

## STRONG LAWS OF R/S STATISTICS WITH A LONG-RANGE MEMORY SAMPLE

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*Abstract:* In this paper, we show a law of the iterated logarithm for range statistics with a long-range memory sample. With the help of this result, using R/S statistics, we give a test for long-range dependence.

*Key words and phrases:* Law of the iterated logarithm, long-range memory, R/S statistics.

### 1. Introduction

Let  $\{X_n, n \geq 1\}$  be a sequence of random variables. Take  $\bar{X}_n = 1/n \sum_{j=1}^n X_j$ ,

$$S^2(n) = \frac{1}{n} \sum_{j=1}^n (X_j - \bar{X}_n)^2, \quad (1.1)$$

the adjusted range of partial sums

$$R(n) = \max_{1 \leq k \leq n} \left\{ \sum_{j=1}^k (X_j - \bar{X}_n) \right\} - \min_{1 \leq k \leq n} \left\{ \sum_{j=1}^k (X_j - \bar{X}_n) \right\}, \quad (1.2)$$

and the self-normalized range

$$Q(n) = R(n)/S(n). \quad (1.3)$$

Then  $Q$  is called the rescaled range or R/S statistic. Introduced by Hurst (1951) when he studied hydrology data of the River Nile, it plays an important role in testing statistical dependence of a sequence. With reference to it, Mandelbrot introduced a class of Gaussian processes—fractional Brownian motions (cf., Mandelbort and Van Ness (1968)). The statistic has been used, for example, in modeling stock prices (cf., e.g., Lo (1991) and Willinger, Taqqu and Teverovsky (1999)). Moreover, it is an important tool in studying fractal theory and chaos phenomena.

Feller (1951) gave the limit distribution of  $R(n)/\sqrt{n}$  for an i.i.d. sequence with  $EX_1^2 < \infty$ . Moran (1964) considered the case of heavy tails. Lin (2001)

studied laws of the iterated logarithm (LIL) for R/S statistics under i.i.d. and mixing samples. Lin and Lee (2002) extended these results to the AR(1) model with an i.i.d. sample. But, for this statistic, one has more interest in a sample with long-range memory. Many practical models, such as economic time series for stock prices, exhibit long-range dependence. Lo (1991) cause a sequence long-range dependent of order  $\alpha$  if the process has an autocovariance function  $\gamma_k$  such that

$$\gamma(k) \sim \begin{cases} k^{2\alpha-2}L(k) & \text{for } \alpha \in (\frac{1}{2}, 1) \\ -k^{2\alpha-2}L(k) & \text{for } \alpha \in (0, \frac{1}{2}) \end{cases} \quad \text{as } k \rightarrow \infty, \quad (1.4)$$

where  $L(k)$  is a slowly varying function at infinity (" $a_k \sim b_k$ " means that  $a_k/b_k \rightarrow 1$  as  $k \rightarrow \infty$ ). An example is the fractional-difference process  $\{X_n, n \geq 1\}$  defined by

$$(1 - L)^d X_n = \varepsilon_n, \quad (1.5)$$

where  $L$  is the lag operator,  $d \in (-1/2, 1/2)$  and  $\varepsilon_n$  is white noise. The autocovariance function here is  $\gamma_k \sim ck^{2d-1}$  as  $k \rightarrow \infty$ . The sequence is stationary and invertible and exhibits a unique kind of dependence that is positive or negative, depending on whether  $d$  is positive or negative (Hosking (1981)). Some methods for detecting long-range memory have been developed. To this end, Lo (1991) showed a functional central limit theorem for R/S statistics (in fact, he considered modified R/S statistics) generated by a zero-mean stationary Gaussian sequence under a long memory assumption using a weak invariance principle due to Taqqu (1975). A test for  $\alpha$  follows: if  $\alpha \in (1/2, 1)$ , the R/S statistics diverge in probability to infinity; if  $\alpha \in (0, 1/2)$ , the statistics converge in probability to zero. In either case, the probability of rejecting the null hypothesis of short-range memory approaches unity for all stationary Gaussian processes satisfying (1.4).

There are many references on long memory, such as papers by Granger and Joyeux (1980), Cox (1984) and Beran (1992), and monographs by Bhattacharya and Waymire (1990) and Beran (1994).

In this paper, we first show a LIL for range statistics  $R(n)$  given a long-range memory sample. Then, with the help of this result, we develop a test for long-range dependence.

## 2. Theorems and Their Proofs

Let  $S_n = \sum_{j=1}^n X_j$ ,  $\sigma_n^2 = \text{Var } S_n$  and  $\alpha' = 2^{-2\alpha} - 2^{-2}$ .

**Theorem 2.1.** *Let  $\{X_n, n \geq 1\}$  be a stationary sequence of Gaussian random variables with mean zero and autocovariance function (1.4). Then*

$$\limsup_{n \rightarrow \infty} \sqrt{\frac{1}{2\sigma_n^2 \log \log n}} R_n \leq \sqrt{\alpha'} \quad a.s.. \quad (2.1)$$

If, in addition,

$$\frac{n^\alpha L(n)^{1/2}}{m^\alpha L(m)^{1/2}} \cdot \left(1 - \frac{L(n-m)}{L(n)}\right) \rightarrow 0, \quad \text{as } n/m \rightarrow \infty, \quad (2.2)$$

$$\limsup_{n \rightarrow \infty} \sqrt{\frac{1}{2\sigma_n^2 \log \log n}} R_n = \sqrt{\alpha'} \quad \text{a.s.} \quad (2.3)$$

**Remark 2.1.** We need a property of slowly varying functions. Let  $l(x)$  be a slowly varying function at infinity with  $b = b(x) \rightarrow 0$ , as  $x \rightarrow \infty$ . Then for any  $\delta > 0$ ,

$$\lim_{x \rightarrow \infty} b^\delta \frac{l(bx)}{l(x)} = \lim_{x \rightarrow \infty} b^\delta \frac{l(x)}{l(bx)} = 0 \quad (2.4)$$

(cf., Hosking (1981)). Using this property, the fact that for some  $\beta > \alpha$ ,

$$1 - \frac{L(n-m)}{L(n)} \leq \left(\frac{m}{n}\right)^\beta \quad (2.5)$$

implies (2.2). Moreover, if there is a  $\beta > 0$  such that  $x^{-\beta}L(x)$  is non-increasing, then (2.5) is satisfied with  $\beta = 1$ . Clearly, (2.5) is quite a weak condition, the common slowly varying functions, such as  $\log x$  and  $1/\log x$ , satisfy it.

Using Theorem 2.1, we give a test for long-range memory.

**Theorem 2.2.** Under (2.2), as  $n \rightarrow \infty$ ,

$$\frac{1}{\sqrt{n \log \log n}} Q_n \xrightarrow{\text{a.s.}} \begin{cases} \infty & \text{for } \alpha \in (\frac{1}{2}, 1), \\ 0 & \text{for } \alpha \in (0, \frac{1}{2}). \end{cases}$$

**Remark 2.2.** This theorem establishes a test for long-range memory in the a.s. convergence sense, following Lo (1991) in the weak convergence sense.

**Remark 2.3.** For some cases of short-range memory, we have established laws of the iterated logarithm for R/S statistics. For instance, for an i.i.d. sequence (corresponding to the case of  $\alpha = 1/2$ ) with  $EX_1 = 0$  and  $EX_1^2 = 1$ , we have

$$\limsup_{n \rightarrow \infty} \frac{1}{\sqrt{2n \log \log n}} Q_n = 1 \quad \text{a.s.}$$

For the AR(1) model with an i.i.d. sample, we have a similar result.

In order to prove Theorem 2.1, we need some preliminaries. The following Fernique-type inequality is a consequence of Lemma 2.1 in Csáki, Csörgő and Shao (1992).

**Lemma 2.1.** Let  $\mathbf{B}$  be a separable Banach space with norm  $\|\cdot\|$  and let  $\{\Gamma(t), -\infty < t < \infty\}$  be a stochastic process with values in  $\mathbf{B}$ . Let  $P$  be the probability measure generated by  $\Gamma(\cdot)$ . Assume that  $\Gamma(\cdot)$  is  $P$ -almost surely separable with respect to  $\|\cdot\|$ , and that for  $|t| \leq t_0$  and  $0 < x^* \leq x$  there exists a non-negative non-decreasing function  $\sigma(x)$  such that  $P\{\|\Gamma(t)\| \geq x\sigma(t)\} \leq K \exp\{-\gamma x^\beta\}$  for some  $K, \gamma, \beta > 0$ . Then

$$P\left\{\sup_{0 \leq t \leq T} \|\Gamma(t)\| \geq x\sigma(t) + \sigma^*(T, k)\right\} \leq 4K2^{2k+1} \exp\{-\gamma x^\beta\}$$

for any  $0 < t \leq t_0$ ,  $x \geq x^*$  and  $k \geq 3$ , where  $\sigma^*(T, k) = 4(14/\gamma)^{1/\beta} \beta \int_{2^{(k-2)/\beta}}^{\infty} \sigma(Te^{-y^\beta}) dy$ .

Using a discrete version of this lemma, we can show the following strong law of large numbers.

**Lemma 2.2.** For  $\{X_n\}$  defined in Theorem 2.1, we have  $(1/n) \sum_{j=1}^n X_j \rightarrow 0$  a.s.

**Proof.** By (1.4) and Lemma 3.1 in Taqqa (1975), we have that for  $\alpha \in (1/2, 1)$ ,

$$\sigma_n^2 = \sum_{i=1}^n \sum_{j=1}^n \gamma(i-j) \sim an^{2\alpha} L(n),$$

where  $a = (2\alpha|2\alpha - 1|)^{-1}$ . Consider the case of  $\alpha \in (0, 1/2)$ . Put  $R = \gamma(0) + 2 \sum_{k=1}^{\infty} \gamma(k)$ . Then  $\sigma_n^2 = \gamma(0) + \sum_{j=1}^{n-1} (R - 2 \sum_{k=n-j+1}^{\infty} \gamma(k))$ . Hence  $R \geq 0$ . If  $R = 0$ , similarly to the proof of Lemma 2.1 in Taqqa (1975), we have also

$$\sigma_n^2 \sim an^{2\alpha} L(n). \quad (2.6)$$

If  $R > 0$ , it is clear that  $\sigma_n^2 \sim Rn$ .

Consider the case of  $\sigma_n^2 \sim an^{2\alpha} L(n)$ . By the property of a slowly varying function, for any  $\varepsilon > 0$ , one was  $\lim_{x \rightarrow \infty} x^\varepsilon L(x) = \infty$ ,  $\lim_{x \rightarrow \infty} x^{-\varepsilon} L(x) = 0$ ,  $n^2/\sigma_n^2 \geq cn^{2(1-\alpha)}/L(n) \geq cn^{1-\alpha}$  for large  $n$ , where  $c$  stands for a positive constant. Hence, using Lemma 2.1 with  $\sigma(n) = \sigma_n$ , for any  $\varepsilon > 0$  there exists a constant  $C = C(\varepsilon) > 0$  such that

$$P\left\{\max_{1 \leq j \leq 2^k} |S_j| \geq 2^k \varepsilon\right\} \leq C \exp\left\{-\frac{2^{2k}}{\sigma_{2^k}^2} \cdot \frac{\varepsilon^2}{3}\right\} \leq C \exp\left\{-\frac{c2^{k(1-\alpha)}\varepsilon^2}{3}\right\},$$

which, in combination with the Borel-Cantelli lemma, implies that as  $k \rightarrow \infty$ ,  $\max_{1 \leq j \leq 2^k} |S_j|/2^k \rightarrow 0$  a.s. Using this result, along the lines of a proof of a strong law of large numbers, we obtain Lemma 2.2. The details are omitted.

As for the case of  $\sigma_n^2 \sim Rn$ , Lemma 2.2 is the Strong Law of Large Numbers under only moment restrictions (cf., Serfling (1970)).

**Lemma 2.3.** For any  $\varepsilon > 0$ , there exists  $C_1 = C_1(\varepsilon) > 0$  such that for any  $x \geq 1$  and large  $n$ ,

$$P\{|R(n)|/\sigma_n > (1 + \varepsilon)\sqrt{\alpha'}x\} \leq C_1 \exp\{- (1 + \varepsilon)x^2/2\}.$$

**Proof.** Rewrite  $R(n)$  as  $R(n) = \max_{1 \leq i < j \leq n} |S_j - S_i - ((j - i)/n)S_n|$ . Consider the case  $\sigma_n^2 \sim an^{2\alpha}L(n)$ . For  $1 \leq i < j \leq n$ , we have  $E(S_j - S_i)^2 = ES_{j-i}^2 \sim a(j - i)^{2\alpha}L(j - i)$  and  $E(S_j - S_i)S_n = (1/2)E(S_{j-i}^2 + S_n^2 - S_{n-j+i}^2)$ . Therefore

$$\begin{aligned} \sigma_{ij}^2 &:= E\left(S_j - S_i - \frac{j-i}{n}S_n\right)^2 \\ &\sim a\left(1 - \frac{j-i}{n}\right)(j-i)^{2\alpha}L(j-i) - a \cdot \frac{j-i}{n}\left(1 - \frac{j-i}{n}\right)n^{2\alpha}L(n) \\ &\quad + a \cdot \frac{j-i}{n}(n-j+i)^{2\alpha}L(n-j+i). \end{aligned}$$

Note that the function  $f_\alpha(x) = (1 - x)x^{2\alpha} - x(1 - x) + x(1 - x)^{2\alpha}$ ,  $0 \leq x \leq 1$ , has maximum value  $\alpha' = 2^{-2\alpha} - 2^{-2}$  at  $x = 1/2$ . For given  $0 < \varepsilon < 1/2$ , there exists  $\delta > 0$  such that

$$\sup_{0 \leq x \leq \varepsilon} f_{\alpha-\delta}(x) \leq \alpha'. \tag{2.7}$$

By the definition of a slowly varying function, for  $0 < \varepsilon < 1/2$ ,  $\max_{\varepsilon n \leq j-i \leq n} L(j-i)/L(n) \rightarrow 1$ , as  $n \rightarrow \infty$ . Hence  $\max_{1 \leq i < j \leq n, j-i \geq \varepsilon n} \sigma_{ij}^2/\sigma_n^2 \sim \sup_{\varepsilon \leq x \leq 1} f_\alpha(x) = \alpha'$ . Moreover by property (2.4), letting  $\varepsilon > 0$  be small enough, for  $n$  large enough we have

$$\begin{aligned} &\max_{1 \leq i < j \leq n, j-i < \varepsilon n} \sigma_{ij}^2/\sigma_n^2 \\ &\leq \max_{1 \leq i < j \leq n, j-i < \varepsilon n} \left\{ \left(1 - \frac{j-i}{n}\right)\left(\frac{j-i}{n}\right)^{2(\alpha-\delta)} - \left(\frac{j-i}{n}\right)\left(1 - \frac{j-i}{n}\right) \right. \\ &\quad \left. + \left(\frac{j-i}{n}\right)\left(1 - \frac{j-i}{n}\right)^{2(\alpha-\delta)} \right\} \leq \alpha', \end{aligned}$$

by (2.7). Therefore

$$\max_{1 \leq i < j \leq n} \sigma_{ij}^2/\sigma_n^2 \sim \alpha'. \tag{2.8}$$

For any positive integers  $i, n$  and  $M$ , with  $M < n/2$ , let  $i_M = [i/[n/m]][n/m]$ , where  $[\cdot]$  denotes the largest integer part. Note that  $0 \leq i - i_M \leq [n/m]$ . Write

$$R(n) \leq \max_{1 \leq i < j \leq n} \left| S_{j_M} - S_{i_M} - \frac{j_M - i_M}{n}S_n \right| + 2 \max_{1 \leq j \leq n} |S_{j-j_M}| + \frac{1}{M}|S_n|. \tag{2.9}$$

By (2.8), we have that for large  $n$ ,

$$\begin{aligned} &P\left\{ \max_{1 \leq i < j \leq n} \left| S_{j_M} - S_{i_M} - \frac{j_M - i_M}{n}S_n \right|/\sigma_n \geq (1 + \frac{\varepsilon}{2})\sqrt{\alpha'}x \right\} \\ &\leq M^2 \exp\{- (1 + \varepsilon)x^2/2\}. \end{aligned} \tag{2.10}$$

Moreover, by Lemma 2.1 and taking  $M$  to be large enough, we have

$$\begin{aligned} & \mathbb{P}\left\{2 \max_{1 \leq j \leq n} |S_{j-j_M}|/\sigma_n \geq \frac{\varepsilon}{4} \sqrt{\alpha'x}\right\} \\ & \leq \sum_{k=0}^M \mathbb{P}\left\{\max_{k[n/m] \leq j \leq (k+1)[n/m]} |S_{j-k[n/m]}|/\sigma_n \geq \frac{\varepsilon}{8} \sqrt{\alpha'x}\right\} \\ & \leq CM \exp\left\{-\frac{\varepsilon^2 \alpha' x^2 n^{2\alpha} L(n)}{2 \cdot 128(n/M)^{2\alpha} L(n/M)}\right\} \\ & \leq CM \exp(-x^2) \end{aligned} \tag{2.11}$$

for some  $C > 0$ , provided  $n$  is large enough. It is clear that for large  $M$

$$\mathbb{P}\left\{\frac{1}{M}|S_n|/\sigma_n \geq \frac{\varepsilon}{4} \sqrt{\alpha'x}\right\} \leq \exp\left\{-\frac{\varepsilon^2 \alpha' M^2 x^2}{32}\right\} \leq \exp(-x^2). \tag{2.12}$$

Combining (2.9)–(2.12) yields  $\mathbb{P}\{R(n)/\sigma_n \geq (1 + \varepsilon)\sqrt{\alpha'x}\} \leq C_1 \exp\{- (1 + \varepsilon)x^2/2\}$  with  $C_1 = M^2 + C + 1$ . The lemma is proved.

We need the well-known Slepian lemma.

**Lemma 2.4.** *Let  $G(t)$  and  $G^*(t)$  be Gaussssian processes on  $[0, T]$  for some  $0 < T < \infty$ , possessing continuous sample path functions with  $\mathbb{E}G(t) = \mathbb{E}G^*(t) = 0$ ,  $\mathbb{E}G^2(t) = \mathbb{E}G^{*2}(t) = 1$ , and let  $\rho(s, t)$  and  $\rho^*(s, t)$  be their respective covariance functions. Suppose that  $\rho(s, t) \geq \rho^*(s, t)$  for  $s, t \in [0, T]$ . Then for any  $x$ ,  $\mathbb{P}\{\sup_{0 \leq t \leq T} G(t) \leq x\} \geq \mathbb{P}\{\sup_{0 \leq t \leq T} G^*(t) \leq x\}$ .*

**Proof of Theorem 2.1.** We first prove (2.1). Let  $\theta > 1$ ,  $k' = [\theta^k]$  and  $L_k = \min_{(k-1)' \leq n \leq k'} L(n)$ . For  $(k-1)' < n \leq k'$ , write

$$\begin{aligned} |R(k') - R(n)| & \leq \max_{(k-1)' \leq j < k'} \left\{2\left|S_j - \frac{j}{k'} S_{k'}\right| + |S_{k'} - S_j|\right\} \\ & \leq 3 \max_{(k-1)' \leq j < k'} |S_{k'} - S_j| + 2(1 - (k-1)'/k')|S_{k'}|, \end{aligned} \tag{2.13}$$

and hence

$$\begin{aligned} & \sqrt{\frac{1}{2\sigma_n^2 \log \log n}} R(n) \\ & \leq \sqrt{\frac{1}{2(k-1)'^{2\alpha} L_k \log \log(k-1)'}} \left\{R(k') + 3 \max_{(k-1)' \leq j < k'} |S_{k'} - S_j| \right. \\ & \quad \left. + 2\left(1 - \frac{(k-1)'}{k'}\right)|S_{k'}|\right\}. \end{aligned} \tag{2.14}$$

Using the property of a slowly varying function we have  $L_k \sim L(k')$ , which implies

$$(k-1)'^{2\alpha} L_k / \sigma_{k'}^2 \sim \theta^{-2\alpha} \quad \text{as } k \rightarrow \infty. \tag{2.15}$$

Then, using Lemma 2.3 and taking  $\theta$  to be near enough to one, we obtain

$$\begin{aligned} & \mathbb{P}\left\{R(k')/\sigma_{k'} \geq \left(1 + \frac{\varepsilon}{2}\right)\sqrt{(k-1)^{2\alpha} L_k/\sigma_{k'}^2} \cdot \sqrt{2\alpha' \log \log(k-1)'}\right\} \\ & \leq \mathbb{P}\left\{R(k')/\sigma_{k'} \geq \left(1 + \frac{\varepsilon}{4}\right)\sqrt{2\alpha' \log \log(k-1)'}\right\} \\ & \leq C_1 \exp\left\{-\left(1 + \frac{\varepsilon}{4}\right)\log \log(k-1)'\right\}. \end{aligned} \tag{2.16}$$

Similarly to (2.15),  $(k-1)^{2\alpha} L_k/\sigma_{k'-(k-1)'}^2 \sim (\theta-1)^{-2\alpha}$ . Then, using Lemma 2.1, we obtain

$$\begin{aligned} & \mathbb{P}\left\{3 \max_{(k-1)' \leq j < k'} \frac{|S_{k'} - S_j|}{\sigma_{k'-(k-1)'}} \geq \frac{\varepsilon}{4} \sqrt{\frac{(k-1)^{2\alpha} L_k}{\sigma_{k'-(k-1)' }^2}} \cdot \sqrt{2\alpha' \log \log(k-1)'}\right\} \\ & \leq C \exp\{-2 \log \log(k-1)'\} \end{aligned} \tag{2.17}$$

for some  $C > 0$ , provided  $\theta$  is near enough to one. Similarly

$$\begin{aligned} & \mathbb{P}\left\{2\left(1 - \frac{(k-1)'}{k'}\right) \frac{|S_{k'}|}{\sigma_{k'}} \geq \frac{\varepsilon}{4} \sqrt{\frac{(k-1)^{2\alpha} L_k}{\sigma_{k'}^2}} \cdot \sqrt{2\alpha' \log \log(k-1)'}\right\} \\ & \leq C \exp\{-2 \log \log(k-1)'\}. \end{aligned} \tag{2.18}$$

Combining (2.16)–(2.18) with (2.14) yields

$$\begin{aligned} & \mathbb{P}\left\{R(n) \geq (1 + \varepsilon)\sqrt{2\alpha'\sigma_n^2 \log \log n}\right\} \\ & \leq (C_1 + 2C) \exp\left\{-\left(1 + \frac{\varepsilon}{4}\right)\log \log(k-1)'\right\} \\ & = (C_1 + 2C)((k-1) \log \theta)^{-(1+\varepsilon/4)}, \end{aligned}$$

which implies (2.1) by the Borel-Cantelli lemma.

In order to prove (2.3), it is enough to show

$$\limsup_{n \rightarrow \infty} \sqrt{\frac{1}{2\sigma_n^2 \log \log n}} R(n) \geq \sqrt{\alpha'} \quad a.s..$$

under (2.2). This is a consequence of

$$\limsup_{n \rightarrow \infty} \sqrt{\frac{1}{2\sigma_{2n}^2 \log \log n}} \left|\frac{1}{2}S_{2n} - S_n\right| \geq \sqrt{\alpha'} \quad a.s.. \tag{2.19}$$

We have  $\mathbb{E}((1/2)S_{2n} - S_n)^2 = (1/4)\mathbb{E}S_{2n}^2 + \mathbb{E}S_n^2 - \mathbb{E}S_{2n}S_n = -(1/4)\mathbb{E}S_{2n}^2 + \mathbb{E}S_n^2 \sim -(1/4)(2n)^\alpha L(2n) + n^\alpha L(n)$ , and hence  $\mathbb{E}((1/2)S_{2n} - S_n)^2/\sigma_{2n}^2 \sim -(1/4) + (1/2)^\alpha = \alpha'$ .

For  $m < n$ , we have

$$\begin{aligned} 2\mathbb{E}\left(\frac{S_m}{\sigma_m} \cdot \frac{S_n}{\sigma_n}\right) &= \frac{\mathbb{E}(S_m^2 + S_n^2 - (S_n - S_m)^2)}{\sigma_m \sigma_n} = \frac{\sigma_m}{\sigma_n} + \frac{\sigma_n}{\sigma_m} - \frac{\sigma_{n-m}^2}{\sigma_m \sigma_n} \\ &\sim \frac{m^\alpha L(m)^{1/2}}{n^\alpha L(n)^{1/2}} + \frac{n^\alpha L(n)^{1/2}}{m^\alpha L(m)^{1/2}} \left(1 - \frac{L(n-m)}{L(n)}\right) \\ &\quad + \frac{n^\alpha L(n-m)}{m^\alpha L(n)^{1/2} L(m)^{1/2}} \left(1 - \frac{(n-m)^{2\alpha}}{n^{2\alpha}}\right) \\ &=: p_1(m, n) + p_2(m, n) + p_3(m, n). \end{aligned}$$

Let  $\theta > 0$  be a large integer and  $n_j = \theta^j$ . By (2.4), for  $i < j$ , we have  $p_1(n_{ki}, n_{kj}) = \theta^{-k(j-i)\alpha} L(\theta^{ki})^{1/2} / L(\theta^{kj})^{1/2} \rightarrow 0$  as  $k \rightarrow \infty$ . Using condition (2.2), we have  $p_2(n_{ki}, n_{kj}) \rightarrow 0$  as  $k \rightarrow \infty$ . Moreover  $p_3(n_{ki}, n_{kj}) \leq 2\alpha\theta^{-k(j-i)(1-\alpha)} \cdot [(L(\theta^{kj}) - \theta^{ki}) / (L(\theta^{kj})L(\theta^{ki}))] \rightarrow 0$  as  $k \rightarrow \infty$ . Hence

$$\mathbb{E}\left(\frac{S_{2n_{ki}}}{\sigma_{2n_{ki}}} \cdot \frac{S_{2n_{kj}}}{\sigma_{2n_{kj}}}\right) \rightarrow 0 \text{ as } k \rightarrow \infty. \quad (2.20)$$

Clearly from (2.20) we have that, as  $k \rightarrow \infty$ ,

$$\mathbb{E}\left(\frac{S_{n_{ki}}}{\sigma_{2n_{ki}}} \cdot \frac{S_{2n_{kj}}}{\sigma_{2n_{kj}}}\right) \rightarrow 0, \quad \mathbb{E}\left(\frac{S_{2n_{ki}}}{\sigma_{2n_{ki}}} \cdot \frac{S_{n_{kj}}}{\sigma_{2n_{kj}}}\right) \rightarrow 0. \quad (2.21)$$

Let  $Z_i = ((1/2)S_{2n_i} - S_{n_i}) / \sigma_{2n_i}$  and  $r_{ij} = \mathbb{E}Z_i Z_j$ . From (2.20) and (2.21), for any given  $\delta > 0$ , we have  $|r_{ki, kj}| \leq \delta$  provided  $k$  is large enough. Now let  $\{\xi_j, j \geq 1\}$  and  $\eta$  be independent normal random variables with means zero,  $\mathbb{E}\xi_j^2 = \mathbb{E}Z_{kj}^2 - \delta$  and  $\mathbb{E}\eta^2 = \delta$ . Define  $\zeta_j = \xi_j + \eta$ . Then  $\mathbb{E}\zeta_j^2 = \mathbb{E}Z_{kj}^2$  and  $\mathbb{E}Z_{ki}Z_{kj} \leq \mathbb{E}\zeta_i\zeta_j$  for  $i \neq j$ . Let  $A_i = \{Z_i / \sqrt{2\alpha' \log \log n_i} \leq 1 - 3\varepsilon\}$ . For small  $\varepsilon > 0$ , taking integer  $k$  large enough and  $\delta > 0$  small enough, with Slepian's lemma we have

$$\begin{aligned} &\mathbb{P}\left\{\bigcap_{i=m}^M A_{ki}\right\} \leq \mathbb{P}\left\{\bigcap_{i=m}^M \left(\frac{\zeta_i}{\sqrt{2\alpha' \log \log n_{ki}}} \leq 1 - 3\varepsilon\right)\right\} \\ &\leq \mathbb{P}\left\{\bigcap_{i=1}^M \left(\frac{\xi_i}{\sqrt{2\alpha' \log \log n_{ki}}} \leq 1 - 2\varepsilon\right)\right\} + \mathbb{P}\{\eta \geq \varepsilon\sqrt{2\alpha' \log \log n_{km}}\} \\ &\leq \prod_{i=1}^M \left(1 - \mathbb{P}\{N(0, 1) \geq (1 - \varepsilon)\sqrt{2 \log \log n_{ki}}\}\right) \\ &\quad + \mathbb{P}\left\{N(0, 1) \geq \frac{\varepsilon}{\sqrt{\delta}}\sqrt{2\alpha' \log \log n_{km}}\right\} \\ &\leq \exp\left\{-\sum_{i=1}^M (\log n_{ki})^{-(1-\varepsilon)}\right\} + \exp\{-2 \log \log n_{km}\}. \end{aligned} \quad (2.22)$$

Note that  $\sum_{i=1}^{\infty} (\log n_{ki})^{-(1-\varepsilon)} = \infty$ . (2.22) implies that, for  $km \geq N$ ,

$$P\left\{\bigcap_{i=N}^{\infty} A_i\right\} \leq P\left\{\bigcap_{i=m}^{\infty} A_{ki}\right\} \leq \exp\{-2 \log \log n_{km}\}.$$

Letting  $k \rightarrow \infty$ , we obtain  $P\{\bigcap_{i=N}^{\infty} A_i\} = 0$ , and hence  $P\{\bigcup_{N=1}^{\infty} \bigcap_{i=N}^{\infty} A_i\} = 0$ , which implies (2.19). This proves Theorem 2.1.

Now we turn to the proof of Theorem 2.2. To this end we need the well-known Borell inequality.

**Lemma 2.5.** *Let  $\{X_t, t \in T\}$  be a centered separable Gaussian process with almost surely bounded sample paths. Let  $\|X\| = \sup_{t \in T} X_t$ . Then for all  $x > 0$ ,*

$$P\{\|\|X\| - E\|X\|\| > x\} \leq 2 \exp\{-x^2/(2\sigma_T^2)\},$$

where  $\sigma_T^2 = \sup_{t \in T} EX_t^2$ .

**Proof of Theorem 2.2.** From Theorem 2.1, to show Theorem 2.2 it suffices to prove that

$$S^2(n)/\sigma_n^2 \rightarrow \begin{cases} 0 & \text{for } \alpha \in (1/2, 1), \\ \infty & \text{for } \alpha \in (0, 1/2). \end{cases} \tag{2.23}$$

Using Lemma 2.2, (2.23) is equivalent to

$$\sum_{j=1}^{\infty} X_j^2/\sigma_n^2 \rightarrow \begin{cases} 0 & \text{for } \alpha \in (1/2, 1), \\ \infty & \text{for } \alpha \in (0, 1/2). \end{cases} \tag{2.24}$$

It is well-known that  $(\sum_{j=1}^n X_j^2)^{1/2} = \sup^* \sum_{j=1}^n a_j X_j$ , where  $\sup^* = \sup_{(a_1, \dots, a_n): \sum_{i=1}^n a_i^2 \leq 1}$ . By Borell's inequality, we have that for any  $x > 0$ ,

$$\begin{aligned} P\left\{\left|\left(\sum_{j=1}^n X_j^2\right)^{1/2} - E\left(\sum_{j=1}^n X_j^2\right)^{1/2}\right| \geq x\right\} &= P\left\{\left|\sup_{j=1}^n a_j X_j - E \sup_{j=1}^n a_j X_j\right| \geq x\right\} \\ &\leq 2 \exp\left\{\frac{-x^2}{2 \sup^* E(\sum_{j=1}^n a_j X_j)^2}\right\}. \end{aligned} \tag{2.25}$$

Putting  $\sigma^2 = EX_j^2$ , we have

$$\begin{aligned} \sup^* E\left(\sum_{j=1}^n a_j X_j\right)^2 &= \sigma^2 + 2 \sup^* \sum_{1 \leq i < j \leq n} a_i a_j EX_i X_j \\ &\sim \sigma^2 + 2 \sup^* \sum_{1 \leq i < j \leq n} a_i a_j (j-i)^{2\alpha-2} L(j-i). \end{aligned} \tag{2.26}$$

Obviously, there is a constant  $c_0$  such that for any  $i < j$ ,  $(j-i)^{2\alpha-2}L(j-i) \leq c_0$ . Put  $n_1 = \lfloor \sqrt{n} \rfloor$  and  $L_1(n) = \max_{n_1 < j \leq n} L(j)$ . Clearly,  $L_1(n) \leq c_1 n^{(1-\alpha)/2}$  for some  $c_1 > 0$ . Therefore

$$\begin{aligned} & \sup_{1 \leq i < j \leq n} \sum_{i=1}^{j-1} a_i a_j (j-i)^{2\alpha-2} L(j-i) \\ & \leq c_0 \sup_{i=1}^{n-1} \sum_{j=i+1}^{i+n_1} a_i a_j + \sup_{i=1}^{n-1} n^{\alpha-1} L_1(n) \sum_{i=1}^{n-n_1} \sum_{j=i+n_1+1}^n a_i a_j \\ & =: \beta_1 + \beta_2, \end{aligned} \tag{2.27}$$

where, using Cauchy-Schwarz's inequality,

$$\beta_1 \leq c_0 \sup_{i=1}^{n-1} \sum_{j=i+1}^{i+n_1} a_i n_1^{1/2} \left( \sum_{j=i+1}^{i+n_1} a_j^2 \right)^{1/2} \leq c_0 n^{1/2} n_1^{1/2} \sup_{i=1}^{n-1} \left( \sum_{i=1}^{n-1} a_i^2 \right)^{1/2} \leq c_0 n^{3/4}, \tag{2.28}$$

and similarly,

$$\beta_2 \leq n^\alpha L_1(n) \leq c_1 n^{(1+\alpha)/2}. \tag{2.29}$$

Combining (2.26)–(2.29) with (2.25) shows that for any  $\varepsilon > 0$ ,

$$\begin{aligned} & \sum_{n=1}^{\infty} \mathbb{P} \left\{ \left| \left( \sum_{j=1}^n X_j^2 \right)^{1/2} - \mathbb{E} \left( \sum_{j=1}^n X_j^2 \right)^{1/2} \right| \geq \sqrt{n} \varepsilon \right\} \\ & \leq 2 \sum_{n=1}^{\infty} \exp \left\{ - \frac{\varepsilon^2}{2c_0 n^{-1/4} + 2c_1 n^{(\alpha-1)/2}} \right\} < \infty, \end{aligned}$$

which, in combination with the Borel-Cantelli lemma, implies that as  $n \rightarrow \infty$ ,

$$\left( \frac{1}{n} \sum_{j=1}^{\infty} X_j^2 \right)^{1/2} - \mathbb{E} \left( \frac{1}{n} \sum_{j=1}^{\infty} X_j^2 \right)^{1/2} \rightarrow 0 \quad \text{a.s.}$$

Hence, noting that  $0 < b := (1/n) \sum_{j=1}^n \mathbb{E} |X_j| \leq ((1/n) \sum_{j=1}^n \mathbb{E} X_j^2)^{1/2} = (\mathbb{E} X_1^2)^{1/2} = \sigma$ , we have  $b_1^2 \leq \liminf_{n \rightarrow \infty} (1/n) \sum_{j=1}^n X_j^2 \leq \limsup_{n \rightarrow \infty} (1/n) \sum_{j=1}^n X_j^2 \leq \sigma^2$  a.s., which yields (2.24) by recalling (2.6). This completes the proof of Theorem 2.2.

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