sum_{n,r}^{*} \exp(a_i z) + \sum_{m,p}^{*} \exp(c_i z),$$

$$A_{2j}(z) = B_j(z) + (n-k+j+1) \exp[(a_r + t) z],$$

and

$$K_2 = \Big\{ \sum_{j=0}^{k-r-1} \binom{k-r-1}{j} \Big[\frac{(-1)^j}{(n-k+j+1)} \Big] \int_0^\infty \frac{z^{r+p-2} \exp(\xi z)}{[A_{1j}(z)]^{r+p}} dz \Big\}^{-1}.$$

For given $\vec{a}$ and $\vec{c}$, one can find t_1 and t_2 from expression (5) such that

$$\Pr[0 < T_k^* < t_1 \mid \vec{a}, \vec{c}] = \alpha_1; \quad \Pr[0 < T_k^* < t_2 \mid \vec{a}, \vec{c}] = 1 - \alpha_2,$$

with $\alpha_1 + \alpha_2 = \alpha$, to get an $100(1-\alpha)\%$ prediction interval for V_k, namely, $[\exp(y_r + t_1 \tilde{b}),\ \exp(y_r + t_2 \tilde{b})]$. Again, this prediction interval has confidence level $1 - \alpha$.

To demonstrate some advantages of using combined data, a simulation study was performed. Two sets of Type II censored data were generated from SEV distributions with $u = 0$ and different scale parameters ($b = 0.5,\ 1,\ 1.6$), under sampling sizes $m = 10$, $p = 8$, $n = 10$, and $r = 6$. The goal is to predict the 7th and the 10th observations of the second sample based on (i) only the first set of data, (ii) only available data of the second sample, and (iii) all available data in both samples. The conditional cdf formula for case (ii) can be obtained from (5) by setting $m = p = 0$. See Hsieh (1996) for more detailed discussion of this particular case.

For each of the 18 ($b \times$ datum type $\times s$) combinations, 200 prediction intervals were generated. Their relative frequencies of covering the corresponding V_k, average interval widths, and standard deviations are given in Table 3a. The first set of sample intervals for each parameter value are given in Table 3b.

Table 3a indicates that the method of using both previous and current data clearly produces shortest average widths and least standard deviations in all cases. Note, however, in some individual situations other methods may provide shorter intervals as shown in Table 3b. But this could not overrule the overall superiority of using combined data as demonstrated in the previous table.

The extremely large values of average widths, SDs and width range for predicting V_{10} under $b = 1.6$ in Tables 3a and 3b are due to high variability of $\tilde{b}$ in T_k^*: the estimator is based on only the first 6 failure times out of 10 samples, and the standard deviation of MLE for b is proportional to b. Besides, here we are making a 4-step ahead prediction. The more steps ahead prediction, the wider the prediction interval would be. These cause high right-skewness of the conditional distribution of T_k^* with $k = 10$, which leads to a large value of the 0.09 upper quantile (t_2). These may explain why those values are so large.

Table 3a. Performance of different 90% prediction intervals $m = 10,\ p = 8,\ n = 10,\ r = 6, u = 0, \alpha_1 = 0.01,\ \alpha_2 = 0.09$

		P.I. for V_7			P.I. for V_{10}		
		Relative	Average		Relative	Average	
b	Data	Frequency	Width	SD	Frequency	Width	SD
0.5	Previous	0.87	1.06	0.42	0.89	2.25	1.39
	Current	0.88	0.49	0.30	0.90	4.03	4.43
	Both	0.86	0.33	0.10	0.92	1.52	0.62
1.0	Previous	0.89	2.26	1.14	0.91	11.43	10.76
	Current	0.87	1.13	1.00	0.88	30.60	64.35
	Both	0.92	0.72	0.27	0.91	6.15	3.89
1.6	Previous	0.90	5.05	5.27	0.95	157.15	1095.44
	Current	0.92	3.11	4.74	0.93	3654.07	22872.76
	Both	0.93	1.37	0.82	0.92	28.56	31.78

Table 3b. Samples of 90% prediction intervals $m = 10,\ p = 8,\ n = 10,\ r = 6,\ u = 0,\ \alpha_1 = 0.01,\ \alpha_2 = 0.09$

b	Data	$\tilde{u}$	$\tilde{b}$	P.I. for V_7	V_7	P.I. for V_{10}	V_{10}
0.5	Previous	−0.03	0.43	[0.45, 1.51]	1.04	[0.87, 2.95]	1.96
	Current	−0.02	0.35	[0.94, 1.32]	1.04	[1.04, 3.14]	1.96
	Both	−0.02	0.40	[0.94, 1.21]*	1.04	[1.07, 2.24]*	1.96
1.0	Previous	0.43	0.59	[0.54, 2.82]	1.65	[1.34, 6.99]*	4.64
	Current	0.60	1.39	[1.48, 5.71]	1.65	[2.26, 176.94]	4.64
	Both	0.43	0.77	[1.48, 2.63]*	1.65	[1.98, 10.94]	4.64
1.6	Previous	0.14	0.98	[0.20, 3.17]	1.64	[0.92, 14.42]*	6.79
	Current	0.45	2.03	[1.28, 8.51]	1.64	[2.32, 1123.74)]	6.79
	Both	0.20	1.33	[1.27, 2.97]*	1.64	[1.93, 25.36]	6.79

Note: '*' indicates the shortest interval in each group of three intervals.

4. Summary

Explicit formulae have been developed for computing conditional quantiles of some useful pivotal statistics. One formula is for that based on only previous independent observations; the other is for that based on both previous and current early-failure data. The quantiles can be used to construct prediction intervals for arbitrary kth failure time in a sample of future observations. The results can be used to predict the total duration time in a Type II censoring life testing experiment, and to predict the lifetime of an k-out-of-n:F system. The computation procedure can be easily programmed and implemented for practical use.

Although the quantiles of the pivotal statistics considered in this paper can be obtained through simulation, as we have demonstrated, simulation results are unstable; they vary from one to another. From theoretical as well as practical points of view, analytical solutions should be used if they are available. The results of this paper provide such analytical solutions.

Furthermore , the techniques used in this paper can be applied to obtaining explicit formulae for computing conditional quantiles relating to prediction intervals for any other location-scale distributions.

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Appendix. Distribution of W_k^* for $p = 2$

Take $\tilde{b} = (X_2 - X_1)/k_1$ and $\tilde{u} = X_2 - k_2\tilde{b}$, where k_1 and k_2 satisfy $E(\tilde{b}) = b$ and $E(\tilde{u}) = u$; explicitly, $k_1 = E(T_2) - E(T_1)$, $k_2 = E(T_2)$, where $E(T_i)$ is the expectation of the ith order statistic of a sample of size m taken from an SEV(0,1). A formula for computing $E(T_i)$ can be found in Mann, Schafer and Singpurwalla (1974), p. 210. The cdf of W_k^* when $p = 2$ is

$$\begin{aligned}\Pr[W_k^* \le t] &= E(\Pr[Y_k \le \tilde{u} + t\,\tilde{b} \mid \tilde{u}, \tilde{b}]) \\ &= \sum_{i=k}^{n} \binom{n}{i} \sum_{j=0}^{i} \binom{i}{j} (-1)^j E([R(\tilde{u} + t\,\tilde{b})]^{n+j-i}),\end{aligned}$$

where

$$E([R(\tilde{u} + t\,\tilde{b})]^l) = \int_0^\infty \frac{m(m-1)\exp(-x)}{\{(m-1) + \exp(-x) + l\,\exp[(t-k_2)x/k_1]\}^2} dx.$$

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