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A NOTE ON MINIMAL ORTHOGONAL PLANS

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Abstract: Sufficient conditions are given for plans, which permit the uncorrelated estimation of all main effects and some specified interaction effects, to have a minimal number of experimental runs.

Key words and phrases: Fractional factorial design, orthogonal main effect plan, resolution-4 plan.

1. Introduction

Orthogonal main effect plans (OMEP's) are plans which allow for the uncorrelated estimation of all main effects, provided that all interactions can be validly assumed to be negligible. In practice, such an assumption may not always be realistic. When an experimenter is not prepared to assume the absence of *all* interactions, a plan which permits estimation of all main effects and all or some specified interaction effects is required. When experimental runs are expensive or time consuming, it is important to know the minimal number of experimental runs necessary to construct such a plan for a given number of factors having a specified number of levels. This problem has been considered by Webb (1968) for resolution-4 plans for 2^n experiments, Margolin (1969a) for resolution-4 plans for 2^n3^m experiments, Margolin (1969b) for general resolution-4 plans, and Jacroux (1992) for orthogonal main effect plans. For resolution-4, -5 and general resolution-r plans, readers are referred to Box and Hunter (1961) and Webb (1965). Briefly, a resolution-4 plan is a plan which permits the estimation of all main effects under the assumption that interactions involving three or more factors are negligible, and a resolution-5 plan is a plan which permits the estimation of all main effects and all two factor interactions under the assumption that interactions involving three or more factors are negligible.

In this article, plans which permit the uncorrelated estimation of all main effects and all i-factor interaction effects among r $(i < r)$ specific factors are considered. They include the OMEP's (when $r = 2$) and some of the class one compromise plans (when $r = 3$) introduced in Addelman (1962b). These plans are more flexible and easier to construct than the orthogonal resolution-4 or -5 plans. They can usually include more factors than an orthogonal resolution-4 or -5 plan with the same number of experimental runs. Sufficient conditions for these plans to have a minimal number of experimental runs are derived.

2. Main Results

For a given plan, the following notations will be used throughout this article: N = total number of observations to be taken; $n(p_1,\ldots,p_m;i_1,\ldots,i_m)$ = number of observations to be taken at the i_1th level of factor $p_1,\ldots,$ the i_mth level of factor p_m. A plan is minimal if it has a minimal number of experimental runs.

Assume that a plan is to be constructed with $k \geq r+1$ factors which permits the uncorrelated estimation of all main effects and all i-factor $(i < r)$ interaction effects among factors $p_1,\ldots,p_r$. It will be called an $O(p_1,\ldots,p_r)$ plan. It is an OMEP when $r = 2$ and a class one compromise plan of Addelman (1962b) when $r = 3$. When $r > 2$, an $O(p_1,\ldots,p_r)$ plan is weaker than an orthogonal resolution-$(r+1)$ plan. It is obvious that an orthogonal resolution-4 plan is an $O(p_1,p_2,p_3)$ plan for any three of its factors, and an orthogonal resolution-5 plan is an $O(p_1,p_2,p_3,p_4)$ plan for any four of its factors. An $O(p_1,\ldots,p_r)$ plan can usually include more factors than an orthogonal resolution-$(r+1)$ plan with the same number of experimental runs. For example, using the lower bound given in Margolin (1969b), we know that a minimal resolution-4 plan (orthogonal or not) for a 4^6 experiment has 64 runs whereas an $O(p_1,p_2,p_3)$ plan with the same number of runs can include 12 factors of four levels each (Addelman (1962b)). Another advantage for $O(p_1,\ldots,p_r)$ plans is that they remain as $O(p_1,\ldots,p_r)$ plans under collapsing (Addelman (1962a)); hence making it very easy to construct $O(p_1,\ldots,p_r)$ plans from existing $O(p_1,\ldots,p_r)$ plans and from orthogonal resolution-$(r+1)$ plans.

We now introduce a method of constructing $O(p_1,\ldots,p_r)$ plans from existing $O(p_1,\ldots,p_r)$ plans. It is related to some of the methods given in Addelman (1962a) and Jacroux (1992). Let $c_j, j = 1,\ldots,k$ be k positive integers and d_0 be an $O(p_1,\ldots,p_r)$ plan with N experimental runs for $k \geq r+1$ factors in which factor j has u_j levels, $j = 1,\ldots,k$. Then an $O(p_1,\ldots,p_r)$ plan d with $N\prod_{j=1}^{k} c_j$ experimental runs for k factors in which factor j has $v_j \leq c_j u_j$ levels, $j = 1,\ldots,k$, can be constructed as follows: Denote the levels of factor j of d_0 by $0,1,\ldots,u_j-1,\ j = 1,\ldots,k$. From d_0, we obtain the plans $d_{i_1,\ldots,i_k},\ i_j = 0,1,\ldots,c_j-1,\ j = 1,\ldots,k$, by replacing the levels of factor j in d_0 with levels $i_ju_j, i_ju_j+1,\ldots,i_ju_j+(u_j-1)$ for $j = 1,\ldots,k$. Let d_c be the plan obtained by combining the runs in $\{d_{i_1,\ldots,i_k},\ i_j = 0,1,\ldots,c_j-1,\ j = 1,\ldots,k\}$, and let d be the plan obtained from d_c by collapsing the levels in d_c to the required number of levels in d (for the collapsing method, see Addelman (1962a)), then d is the desired plan.

From Table 1 in Addelman (1962b), we know that for experiments 2^4, 3^7, 4^{12} and 5^{19}, there exist $O(p_1, p_2, p_3)$ plans in $8, 27, 64$ and 125 runs respectively. Using the construction method mentioned above, many $O(p_1, \ldots, p_r)$ plans can be constructed. For example, by the $O(p_1, p_2, p_3)$ plan in 8 runs, an $O(p_1, p_2, p_3)$ plan d can be constructed for experiment $4 \cdot 3 \cdot 2^2$ in 32 runs where p_1, p_2 are the factors with 4 and 3 levels and p_3 is a factor with 2 levels. Let d_0 be the $O(p_1, p_2, p_3)$ plan in 8 runs. We obtain the plans $d_{i_1,i_2},\ i_j = 0, 1,\ j = 1, 2$, by replacing the levels of factor j in d_0 with levels $2i_j, 2i_j + 1$ for $j = 1, 2$. Let d_c be the plan obtained by combining the runs in $\{d_{i_1,i_2},\ i_j = 0, 1,\ j = 1, 2\}$, then d can be obtained from d_c by collapsing the levels in d_c to the required number of levels in d. Similarly, there exist $O(p_1, p_2, p_3)$ plans, for example, for experiment $5 \cdot 3 \cdot 2^5$ in 54 runs where p_1, p_2 are the factors with 5 and 3 levels and p_3 is a factor with 2 levels; for experiment $4 \cdot 3^2 \cdot 2^9$ in 64 runs where p_1 is the factor with 4 levels and p_2, p_3 are the factors with 3 levels; and for experiment $5 \cdot 4^2 \cdot 3^4 \cdot 2^{12}$ in 125 runs where p_1 is the factor with 5 levels and p_2, p_3 are the factors with 4 levels. Using Theorem 1 which will be stated and proved shortly, we see that all these plans are minimal $O(p_1, p_2, p_3)$ plans.

Using the definition of the least common multiple and the greatest common divisor, the following result can be obtained easily.

Lemma 1. *Let $x_1, \ldots, x_n$ be n positive integers and $x = \prod_{i=1}^n x_i$. Then $x = KL$ where K is the least common multiple for $x/x_1, \ldots, x/x_n$, and L is the greatest common divisor for $x_1, \ldots, x_n$.*

Now we state the main result.

Theorem 1. *suppose d is an $O(p_1, \ldots, p_r)$ plan with N experimental runs for $k \geq r + 1$ factors in which factor i has s_i levels, $i = 1, \ldots, k$ and let $s_e = \max_{j \neq p_1, \ldots, p_r} s_j$. If $N = \prod_{i=1}^r t_i$ for $t_1, \ldots, t_r$ satisfying*

$$\prod_{i=1}^r t_i = \min\Big\{\prod_{i=1}^r x_i; x_i \geq s_{p_i}, i = 1, \ldots, r, g(x_1, \ldots, x_r) \geq s_e, \prod_{i=1}^r x_i < 2\prod_{i=1}^r s_{p_i}\Big\},$$

where $g(x_1, \ldots, x_r)$ is the greatest common divisor for $x_1, \ldots, x_r$, then d is a minimal $O(p_1, \ldots, p_r)$ plan.

Proof. A necessary and sufficient condition for a plan to be an $O(p_1, \ldots, p_r)$ plan is that for any integers $h_1, \ldots, h_r \leq k$, $h_i \neq h_j$ when $i \neq j$, at least $r - 1$ of them belong to the set $\{p_1, \ldots, p_r\}$, and $i_j \leq s_{h_j}$, $j = 1, \ldots, r$, we have $N^{r-1} n(h_1, \ldots, h_r; i_1, \ldots, i_r) = \prod_{j=1}^r n(h_j; i_j)$ (for example, see Addelman (1962b)). Therefore for any $h_1, \ldots, h_{r-1}$ which belong to the set $\{p_1, \ldots, p_r\}$, and $i_j \leq s_{h_j}$, $j = 1, \ldots, r-1$, we have $N^{r-2} n(h_1, \ldots, h_{r-1}; i_1, \ldots, i_{r-1}) = \prod_{j=1}^{r-1} n(h_j; i_j)$. For an $O(p_1, \ldots, p_r)$ plan, $n(p_1, \ldots, p_r; i_1, \ldots, i_r) \geq 1$ for any $i_j \leq s_{p_j}$, $j = 1, \ldots, r$.

Since $N < 2\prod_{i=1}^{r} s_{p_i}$, there exist positive integers $a_i,\ i = 1,\ldots,r$ such that $N^{-(r-1)}\prod_{j=1}^{r} n(p_j;a_j) = n(p_1,\ldots,p_r;a_1,\ldots,a_r) = 1$. Let

$$y_i = n(p_1,\ldots,p_{i-1},p_{i+1},\ldots,p_r;a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_r),\ i = 1,\ldots,r.$$

Then $y_i = \sum_{j=1}^{s_{p_i}} n(p_1,\ldots,p_r;a_1,\ldots,a_{i-1},j,a_{i+1},\ldots,a_r) \geq s_{p_i},\ i = 1,\ldots,r$ and $\prod_{i=1}^{r} y_i = N^{-r(r-2)}(\prod_{j=1}^{r} n(p_j;a_j))^{r-1} = N^{-r(r-2)}N^{(r-1)^2} = N$. Furthermore for any $i \leq s_e$ and $j \leq r$ we have

$$\begin{aligned} n(e;i) &= N^{-(r-1)}n(e;i)\prod_{l=1}^{r} n(p_l;a_l) \\ &= n(p_j;a_j)n(p_1,\ldots,p_{j-1},e,p_{j+1},\ldots,p_r;a_1,\ldots,a_{j-1},i,a_{j+1},\ldots,a_r). \end{aligned}$$

Hence $n(e;i)$ is a multiple of $n(p_j;a_j), j = 1,\ldots,r$. Therefore $N = \sum_{i=1}^{s_e} n(e;i) \geq s_eK$ where K is the least common multiple for $n(p_j;a_j), j = 1,\ldots,r$. By Lemma 1, $L \geq s_e$ where L is the greatest common divisor for $y_i, i = 1,\ldots,r$. Since $N = \prod_{i=1}^{r} t_i$ where $t_i, i = 1,\ldots,r$ satisfy the condition in Theorem 1, d is minimal. This completes the proof.

The following result is a direct consequence of Theorem 1.

Corollary 1. *Let d and s_e be as in Theorem 1. If $s_e^r = N < 2\prod_{i=1}^{r} s_{p_i}$ and $s_{p_i} \leq s_e, i = 1,\ldots,r$ then d is minimal.*

3. Discussion

In this article, we considered certain plans which permit the uncorrelated estimation of all main effects and some specified interaction effects. They are more flexible and easier to construct than orthogonal resolution-4 or -5 plans. Sufficient conditions are given for these plans to have a minimal number of experimental runs. This minimal number is useful to the experimenters for constructing minimal orthogonal designs. For example, for a $10\cdot 7\cdot 4^2\cdot 3^8$ experiment, a plan which permits the uncorrelated estimation of all main effects and two-factor interactions among the 10-, 7- and a 3-level factors requires at least 384 experimental runs. If this is considered too expensive or time consuming to perform, then either some factors should be dropped, or the levels of some factors should be redefined, or a non-orthogonal plan should be considered.

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