The bias mapping of the Yule–Walker estimator is a contraction

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Supplementary Material

S1. Proof of Proposition 2

Proof. We first simplify the matrix $\frac{1}{T}\Gamma^{-1}\mathbf{c}$ as follows:

$$\frac{1}{T}\Gamma^{-1}\mathbf{c} = \frac{1}{T(\gamma_0^2-\gamma_1^2)}\begin{pmatrix}\gamma_0 & -\gamma_1\\ -\gamma_1 & \gamma_0\end{pmatrix}\begin{pmatrix}\gamma_1(1+a_2)\\ 2\gamma_2+\gamma_1 a_1\end{pmatrix} = \frac{1}{T(\gamma_0^2-\gamma_1^2)}\begin{pmatrix}\gamma_0\gamma_1 - 2\gamma_1\gamma_2 - \gamma_1^2 a_1 + \gamma_0\gamma_1 a_2\\ 2\gamma_0\gamma_2 - \gamma_1^2 + \gamma_0\gamma_1 a_1 - \gamma_1^2 a_2\end{pmatrix}.$$

Thus

$$\begin{pmatrix}0\\ \frac{1}{T}\Gamma^{-1}\mathbf{c}\end{pmatrix} = \frac{1}{T(\gamma_0^2-\gamma_1^2)}\begin{pmatrix}0 & 0 & 0\\ \gamma_0\gamma_1 - 2\gamma_1\gamma_2 & -\gamma_1^2 & \gamma_0\gamma_1\\ 2\gamma_0\gamma_2-\gamma_1^2 & \gamma_0\gamma_1 & -\gamma_1^2\end{pmatrix}\begin{pmatrix}1\\ \mathbf{a}\end{pmatrix}.$$

Plugging the above expressions into equation (2.4) of the main file, we end up with the formula for the Yule–Walker bias mapping,

$$\begin{pmatrix} 1 \\ E(\hat{\mathbf{a}}) \end{pmatrix} = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{T} \begin{pmatrix} 0 & 0 & 0 \\ -k & 1 & k \\ -(1+k) & 0 & 3+k \end{pmatrix} + \frac{1}{T(\gamma_0^2 - \gamma_1^2)} \begin{pmatrix} 0 & 0 & 0 \\ \gamma_0\gamma_1 - 2\gamma_1\gamma_2 & -\gamma_1^2 & \gamma_0\gamma_1 \\ 2\gamma_0\gamma_2 - \gamma_1^2 & \gamma_0\gamma_1 & -\gamma_1^2 \end{pmatrix} \right] \begin{pmatrix} 1 \\ \mathbf{a} \end{pmatrix}$$
$$+ o\left(\frac{1}{T}\right). \tag{S1.1}$$

Using elementary matrix algebra, the above equation reduces to the expression given in equation (2.6) in the main file. This completes the proof. □

S2. Proof of Proposition 3

Proof. Denote $\mathbf{g}(\mathbf{a}) = (g_1(\mathbf{a}), g_2(\mathbf{a}))'$, as defined in (2.11) in the main file. Then

$$\frac{\partial g_1(\mathbf{a})}{\partial a_1} = 1 - \frac{1}{T} + \frac{[3a_1^2 - 4a_2 - 3a_2^2 - 1][(1+a_2)^2 - a_1^2] + 2a_1^2(a_1^2 - 4a_2 - 3a_2^2 - 1)}{T\,[(1+a_2)^2 - a_1^2]^2}$$

$$= 1 - \frac{1}{T} + \frac{[-3[(1+a_2)^2 - a_1^2] + 2(1+a_2)][(1+a_2)^2 - a_1^2] + 2a_1^2[a_1^2 - (1+a_2)(1+3a_2)]}{T\,[(1+a_2)^2 - a_1^2]^2}$$

$$= 1 - \frac{4}{T} + \frac{2(1+a_2 - a_1^2)}{T\,[(1+a_2)^2 - a_1^2]} - \frac{4a_1^2 a_2(1+a_2)}{T\,[(1+a_2)^2 - a_1^2]^2}$$

and

$$\frac{\partial g_1(\mathbf{a})}{\partial a_2} = -\frac{k}{T} + \frac{-2a_1[2+3a_2][(1+a_2)^2 - a_1^2] - 2a_1(1+a_2)(a_1^2 - 4a_2 - 3a_2^2 - 1)}{T\left[(1+a_2)^2 - a_1^2\right]^2}$$

$$= -\frac{k}{T} - \frac{2a_1[(1+a_2)^2 - a_1^2(1+2a_2)]}{T\left[(1+a_2)^2 - a_1^2\right]^2}$$

$$= -\frac{k}{T} - \frac{2a_1}{T\left[(1+a_2)^2 - a_1^2\right]} + \frac{4a_1^3 a_2}{T\left[(1+a_2)^2 - a_1^2\right]^2}.$$

For the second coordinate of the vector $\mathbf{g}$ we obtain

$$\frac{\partial g_2(\mathbf{a})}{\partial a_1} = -\frac{4a_1 a_2 (1+a_2)^2}{T\left[(1+a_2)^2 - a_1^2\right]^2}$$

and

$$\frac{\partial g_2(\mathbf{a})}{\partial a_2} = 1 - \frac{k+3}{T} + \frac{[-2(1+a_2)^2 - 4a_2(1+a_2)][(1+a_2)^2 - a_1^2] + 4a_2(1+a_2)^3}{T\left[(1+a_2)^2 - a_1^2\right]^2}$$

$$= 1 - \frac{k+3}{T} - \frac{2(1+a_2)^2}{T\left[(1+a_2)^2 - a_1^2\right]} + \frac{4a_1^2 a_2 (1+a_2)}{T\left[(1+a_2)^2 - a_1^2\right]^2}.$$

This completes the proof. □

S3. For $p = 2$, the bias mapping is a contraction

In the remainder of the Appendix, we prove that for $p = 2$, the bias mapping is a contraction.

We begin below by working with the eigenvalues of (3.3) in the main file.

S3.1 The characteristic polynomial and its discriminant

The eigenvalues of (3.3) in the main file are determined by solving

$$|\mathbf{g}'(a_1, a_2) - \lambda I_2| = 0,$$

which is equivalent to

$$\lambda^2 - 2\lambda\left[1 - \frac{a_2(1+a_2)}{T\left[(1+a_2)^2 - a_1^2\right]} - \frac{a_1^2}{T\left[(1+a_2)^2 - a_1^2\right]} - \frac{k+7}{2T}\right]$$

$$+ 1 - \frac{k+7}{T} - \frac{2a_2(1+a_2)}{T\left[(1+a_2)^2 - a_1^2\right]} - \frac{2a_1^2}{T\left[(1+a_2)^2 - a_1^2\right]} + \frac{4(k+3)}{T^2} + \frac{2(1+a_2)(1+4a_2)}{T^2\left[(1+a_2)^2 - a_1^2\right]}$$

$$+ \frac{6a_1^2}{T^2\left[(1+a_2)^2 - a_1^2\right]} - \frac{2k(1+a_2)}{T^2\left[(1+a_2)^2 - a_1^2\right]} + \frac{2ka_1^2}{T^2\left[(1+a_2)^2 - a_1^2\right]} + \frac{4a_1^2 a_2(1+a_2)}{T^2\left[(1+a_2)^2 - a_1^2\right]^2}$$

$$- \frac{4(1+a_2)^3}{T^2\left[(1+a_2)^2 - a_1^2\right]^2} + \frac{4a_1^2(1+a_2)^2}{T^2\left[(1+a_2)^2 - a_1^2\right]^2} + \frac{4ka_1^2 a_2(1+a_2)}{T^2\left[(1+a_2)^2 - a_1^2\right]^2}$$

$$- \frac{4ka_1 a_2(1+a_2)^2}{T^2\left[(1+a_2)^2 - a_1^2\right]^2} = 0. \tag{S3.1}$$

The discriminant of (S3.1), after some algebraic manipulation, is given by

$$\Delta = \left[\frac{k-1}{T} + \frac{2[a_2(1+a_2) - a_1^2]}{T\left[(1+a_2)^2 - a_1^2\right]}\right]^2 - \frac{8(k-1)a_1^2(1+a_2)}{T^2\left[(1+a_2)^2 - a_1^2\right]^2} + \frac{8(k+1)(1+a_2)^3}{T^2\left[(1+a_2)^2 - a_1^2\right]^2}$$

$$- \frac{16a_1^2(1+a_2)[1+a_2+ka_2]}{T^2\left[(1+a_2)^2 - a_1^2\right]^2} + \frac{16ka_1 a_2(1+a_2)^2}{T^2\left[(1+a_2)^2 - a_1^2\right]^2}. \tag{S3.2}$$

We proceed to calculate the discriminant (S3.2) at the fixed points (a_1^*, a_2^*) and investigate its sign. We first consider a simpler expression for a_1^* from (3.2) in the main file. We set

$$\begin{aligned} A &\triangleq (k+1)(1-a_2^*)+2 \\ B &\triangleq (a_2^*-3)\left[(2k+1)a_2^*-(2k+3)\right] = (a_2^*)^2(2k+1)-2a_2^*(4k+3)+3(2k+3) \end{aligned} \tag{S3.3}$$

and we notice that

$$A^2 = k^2(1-a_2^*)^2 + B \tag{S3.4}$$

holds. After some algebra the discriminant Δ is given by the following expression:

$$\Delta = \left[\frac{k+1}{T} + \frac{2A^2}{TB(1+a_2^*)}\right]^2 + \frac{32A^2k^2a_2^*(1-a_2^*)}{T^2B^2(1+a_2^*)}. \tag{S3.5}$$

According to (S3.1), if the discriminant given by (S3.5) is positive, then the eigenvalues are given by

$$\lambda_{1,2} = 1 - \frac{A^2}{TB(1+a_2^*)}(1+2a_2^*) - \frac{k+5}{2T} \pm \frac{1}{2}\sqrt{\Delta}\,. \tag{S3.6}$$

If the discriminant given by (S3.5) is negative, then the eigenvalues are given by

$$\lambda_{1,2} = 1 - \frac{A^2}{TB(1+a_2^*)}(1+2a_2^*) - \frac{k+5}{2T} \pm \frac{1}{2}i\sqrt{-\Delta}\,. \tag{S3.7}$$

S3.2 Analysis of Δ

For the fixed points determined by solving the system of equations (3.1) and (3.2) in the main file, the inequality

$$\frac{k+1-(k+5)a_2^*}{k+1-(k+3)a_2^*} \geq 0 \tag{S3.8}$$

must hold. For $k \geq 1$, we proceed to investigate the following two cases arising from (S3.8).

Case 1: $k+1-(k+5)a_2^* \geq 0$ and $k+1-(k+3)a_2^* > 0$

This region for the values of a_2^* is displayed below in Figure 1.

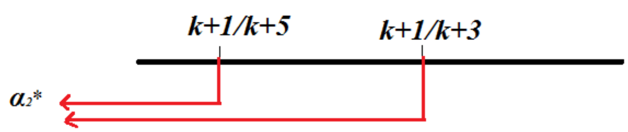

Figure 1: Values of a_2^* for Case 1.

The figure shows that the inequalities in Case 1 hold simultaneously if $a_2^* \leq (k+1)/(k+5)$.

Case 2: $k+1-(k+5)a_2^* \leq 0$ and $k+1-(k+3)a_2^* < 0$. This region for the values of a_2^* is depicted below in Figure 2.

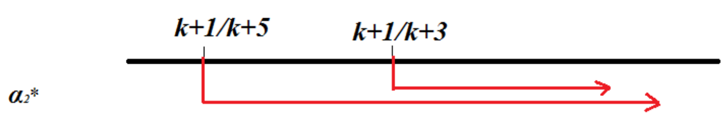

Figure 2: Values of a_2^* for Case 2.

Thus the inequalities in Case 2 hold simultaneously if $a_2^* > (k+1)/(k+3)$.

Equation (S3.5) shows that the sign of Δ directly depends on the sign of the fraction $a_2^*(1-a_2^*)/(1+a_2^*)$. Combining the results from Cases 1-2, we consider the following table of signs for the discriminant Δ.

		-1	0	k+1/k+5	k+1/k+3	1
a_2	-	-	+	+	+	+
$1-a_2$	+	+	+	+	+	-
$1+a_2$	-	+	+	+	+	+
$a_2(1-a_2)/(1+a_2)$	+	-	+	+	+	-

Table 1: Sign of Δ.

Table 1 shows that the discriminant Δ is positive if $a_2^*(1 - a_2^*)/(1 + a_2^*)$ is positive, but if this fraction is negative we shall need to investigate further its sign. We proceed with our analysis by considering subcases of Cases 1-2.

Subcases of Case 1

Figure 1 shows that the inequalities in Case 1 hold simultaneously if $a_2^* \leq (k+1)/(k+5)$. We now consider the following subcases of Case 1:

(a) $a_2^* \in (-\infty, -1)$

(b) $a_2^* \in (-1, 0)$

(c) $a_2^* \in [0, \frac{k+1}{k+5}]$.

Subcase (a) of Case 1.

Proposition 1. *Let* $a_2^* \in (-\infty, -1)$. *In this range, there is a contraction if and only if the following inequality holds:*

$$T > \frac{A^2(1+2a_2^*)}{2B(1+a_2^*)} + \frac{k+5}{4} + \frac{1}{4}\sqrt{\left[k+1+\frac{2A^2}{B(1+a_2^*)}\right]^2 + \frac{32k^2A^2a_2^*(1-a_2^*)}{B^2(1+a_2^*)}}. \quad \text{(S3.9)}$$

Proof. From Table 1 we see that, for $a_2^* \in (-\infty, -1), \Delta \geq 0$. To prove a contraction result we must prove that the eigenvalues $\lambda_{1,2}$, given by (S3.6), are less than 1 in absolute value. We find:

$$|\lambda_{1,2}| < 1$$

$$\Leftrightarrow T > \frac{A^2(1+2a_2^*)}{2B(1+a_2^*)} + \frac{k+5}{4} \mp \frac{1}{4}\sqrt{\left[k+1+\frac{2A^2}{B(1+a_2^*)}\right]^2 + \frac{32k^2A^2a_2^*(1-a_2^*)}{B^2(1+a_2^*)}} > 0.$$

Hence the following two inequalities must be valid simultaneously:

$$\begin{aligned} &T > \frac{A^2(1+2a_2^*)}{2B(1+a_2^*)} + \frac{k+5}{4} + \frac{1}{4}\sqrt{\left[k+1+\frac{2A^2}{B(1+a_2^*)}\right]^2 + \frac{32k^2A^2a_2^*(1-a_2^*)}{B^2(1+a_2^*)}}, \\ &\frac{A^2(1+2a_2^*)}{2B(1+a_2^*)} + \frac{k+5}{4} - \frac{1}{4}\sqrt{\left[k+1+\frac{2A^2}{B(1+a_2^*)}\right]^2 + \frac{32k^2A^2a_2^*(1-a_2^*)}{B^2(1+a_2^*)}} > 0. \end{aligned} \quad \text{(S3.10)}$$

We now further analyze the second inequality in (S3.10). For $A^2(1+2a_2^*)/(2B(1+a_2^*)) + (k+5)/4 > 0$, we must show that

$$\frac{A^2(1+2a_2^*)}{2B(1+a_2^*)} + \frac{k+5}{4} > \frac{1}{4}\sqrt{\left[k+1+\frac{2A^2}{B(1+a_2^*)}\right]^2 + \frac{32k^2A^2a_2^*(1-a_2^*)}{B^2(1+a_2^*)}}$$

$$\Leftrightarrow \quad \frac{k+3}{2} + \frac{A^2\big(A^2 - k^2(1-a_2^*)\big)}{B^2} + \frac{A^2(k+5)a_2^*}{2B(1+a_2^*)} > 0.$$

The last inequality indeed holds for $a_2^* \in (-\infty, -1)$. Indeed, the sign of the last fraction depends on B and on $a_2^*/(1+a_2^*)$, both of which are positive in this case. The sign of B is provided in Figure 3 below, where the axis gives values of a_2^*.

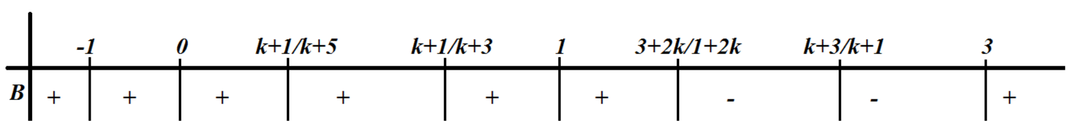

Figure 3: Sign of B.

It remains to prove that the polynomial $A^2 - k^2(1-a_2^*)$ is positive. To do so, we compute

$$\Pi_1 = A^2 - k^2(1-a_2^*) \overset{(S3.3)}{=} (k+1)^2(1-a_2^*)^2 + 4 + 4(k+1)(1-a_2^*) - k^2(1-a_2^*)$$
$$= (k+1)^2\,(a_2^*)^2 - [k^2+8k+6]a_2^* + 3[2k+3]. \qquad \text{(S3.11)}$$

The discriminant of the above quadratic polynomial in terms of a_2^* is given by

$$\Delta_1 = [k^2+8k+6]^2 - 12(k+1)^2(2k+3)$$
$$= k^4 - 8k^3 - 8k^2 = k^2(k^2-8k-8), \qquad \forall\, k > 0.$$

Thus the sign of the polynomial (S3.11) depends on the sign of the polynomial k^2-8k-8. The discriminant of the latter is given by

$$\Delta_{11} = 8^2 - 4(-8) = 96 > 0.$$

The two corresponding real roots are

$$k_1 = \frac{8-\sqrt{96}}{2} \simeq -0.899, \qquad k_2 = \frac{8+\sqrt{96}}{2} = 8.899 \overset{k=1,2,\ldots}{\equiv} 9.$$

The sign of Δ_1 is presented below:

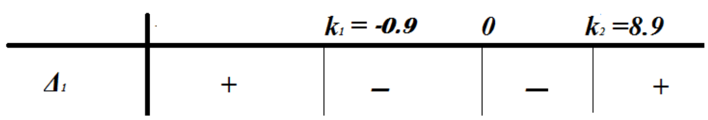

Figure 4: Sign of Δ_1.

Figure 4 shows that for $1 \leq k \leq 8$, $\Delta_1 < 0$. For $k \geq 9$, $\Delta_1 > 0$, and thus the polynomial (S3.11) has the two real roots

$$r_1 = \frac{k^2+8k+6-k\sqrt{k^2-8k-8}}{2(k+1)^2}, \qquad r_2 = \frac{k^2+8k+6+k\sqrt{k^2-8k-8}}{2(k+1)^2}. \qquad \text{(S3.12)}$$

Simple algebra shows that $-1 < r_1$. The sign of the polynomial given in (S3.11) for $k \geq 9$ is presented below.

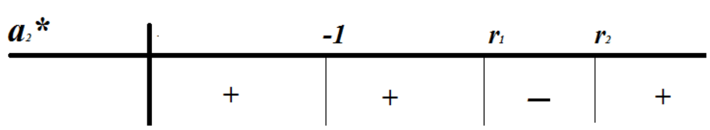

Figure 5: Sign of Π_1 .

In the case $a_2^* \in (-\infty, -1)$ and for every $k > 0$, we have shown the polynomial (S3.11) is positive, and thus the second inequality of (S3.10) holds. Thus, for $a_2^* \in (-\infty, -1)$ we have a contraction subject to the first inequality of (S3.10). This finishes the proof. □

Subcase (b) of Case 1: Let $a_2^* \in (-1, 0)$.

Proposition 2. *If $k = 1, 2$ and $a_2^* \in (-1/2, 0)$, then there is a contraction if and only if the following inequalities hold:*

$$T > \frac{A^2(1+2a_2^*)}{2B(1+a_2^*)} + \frac{k+5}{4} + \frac{1}{4}\sqrt{\left[k+1+\frac{2A^2}{B(1+a_2^*)}\right]^2 + \frac{32k^2A^2a_2^*(1-a_2^*)}{B^2(1+a_2^*)}},$$

$$\frac{k+3}{2} + \frac{A^2\big(A^2 - k^2(1-a_2^*)\big)}{B^2} + \frac{A^2(k+5)a_2^*}{2B(1+a_2^*)} > 0. \tag{S3.13}$$

Proof. We rewrite (S3.5) as

$$\Delta = \frac{(k+1)^2}{T^2} + \frac{4A^4}{T^2B^2(1+a_2^*)^2} + \frac{4A^2(k+1)}{T^2B(1+a_2^*)} + \frac{32k^2A^2a_2^*(1-a_2^*)}{T^2B^2(1+a_2^*)}$$

$$= \frac{(k+1)^2}{T^2} + \frac{4A^2\big[A^2+8k^2a_2^*(1-a_2^*)(1+a_2^*)\big]}{T^2B^2(1+a_2^*)^2} + \frac{4A^2(k+1)}{T^2B(1+a_2^*)}.$$

Here the sign of the discriminant Δ depends on the polynomial $A^2+8k^2a_2^*(1-a_2^*)(1+a_2^*)$. This follows since $B > 0$ and $1 + a_2^* > 0$ for $a_2^* \in (-1, 0)$. Then

$$A^2 + 8k^2a_2^*(1-a_2^*)(1+a_2^*) \overset{(S3.3)}{=} (k+1)^2(1-a_2^*)^2 + 4 + 4(k+1)(1-a_2^*) + 8k^2a_2^*(1-a_2^*)(1+a_2^*)$$

$$= (k+1)^2(1-a_2^*)^2 + 4 + 4(1-a_2^*)[2k^2(a_2^*)^2 + 2k^2a_2^* + k + 1].$$

We investigate the sign of the polynomial $2k^2(a_2^*)^2+2k^2a_2^*+k+1$, which has discriminant $\Delta_2 = 4k^4 - 8k^2(k+1) = 4k^2[k^2-2k-2]$. This discriminant is negative for $k=1,2$ and thus the polynomial is positive for $k=1,2$. For $k \geq 3, \Delta > 0$. Furthermore, for $a_2^* \in (-1/2, 0)$, the constraint

$$\frac{A^2(1+2a_2^*)}{2B(1+a_2^*)} + \frac{k+5}{4} > 0 \tag{S3.14}$$

for the contraction holds since both $B > 0$ and $\frac{1+2a_2^*}{1+a_2^*} > 0$. The second inequality in (S3.10) holds if and only if the second inequality of (S3.13) holds. Thus, for $k=1,2$ and $a_2^* \in (-1/2, 0)$, there is a contraction mapping if and only if the inequalities in (S3.13) hold. This finishes the proof. □

Proposition 3. *If $k=1,2$ and $a_2^* \in (-1,-1/2]$, then there is a contraction if and only if the following inequalities hold:*

$$\begin{aligned} &T > \frac{A^2(1+2a_2^*)}{2B(1+a_2^*)} + \frac{k+5}{4} + \frac{1}{4}\sqrt{\left[k+1+\frac{2A^2}{B(1+a_2^*)}\right]^2 + \frac{32k^2A^2a_2^*(1-a_2^*)}{B^2(1+a_2^*)}}, \\ &\frac{A^2(1+2a_2^*)}{2B(1+a_2^*)} + \frac{k+5}{4} > 0, \\ &\frac{k+3}{2} + \frac{A^2\left(A^2-k^2(1-a_2^*)\right)}{B^2} + \frac{A^2(k+5)a_2^*}{2B(1+a_2^*)} > 0. \end{aligned} \tag{S3.15}$$

Proof. The proof is the same as that of Proposition 2. □

We now present results for $k \geq 3$.

Proposition 4. *If $k \geq 3$, $a_2^* \in (-1/2, 0)$ and $\Delta > 0$, then there is a contraction if and only if the inequalities in (S3.13) hold.*

Proof. Using the hypothesis that $\Delta > 0$, the proof is nearly identical to that of Proposition 2. □

Proposition 5. *If $k \geq 3$, $a_2^* \in (-1, -1/2]$ and $\Delta > 0$, then there is a contraction if and only if the inequalities of (S3.15) hold.*

Proof. Using the hypothesis that $\Delta > 0$, the proof is nearly identical to that of Proposition 3. □

Proposition 6. *If $k \geq 3$, $a_2^* \in (-1, 0)$ and $\Delta < 0$, then there is a contraction if and only if the following inequality holds*

$$\frac{4A^4 a_2^*(1+a_2^*)}{B^2(1+a_2^*)^2} + 2k + 6 + \frac{2A^2(ka_2^* + 5a_2^* + 2)}{B(1+a_2^*)} - \frac{8k^2A^2a_2^*(1-a_2^*)}{B^2(1+a_2^*)} < T\left[\frac{2A^2(1+2a_2^*)}{B(1+a_2^*)} + k + 5\right]. \tag{S3.16}$$

Proof. To obtain a contraction result for $\Delta < 0$, one must prove that the complex eigenvalues $\lambda_{1,2}$ given by (S3.7) have modulus less than 1. We find:

$$|\lambda_{1,2}| < 1$$

$$\Leftrightarrow \quad \frac{A^4(1+2a_2^*)^2}{T^2B^2(1+a_2^*)^2} + \frac{(k+5)^2}{4T^2} - \frac{2A^2(1+2a_2^*)}{TB(1+a_2^*)} - \frac{k+5}{T} + \frac{A^2(k+5)(1+2a_2^*)}{T^2B(1+a_2^*)}$$

$$- \frac{(k+1)^2}{4T^2} - \frac{A^4}{T^2B^2(1+a_2^*)^2} - \frac{A^2(k+1)}{T^2B(1+a_2^*)} - \frac{8k^2A^2a_2^*(1-a_2^*)}{T^2B^2(1+a_2^*)} < 0,$$

$$\Leftrightarrow \frac{4A^4 a_2^*(1+a_2^*)}{B^2(1+a_2^*)^2} + 2k + 6 + \frac{2A^2(ka_2^* + 5a_2^* + 2)}{B(1+a_2^*)} - \frac{8k^2A^2a_2^*(1-a_2^*)}{B^2(1+a_2^*)} < T\left[\frac{2A^2(1+2a_2^*)}{B(1+a_2^*)} + k + 5\right],$$

which is (S3.16). This finishes the proof. □

Subcase (c) of Case 1: Let $a_2^* \in [0, \frac{k+1}{k+5}]$.

Proposition 7. *If $1 \leq k \leq 9$ and $a_2^* \in [0, (k+1)/(k+5)]$, then there is a contraction if and only if the first inequality of (S3.10) holds.*

Proof. For $a_2^* \in [0, (k+1)/(k+5)]$, Table 1 shows that $\Delta > 0$. The proof of Proposition 1 tells us that there is a contraction if and only if the inequalities of (S3.10) hold. For $a_2^* \in [0, (k+1)/(k+5)]$, the constraint

$$\frac{A^2(1+2a_2^*)}{2B(1+a_2^*)} + \frac{k+5}{4} > 0 \tag{S3.17}$$

holds since both $B > 0$ and $\frac{1+2a_2^*}{1+a_2^*} > 0$. For $1 \leq k \leq 8$, $\Delta_1 < 0$, and thus $\Pi_1 > 0$. Further, $\frac{a_2^*}{1+a_2^*} > 0$. Thus, the second inequality of (S3.10) holds. For $k = 9$, $\Delta_1 = 81 > 0$, and thus the polynomial Π_1 has the two real roots r_1 and r_2 of (S3.12) with $(k+1)/(k+5) < r_1 < r_2$. Figure 5 shows that the polynomial Π_1 remains positive. This finishes the proof. □

For $k \geq 10$, simple algebra shows that $0 < r_1 < \frac{k+1}{k+5} < r_2$, where r_1 and r_2 are given by (S3.12). This leads us to the following two propositions.

Proposition 8. *If $k \geq 10$ and $a_2^* \in [0, r_1]$, where r_1 is given by (S3.12), then there is a contraction if and only if the first inequality of (S3.10) holds.*

Proof. By the hypothesis and Figure 5, we see that the polynomial Π_1 remains positive. The proof is thus the same as that of Proposition 7. □

Proposition 9. *If $k \geq 10$ and $a_2^* \in [r_1, (k+1)/(k+5)]$, where r_1 is given by (S3.12), then there is a contraction if and only if the inequalities of (S3.13) hold.*

Proof. By the hypothesis and Figure 5, the polynomial $\Pi_1 < 0$. For a contraction result, the second inequality of (S3.10) necessitates the second inequality of (S3.13). The proof is now the same as that of Proposition 7. □

Subcases of Case 2

We now consider the following subcases of Case 2:

(a) $a_2^* \in (\frac{k+1}{k+3}, 1)$

(b) $a_2^* \in (1, \frac{3+2k}{1+2k}) \cup (\frac{3+2k}{1+2k}, \frac{k+3}{k+1}) \cup (\frac{k+3}{k+1}, 3) \cup (3, \infty)$.

Subcase (a) of Case 2:

Proposition 10. *If $1 \leq k \leq 8$ and $a_2^* \in \left(\frac{k+1}{k+3}, 1\right)$, then there is a contraction if and only if the first inequality of (S3.10) holds.*

Proof. From the conditions of the hypothesis, we obtain $\Delta > 0$. We also have that $\frac{A^2(1+2a_2^*)}{2B(1+a_2^*)} + \frac{k+5}{4} > 0$ since $B > 0$. Now the proof is the same as that of Proposition 7. □

For $k \geq 9$, simple algebra shows that $r_1 < \frac{k+1}{k+3} < r_2 < 1$, where r_1 and r_2 are given by (S3.12). This gives the following results.

Proposition 11. *If $k \geq 9$ and $a_2^* \in (r_2, 1)$, where r_2 is given by (S3.12), then there is a contraction if and only if the first inequality of (S3.10) holds.*

Proof. The proof is the same as that of Proposition 8. □

Proposition 12. *If $k \geq 9$ and $a_2^* \in \left(\frac{k+1}{k+3}, r_2\right]$, where r_2 is given by (S3.12), then there is a contraction if and only if the inequalities of (S3.13) hold.*

Proof. The proof is the same as that of Proposition 9. □

Subcase (b) of Case 2:

Proposition 13. *If* $a_2^* \in (1, \frac{3+2k}{1+2k}) \cup (3, \infty)$ *and* $\Delta > 0$*, then there is a contraction if and only if the first inequality of (S3.10) holds.*

Proof. By hypothesis, $\Delta > 0$. Recall that the proof of Proposition 1 shows that there is a contraction if and only if the inequalities of (S3.10) hold.

We have that $\frac{A^2(1+2a_2^*)}{2B(1+a_2^*)} + \frac{k+5}{4} > 0$, since both $B > 0$ and $\frac{1+2a_2^*}{1+a_2^*} > 0$. For $1 \leq k \leq 8$, $\Delta_1 < 0$, which gives $\Pi_1 > 0$. Further, $\frac{a_2^*}{1+a_2^*} > 0$. We thus obtain that the second inequality of (S3.10) holds. If $k \geq 9$, $\Delta_1 > 0$, and thus the polynomial Π_1 has two real roots $r_1 < r_2 < 1$. The conditions of the hypothesis in conjunction with Figure 5 ensure that Π_1 remains positive. This finishes the proof. □

Proposition 14. *If* $a_2^* \in (\frac{3+2k}{1+2k}, \frac{k+3}{k+1}) \cup (\frac{k+3}{k+1}, 3)$ *and* $\Delta > 0$*, then there is a contraction if and only if the inequalities in (S3.15) hold.*

Proof. By hypothesis, $\Delta > 0$. The proof of Proposition 1 shows that there is a contraction if and only if the inequalities of (S3.10) hold. However, since $a_2^* \in (\frac{3+2k}{1+2k}, \frac{k+3}{k+1}) \cup (\frac{k+3}{k+1}, 3)$, we have from Figure 3 that $B < 0$. According to the analysis of Proposition 1, the second inequality of (S3.10) necessitates the second and third inequalities of (S3.15). This finishes the proof. □

Proposition 15. *If* $a_2^* \in (1, \frac{3+2k}{1+2k}) \cup (\frac{3+2k}{1+2k}, \frac{k+3}{k+1}) \cup (\frac{k+3}{k+1}, 3) \cup (3, \infty)$ *and* $\Delta < 0$*, then there is a contraction if and only if the inequality (S3.16) holds.*

Proof. The proof is the same as that of Proposition 6. □