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SIMULTANEOUS INSPECTION FOR VARIABLE SAMPLING ACCEPTANCE

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Abstract: We investigate the problem of simultaneously inspecting k shipments for variable sampling acceptance. Our goal is to simultaneously select all good shipments and exclude all bad shipments. By incorporating information from the samples taken from each of the k shipments, an empirical Bayes simultaneous inspection procedure $\underset{\sim}{\delta}^*$ is proposed. The relative regret Bayes risk of $\underset{\sim}{\delta}^*$ is used as a measure of its performance. We have proved that the simultaneous inspection procedure $\underset{\sim}{\delta}^*$ is asymptotically optimal, and its relative regret Bayes risk converges to zero at rate $O\left(k^{-1}\ell n^2 k\right)$.

Key words and phrases: Asymptotically optimal, empirical Bayes, rate of convergence, regret Bayes risk, simultaneous inspection, variable sampling acceptance.

1. Introduction

The exponential distribution has played an important role in modeling the lifetime of random phenomena. It arises in many areas of applications, including reliability, life testing and survival analysis. Johnson, Kotz and Balakrishnan (1994) present an introduction to the exponential distribution. More applications of the exponential distribution are given in Balakrishnan and Basu (1995).

Suppose that a batch of components is presented for acceptance sampling. The quality of a component is measured by its lifetime X. In order to estimate the quality of the batch, a sample of m items is put on life test and not replaced on failure. Then one decides whether to accept the batch based on the observed lifetimes of the sampled components. An introduction to sampling inspection and quality control can be seen, for example, in Wetherill (1977). Interested readers are referred to Lam (1994) and Lam and Choy (1995) for the recent development of variable sampling plans for exponential distributions.

Consider k shipments (populations) $\pi_1, \ldots, \pi_k$, each consisting of M components. For each $i = 1, \ldots, k$, let X_{ij} denote the lifetime of component j of π_i. Suppose that $X_{i1}, \ldots, X_{iM}$ are identically and independently distributed, having an exponential distribution with mean lifetime θ_i, and that $\underset{\sim}{X_i} = (X_{i1}, \ldots, X_{iM})$, $i = 1, \ldots, k$, are mutually independent. Thus, $Y_i = \sum_{j=1}^{M} X_{ij}$ is the total lifetime of the M items of π_i. Let θ_0 be a specified positive value. Shipment π_i is said to be acceptable if $Y_i \geq \theta_0$; π_i is said to be rejected if $Y_i < \theta_0$. We consider the problem of simultaneously inspecting k shipments. Our goal is to accept all good shipments and to reject all bad shipments.

In the literature, the problem of comparing populations with a control has been extensively studied by many authors. We mention a few here. Huang (1975) derived Bayes selection procedures to partition k normal populations; Gupta and Hsiao (1981) derived Bayes Γ-minimax and minimax procedures for selecting populations close to a control; Mee, Shah and Lefante (1987) developed multiple testing procedures to compare the means of k normal populations with respect to a control; Miescke (1981) and Gupta and Miescke (1985) derived optimal selection procedures via Γ-minimax and minimax approaches for selecting good populations. Interested readers are referred to Bechhofer, Santner and Goldsman (1995) for an overview of the area of ranking and selection. See Gupta and Panchapakesan (1988) for a comprehensive survey of selection procedures in exponential distributions and other reliability models.

In this paper, it is assumed that the mean lifetime parameter θ_i is a realization of a positive random variable Θ_i, $i = 1, \ldots, k$, and that $\Theta_1, \ldots, \Theta_k$ are independently and identically distributed, following an unknown but nondegenerate prior distribution $H(\cdot)$ over the interval $(0, \infty)$. Under our assumptions, $\underset{\sim}{X}_i$, $i = 1, \ldots, k$, are mutually independent and marginally identically distributed. Thus, the empirical Bayes approach is adopted to incorporate information from each of the k sources for constructing simultaneous inspection procedures.

The paper is organized as follows. In Section 2, the inspection problem for variable sampling acceptance is formulated and a Bayes inspection procedure $\underset{\sim}{\delta}^H$ is derived. In Section 3, an empirical Bayes simultaneous inspection procedure $\underset{\sim}{\delta}^*$ is constructed by mimicking the behavior of the Bayes inspection procedure $\underset{\sim}{\delta}^H$. Relative regret Bayes risk is used as a measure of performance and the asymptotic optimality of $\underset{\sim}{\delta}^*$ is established in Section 4 and Section 5. An upper bound of order $O(k^{-1} \ln^2 k)$ is established for the convergence rate of the relative regret Bayes risk of $\underset{\sim}{\delta}^*$. Then, a lower bound of the same order is established These results together show that the empirical Bayes simultaneous inspection procedure $\underset{\sim}{\delta}^*$ is asymptotically optimal, relative to the prior distribution H, at a convergence rate of order $k^{-1} \ln^2 k$.

Readers are referred to Balakrishnan and Ma (1996, 1997), Huang and Lai (1998) and Liang (1997a, b), and the references cited there, for recent development of empirical Bayes procedures in the area of ranking and selection.

2. Formulation of the Problem and a Bayes Inspection Procedure

In this section, we provide a decision-theoretic formulation of the sampling inspection problem and derive a Bayes inspection procedure based on which an empirical Bayes simultaneous inspection procedure will be developed.

Let $\underset{\sim}{a} = (a_1, \ldots, a_k)$ be an action, where $a_i = 0, 1,\ i = 1, \ldots, k$. Shipment π_i is accepted if $a_i = 1$, and rejected if $a_i = 0$. For action $\underset{\sim}{a}$ and observation $\underset{\sim}{Y} = (Y_1, \ldots, Y_k)$, the loss $L(\underset{\sim}{Y},\ \underset{\sim}{a})$ is defined to be

$$L(\underset{\sim}{Y}, \underset{\sim}{a}) = \sum_{i=1}^{k} \ell\,(Y_i,\ a_i)\,, \tag{2.1}$$

where

$$\ell\,(Y_i,\ a_i) = a_i\,(\theta_0 - Y_i)\,I\,(\theta_0 - Y_i) + (1 - a_i)\,(Y_i - \theta_0)\,I\,(Y_i - \theta_0)\,, \tag{2.2}$$

and $I\,(x) = 1\,(0)$ if $x > 0\,(x \leq 0)$. In (2.2), the first term is the loss of wrongly accepting a bad shipment, and the second term is the loss of wrongly excluding a good shipment.

Given the values of $\underset{\sim}{X}_i$, $i = 1, \ldots, k$, we can always reach the best decision. However, it is time-consuming to have a life test for each of the M items in π_i. Also, at the end of the life test, the items put on life test are destroyed. Therefore, in order to implement a decision, a sample of m items, $1 \leq m < M$, is taken from each shipment, and put on life test. At the end of the life test, the corresponding lifetimes are observed. We denote the lifetimes of the m items sampled from π_i by $\underset{\sim}{X}_i(m) = (X_{i1}, \ldots, X_{im})$. Let $\underset{\sim}{X}(m) = (\underset{\sim}{X}_1(m), \ldots, \underset{\sim}{X}_k(m))$ and χ denote the sample space of $\underset{\sim}{X}(m)$. An inspection procedure $\underset{\sim}{\delta} = (\delta_1, \ldots, \delta_k)$ is defined to be a measurable mapping from the sample space χ into the product space $[0, 1]^k$, such that for each $\underset{\sim}{x}(m) \in \chi$, $\underset{\sim}{\delta}(\underset{\sim}{x}(m)) = (\delta_1(\underset{\sim}{x}(m)), \ldots, \delta_k(\underset{\sim}{x}(m)))$, and $\delta_i(\underset{\sim}{x}(m))$ is the probability of accepting π_i when $\underset{\sim}{X}(m) = \underset{\sim}{x}(m)$ is observed.

Let $\mathcal{C}$ be the class of all inspection procedures. For each $\underset{\sim}{\delta}$ in $\mathcal{C}$ and a prior distribution H, let $R(H, \underset{\sim}{\delta})$ denote the associated Bayes risk. Then $R(H) = \inf_{\underset{\sim}{\delta} \in \mathcal{C}} R(H, \underset{\sim}{\delta})$ is the minimum Bayes risk among the class $\mathcal{C}$ and an inspection procedure $\underset{\sim}{\delta}^H$ such that $R(H, \underset{\sim}{\delta}^H) = R(H)$ is called a Bayes inspection procedure.

Let $f_i(\underset{\sim}{x}_i|\theta_i)$ and $f_{im}(\underset{\sim}{x}_i(m)|\theta_i)$ be the conditional probability densities of $\underset{\sim}{X}_i$ and $\underset{\sim}{X}_i(m)$, respectively. Then

$$f_i(\underset{\sim}{x}_i|\theta_i) = \frac{1}{\theta_i^M} \exp\{-\sum_{j=1}^{M} x_{ij}/\theta_i\},\ f_{im}(\underset{\sim}{x}_i(m)|\theta_i) = \frac{1}{\theta_i^m} \exp\{-\sum_{j=1}^{m} x_{ij}/\theta_i\}.$$

Let $f_i(\underset{\sim}{x}_i) = \int f_i(\underset{\sim}{x}_i|\theta_i) dH(\theta_i)$, and $f_{im}(\underset{\sim}{x}_i(m)) = \int f_{im}(\underset{\sim}{x}_i(m)|\theta_i) dH(\theta_i)$. Then $f_i(\underset{\sim}{x}_i)$ and $f_{im}(\underset{\sim}{x}_i(m))$ are the marginal probability densities of $\underset{\sim}{X}_i$ and $\underset{\sim}{X}_i(m)$, respectively. Note that $\underset{\sim}{X}_i$, $i = 1, \ldots, k$, are mutually independent and marginally identically distributed. Thus, $f_1 = \cdots = f_k$ and $f_{1m} = \cdots = f_{km}$. Let $f_i(\underset{\sim}{x}_i|\underset{\sim}{x}_i(m)) = f_i(\underset{\sim}{x}_i)/f_{im}(\underset{\sim}{x}_i(m))$, the marginal conditional probability density of $\underset{\sim}{X}_i$ given $\underset{\sim}{X}_i(m) = \underset{\sim}{x}_i(m)$.

It is assumed that $\int_0^\infty \theta dH(\theta) < \infty$. With the loss function given in (2.1)-(2.2), and by Fubini's theorem, the Bayes risk associated with an inspection procedure $\underset{\sim}{\delta} = (\delta_1, \ldots, \delta_k)$ is

$$R(H, \underset{\sim}{\delta}) = \sum_{i=1}^{k} R_i\left(H, \delta_i\right), \tag{2.3}$$

where

$$\begin{aligned} R_i\left(H, \delta_i\right) &= \int_{\underset{\sim}{x}} \delta_i(\underset{\sim}{x}\,(m))\left[\theta_0 - y_i\right] \prod_{j=1}^{k} f_j(\underset{\sim}{x}_j) d\underset{\sim}{x} + D_H \qquad (2.4)\\ &= \int_{\chi} \delta_i(\underset{\sim}{x}\,(m))\{\theta_0 - E[Y_i|\underset{\sim}{X}_i\,(m) = \underset{\sim}{x}_i\,(m)]\} \prod_{j=1}^{k} f_j(\underset{\sim}{x}_j\,(m)) d\underset{\sim}{x}(m) + D_H. \end{aligned}$$

Here $y_i = \sum_{j=1}^{M} x_{ij}, D_H = \int (y_i - \theta_0)\, I\,(y_i - \theta_0)\ \ f_i(\underset{\sim}{x}_i) d\underset{\sim}{x}_i$, which depends on the prior distribution H but is independent of i (since $f_1 = \cdots = f_k$), and $E[Y_i|\underset{\sim}{X}_i\,(m) = \underset{\sim}{x}_i\,(m)]$ is the marginal posterior mean of Y_i given $\underset{\sim}{X}_i\,(m) = \underset{\sim}{x}_i\,(m)$.

From (2.4), a Bayes inspection procedure $\underset{\sim}{\delta}^H = (\delta_1^H, \ldots, \delta_k^H)$ is clearly given by: for each $\underset{\sim}{x}(m)$ in χ, and each $i = 1, \ldots, k$,

$$\delta_i^H(\underset{\sim}{x}\,(m)) = \begin{cases} 1, & \text{if } \theta_0 \le E[Y_i|\underset{\sim}{X}_i\,(m) = \underset{\sim}{x}_i\,(m)], \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

From (2.5), we see that the component inspection procedure δ_i^H depends on $\underset{\sim}{x}\,(m)$ only through $\underset{\sim}{x}_i(m)$ and is independent of $\underset{\sim}{x}_j\,(m)$, for $j \neq i$.

Let $V_i = \sum_{j=1}^{m} X_{ij}$. Conditioning on θ_i, V_i has a probability density $g_i\,(v_i|\theta_i) = \frac{u(v_i)}{\theta_i^m} \exp(-\frac{v_i}{\theta_i})$, where $u\,(v_i) = v_i^{m-1}/\Gamma\,(m)$. Marginally, V_i has a probability density $g_i\,(v_i) = \displaystyle\int g_i\,(v_i|\theta_i)\, dH\,(\theta_i)$. Note that $Y_i = \sum_{j=1}^{M} X_{ij}$. A straightforward computation yields

$$\begin{aligned} E[Y_i|\underset{\sim}{X}_i\,(m) = \underset{\sim}{x}_i\,(m)] &= \sum_{j=1}^{m} x_{ij} + \sum_{\ell=m+1}^{M} E[X_{i\ell}|\underset{\sim}{X}_i\,(m) = \underset{\sim}{x}_i\,(m)] \\ &= v_i + (M - m)\, \psi_i\,(v_i)\, / g_i\,(v_i)\,, \end{aligned} \tag{2.6}$$

where $v_i = \sum_{j=1}^{m} x_{ij}$ and $\psi_i\,(v_i) = \displaystyle\int_0^\infty \frac{u(v_i)}{\theta^{m-1}} \exp(-\frac{v_i}{\theta}) dH\,(\theta)$.

Let $\alpha\ (v_i) = v_i - \theta_0 + (M-m)\,\psi_i\,(v_i)\,/g_i\,(v_i)$ and $W\,(v_i) = (v_i-\theta_0)\,g_i\,(v_i) + (M-m)\,\psi_i\,(v_i)$. From (2.5) - (2.6), the Bayes inspsection procedure $\underset{\sim}{\delta}^H = (\delta_1^H,\ldots,\delta_k^H)$ can be written as: for each $i=1,\ldots,k$, and each $\underset{\sim}{x}(m)$ in χ,

$$\delta_i^H(\underset{\sim}{x}(m)) = \begin{cases} 1, & \text{if } \alpha\,(v_i)\geq 0,\\ 0, & \text{otherwise;}\end{cases}$$
$$= \begin{cases} 1, & \text{if } (v_i-\theta_0\geq 0) \text{ or } (v_i-\theta_0<0 \text{ and } W\,(v_i)\geq 0)\,,\\ 0, & \text{otherwise.}\end{cases} \tag{2.7}$$

We see that δ_i^H depends on $\underset{\sim}{x}(m)$ only through $v_i = \sum_{j=1}^m x_{ij}$.

Let $A = \{v>0|\alpha\,(v)\geq 0\}$. Note that $\alpha\,(v) = v-\theta_0+(M-m)\,\psi_i\,(v)\,/g_i\,(v)$ is continuous and strictly increasing in v for $v>0$. Also, $\alpha\,(\theta_0) = (M-m)\,\psi_i\,(\theta_0)\,/ g_i\,(\theta_0) > 0$. So θ_0 is in A, and hence A is not an empty set. Define $a_H = \inf A$. Then $0\leq a_H<\theta_0$. If $a_H>0$ then $\alpha\,(a_H)=0$. If $a_H=0$, then either $\alpha\,(0)=0$ or $\alpha\,(0)>0$. Note that a_H can be viewed as the critical point of the Bayes inspection procedure $\underset{\sim}{\delta}^H$. In terms of a_H, the Bayes inspection procedure $\underset{\sim}{\delta}^H$ can be expressed as: for each $i=1,\ldots,k$, and $\underset{\sim}{v} = (v_1,\ldots,v_k)$,

$$\delta_i^H(\underset{\sim}{x}\,(m)) \equiv \delta_i^H(\underset{\sim}{v}) = \delta_i^H\,(v_i) = \begin{cases} 1, & \text{if } v_i\geq a_H,\\ 0, & \text{otherwise.}\end{cases} \tag{2.8}$$

The minimum Bayes risk $R(H,\underset{\sim}{\delta}^H)$ is

$$R(H,\underset{\sim}{\delta}^H) = \sum_{i=1}^k R_i(H,\delta_i^H), \tag{2.9}$$

where

$$R_i(H,\delta_i^H) = \int_0^\infty \delta_i^H\,(v_i)\,[-\alpha\,(v_i)]\,g_i\,(v_i)\,dv_i + D_H$$
$$= \int_0^\infty \delta_i^H\,(v_i)\,[-W\,(v_i)]\,dv_i + D_H. \tag{2.10}$$

3. An Empirical Bayes Simultaneous Inspection Procedure

It can be seen that the Bayes inspection procedure $\underset{\sim}{\delta}^H$ depends on the prior distribution H. When H is unknown, it is not possible to implement $\underset{\sim}{\delta}^H$. However, according to the model described previously, the $\underset{\sim}{X}_i\,(m)$, $i=1,\ldots,k$, are marginally identically distributed, and mutually independent. Therefore, the empirical Bayes approach is employed to combine information from the k observations $\underset{\sim}{X}_i\,(m)$, $i=1,\ldots,k$, to construct robust inspection procedures for each of the k variable sampling acceptance problems.

The proposed empirical Bayes inspection procedure resembles $\underset{\sim}{\delta}^H$. For this, (2.7) provides important motivation for the construction. To construct an empirical Bayes simultaneous inspection procedure, we need to have estimates for $g_i(v_i)$ and $\psi_i(v_i)$.

Let $K(t) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell t^\ell}{\ell!(\ell+1)!} I(t)$ (the kernel $K(t)$ has been used by Pensky and Singh (1995) for some empirical Bayes estimation problem). For each $i = 1, \ldots, k$, define

$$\begin{cases} \psi_{ik}(V_i) = \frac{1}{k-1} \sum_{\substack{j \neq i \\ j=1}}^{k} \frac{u(V_i) I(V_j - V_i)}{u(V_j)}, \\ g_{ik}(V_i) = \frac{1}{(k-1)h(k,V_i)} \sum_{\substack{j \neq i \\ j=1}}^{k} \frac{u(V_i) I(V_j - V_i)}{u(V_j)} K\left(\frac{V_j - V_i}{h(k,V_i)}\right), \end{cases} \tag{3.1}$$

where $h = h(k,v) = v/(\ln k)^2$. In the following, for convenience, we use h instead of $h(k,v)$. We show that $\psi_{ik}(v)$ is an unbiased, consistent estimator of $\psi_i(v)$. Also, by choice of $h = v/(\ln\ k)^2$, we can show that $g_{ik}(v)$ is an asymptotically unbiased, consistent estimator of $g_i(v)$, having bias converging to zero at a rate of order $O(k^{-1})$. A straightforward computation leads to

$$\begin{cases} E_i[\psi_{ik}(v_i)] = \psi_i(v_i), \\ E_i[g_{ik}(v_i)] = g_i(v_i) - \int \frac{u(v_i)}{\theta^m} \exp(-\frac{v_i}{\theta}) \exp(-\frac{\theta}{h}) dH(\theta) < g_i(v_i), \end{cases} \tag{3.2}$$

where the expectation E_i is with respect to the probability measure generated by $\underset{\sim}{V}(i) = (V_1, \ldots, V_{i-1}, V_{i+1}, \ldots, V_k)$, $i = 1, \ldots, k$; the bias $0 < B_H(v_i, k) = \int \frac{u(v_i)}{\theta^m} \exp(-\frac{v_i}{\theta}) \exp(-\frac{\theta}{h}) dH(\theta) < \frac{u(v_i)}{k} \int_0^\infty \theta^{-m} dH(\theta)$ (see (6.5)) and $\to 0$ as $k \to \infty$. By noting that the function $u(v) = v^{m-1}/\Gamma(m)$ is increasing in v, and that $\int_0^\infty K^2(t)\,dt = \frac{1}{2}$, (see Pensky and Singh (1995)), we find

$$\begin{cases} \mathrm{Var}\,(u(v_i) I(V_j - v_i)/u(V_j)) \le \psi_i(v_i), \\ \mathrm{Var}\,(u(v_i) I(V_j - v_i) K((V_j - v_i)/h)/u(V_j)) \le \frac{h g_i(v_i)}{2}. \end{cases} \tag{3.3}$$

Therefore,

$$\begin{cases} \mathrm{Var}\ (\psi_{ik}(v_i)) \le \psi_i(v_i) / (k-1), \\ \mathrm{Var}\ (g_{ik}(v_i)) \le g_i(v_i) / [2(k-1)h]. \end{cases} \tag{3.4}$$

Hence, $\psi_{ik}(v_i)$ and $g_{ik}(v_i)$ are consistent estimators of $\psi_i(v_i)$ and $g_i(v_i)$, respectively. Define

$$W_{ik}(v_i) = (v_i - \theta_0) g_{ik}(v_i) + (M - m) \psi_{ik}(v_i). \tag{3.5}$$

We have

$$\begin{cases} E_i[W_{ik}(v_i)] = W(v_i) + (\theta_0 - v_i) B_H(v_i, k), \\ \text{Var } (W_{ik}(v_i)) \le \frac{(\theta_0 - v_i)^2 g_i(v_i)}{(k-1)h} + \frac{2(M-m)^2 \psi_i(v_i)}{k-1}. \end{cases} \tag{3.6}$$

Thus, $W_{ik}(v_i)$ is a consistent estimator of $W(v_i)$. Based on $W_{ik}(v_i)$, $i = 1, \ldots, k$, and akin to $\underset{\sim}{\delta}^H$ at (2.7), we propose an empirical Bayes simultaneous inspection procedure $\underset{\sim}{\delta}^* = (\delta_1^*, \ldots, \delta_k^*)$ as follows: for each $i = 1, \ldots, k$, define

$$\delta_i^*(\underset{\sim}{v}) = \delta_i^*(v_i, \underset{\sim}{v}(i)) = \begin{cases} 1, & \text{if } (v_i \ge \theta_0) \text{ or } (v_i < \theta_0 \text{ and } W_{ik}(v_i) \ge 0), \\ 0, & \text{otherwise.} \end{cases} \tag{3.7}$$

The Bayes risk of $\underset{\sim}{\delta}^*$ is

$$R(H, \underset{\sim}{\delta}^*) = \sum_{i=1}^{k} R_i(H, \delta_i^*), \tag{3.8}$$

where

$$R_i(H, \delta_i^*) = \int_0^\infty E_i[\delta_i^*(v_i, \underset{\sim}{V}(i))] [-\alpha(v_i)] g_i(v_i) dv_i + D_H. \tag{3.9}$$

Since $\underset{\sim}{\delta}^H$ is the Bayes inspection procedure, for any inspection procedure $\underset{\sim}{\delta} = (\delta_1, \ldots, \delta_k)$, $R_i(H, \delta_i) - R_i(H, \delta_i^H) \ge 0$ for each $i = 1, \ldots, k$, and therefore $R(H, \underset{\sim}{\delta}) - R(H, \underset{\sim}{\delta}^H) \ge 0$. Define $\rho(H, \underset{\sim}{\delta}) = [R(H, \underset{\sim}{\delta}) - R(H, \underset{\sim}{\delta}^H)]/R(H, \underset{\sim}{\delta}^H)$. Then $\rho(H, \underset{\sim}{\delta})$ is called the relative regret Bayes risk of $\underset{\sim}{\delta}$.

An inspection procedure $\underset{\sim}{\delta}$ is said to be asymptotically optimal relative to H, if $\rho(H, \underset{\sim}{\delta}) \to 0$ as $k \to \infty$. An inspection procedure $\underset{\sim}{\delta}$ is said to be asymptotically optimal, relative to H, at a rate of order $O(\beta_k)$, if $\rho(H, \underset{\sim}{\delta}) = O(\beta_k)$, where $\{\beta_k\}$ is a sequence of decreasing, positive numbers such that $\lim_{k \to \infty} \beta_k = 0$.

In the following sections, we investigate the asymptotic optimality and rate of convergence of the empirical Bayes simultaneous inspection procedure $\underset{\sim}{\delta}^*$.

Before ending this section, we provide a numerical example to illustrate the implementation of $\underset{\sim}{\delta}^*$.

An illustrative numerical example

Suppose that $k = 10$ shipments, each consisting of $M = 30$ components, are presented for acceptance sampling. Let Y_i denote the total lifetime of the M items of shipment π_i. Shipment π_i is accepied if $Y_i \ge \theta_0 = 35; \pi_i$ is rejected if $Y_i < \theta_0$. In order to implement the empirical Bayes simultaneous inspection procedure $\underset{\sim}{\delta}^*$, a sample of five components are selected at random from each π_i and put on life test. We denote the total lifetime of the five sampled item from π_i by v_i. The following data are observed.

i	1	2	3	4	5	6	7	8	9	10
v_i	2.4	3.2	4.0	4.8	7.8	12.5	18.2	24.3	28.1	32.4

Using (3.1), (3.5), and (3.7), the values of $g_{ik}(v_i), \psi_{ik}(v_i), W_{ik}(v_i)$ and $\delta_i^*(\underset{\sim}{v})$ are computed and tabulated in the following. Note that $\delta_i^*(\underset{\sim}{v}) = 1$ means that π_i is accepted while $\delta_j^*(\underset{\sim}{v}) = 0$ means that π_j is rejected. According to the numerical results, shipments π_1, π_2 and π_3 are rejected and the others are accepted.

Numerical result for the procedure $\underset{\sim}{\delta}^*$ based on $(v_1, \ldots, v_{10})$ with $k = 10, M = 30, m = 5$ and $\theta_0 = 35$.

i	$g_{ik}(v_i)$	$psi_{ik}(v_i)$	$W_{ik}(v_i)$	$\delta_i^*(\underset{\sim}{v})$
1	0.05220	0.05770	-0.25920	0
2	0.09756	0.07125	-1.32108	0
3	0.05965	0.06285	-0.27800	0
4	0.00031	0.01921	0.47076	1
5	-0.00030	0.02281	0.57834	1
6	-0.00090	0.03932	1.00315	1
7	0.00963	0.06558	1.47772	1
8	0.01964	0.09729	2.22220	1
9	0.01174	0.06286	1.49059	1
10	0.00000	0.00000	0.00000	1

4. Asymptotic Optimality and Rate of Convergence

4.1. Asymptotic Optimality of $\underset{\sim}{\delta}^*$

Under the model described previously, one can see that for the Bayes inspection procedure $\underset{\sim}{\delta}^H, R_1(H, \delta_1^H) = \cdots = R_k(H, \delta_k^H)$. Also, by the symmetric property of the empirical Bayes simultaneous inspection procedure $\underset{\sim}{\delta}^*$, we have $R_1(H, \delta_1^*) = \cdots = R_k(H, \delta_k^*)$. Therefore, $\rho(H, \delta^*) = D_1(H, \delta_1^*)/R_1(H, \delta_1^H)$ where $D_1(H_1, \delta_1^*) = R_1(H, \delta_1^*) - R_1\left(H, \delta_1^H\right)$. Note that $R_1(H, \delta_1^H)$ is a constant, independent of the number of shipments k. So, to study the asymptotic optimality of $\underset{\sim}{\delta}^*$, it suffices to investigate the asymptotic behavior of the regret Bayes risk $D_1(H, \delta_1^*)$ for sufficiently large k. From (2.8), (2.10), (3.7) and (3.9), the regret Bayes risk $D_1(H, \delta_1^*)$ can be expressed as

$$D_1(H, \delta_1^*) = \int_0^{a_H} P\{W_{1k}(v) \geq 0\}[-W(v)]dv + \int_{a_H}^{\theta_0} P\{W_{1k}(v) < 0\} W(v)\, dv. \tag{4.1}$$

Note that $\int_0^{\theta_0} |W(v)|\, dv < \infty$. Therefore, from Corollary 2 of Robbins (1964), to show the asymptotic optimality of δ_1^* it suffices to show that $P\{W_{1k}(v) \geq 0\} \to 0$ for each v in $(0, a_H)$ and $P\{W_{1k}(v) < 0\} \to 0$ for each v in (a_H, θ_0). By

Markov's inequality, for v in $(0, a_H)$,

$$P\{W_{1k}(v)\geq 0\}=P\{W_{1k}(v)-W(v)\geq -W(v)\}\leq E_1[W_{1k}(v)-W(v)]^2/[W(v)]^2.$$

From (3.6),

$$\begin{aligned}&E_1[W_{1k}(v)-W(v)]^2\\&=Var(W_{1k}(v))+[E_1W_{1k}(v)-W(v)]^2\\&\leq\frac{(\theta_0-v)^2g_1(v)}{(k-1)h}+\frac{2(M-m)^2\psi_1(v)}{k-1}+(\theta_0-v)^2[B_H(v,k)]^2,\end{aligned}$$

which tends to 0 as $k\to\infty$. Therefore, for each v in $(0,a_H)$, $P\{W_{1k}(v)\geq 0\}\to 0$ as $k\to\infty$. Similarly, for each v in (a_H,θ_0), we can obtain that $P\{W_{1k}(v)<0\}\to 0$ as $k\to\infty$.

The preceding result is summarized as a theorem.

Theorem 4.1. *Let $\underset{\sim}{\delta}^*$ be the empirical Bayes simultaneous inspection procedure constructed in Section 3. Assume that $\int_0^\infty\theta dH(\theta)<\infty$. Then, $\underset{\sim}{\delta}^*$ is asymptotically optimal in the sense that $\rho(H,\underset{\sim}{\delta}^*)\to 0$ as $k\to\infty$.*

4.2. Rate of convergence

We investigate the rate of convergence of the empirical Bayes simultaneous inspection procedure $\underset{\sim}{\delta}^*$by establishing an upper bound on the regret Bayes risk $D_1(H,\delta_1^*)$. Note that as $a_H=0$, the first term in the RHS of (4.1) equals zero. Without loss of generality, in the following, we assume that $a_H>0$.

Since $\alpha(v)$ is strictly increasing in v and $\alpha(a_H)=0$, for sufficiently large k, there exists a point $a_H(k)$ between $a_H/2$ and a_H such that

$$\alpha(a_H(k))=-\ln^2 k/k. \tag{4.2}$$

The regret Bayes risk $D_1(H,\delta_1^*)$ can be expressed as

$$D_1(H,\delta_1^*)=I_k+II_k+III_k+IV_k, \tag{4.3}$$

where

$$I_k=\int_0^{a_H/2}P\{W_{1k}(v)\geq 0\}[-W(v)]dv, \tag{4.4}$$

$$II_k=\int_{a_H/2}^{a_H(k)}P\{W_{1k}(v)\geq 0\}[-W(v)]dv, \tag{4.5}$$

$$III_k=\int_{a_H(k)}^{a_H}P\{W_{1k}(v)\geq 0\}[-W(v)]dv, \tag{4.6}$$

$$IV_k=\int_{a_H}^{\theta_0}P\{W_{1k}(v)<0\}W(v)dv. \tag{4.7}$$

To investigate the asymptotic behavior of the regret Bayes risk $D_1(H, \delta_1^*)$, it suffices to investigate the asymptotic behavior of each of the terms (4.4) - (4.7). Let $C_H(m, v) = \int_0^\infty \frac{1}{\theta^m} \exp(-v/\theta) dH(\theta)$. We have the following lemmas (proofs are provided in Section 6).

Lemma 4.1. *Assume that* $C_H(m, 0) < \infty$. *Then*

$$I_k \leq \frac{8\theta_0 C_H(m,0)\,[5\theta_0^2 (\ln k)^2 + 3a_H M^2 \beta(\frac{a_H}{2})]}{3m(k-1)\, C_H(m, \frac{a_H}{2})\alpha^2(\frac{a_H}{2})} = O(\frac{(\ln k)^2}{k}),$$

where $\beta(v) = \frac{\psi_1(v)}{g_1(v)}$.

Lemma 4.2. *Assume that* $C_H(m, 0) < \infty$. *Then*

$$II_k \leq \frac{4\tau_2\,[5\theta_0^2 + 6h(k, a_H)\, M^2 \beta(a_H)]}{3(k-1)\, h(k, \frac{a_H}{2}) d_2} = O(\frac{(\ln k)^2}{k}),$$

where $d_2 = \min_{a_H/2 \leq v \leq a_H} g_1(v) > 0$ *and* $0 < \tau_2 = \max_{a_H/2 \leq v \leq a_H} \frac{g_1(v)}{\alpha^{(1)}(v)} < \infty$.

Lemma 4.3. $III_k \leq \frac{(\ln k)^2}{k}$.

Lemma 4.4. $IV_k \leq \frac{\tau_4\left[3\theta_0^2 + 6h(k,\theta_0)M^2\beta(\theta_0) + 4\theta_0\alpha(\theta_0)\right]}{3(k-1)h(k,a_H)d_4} = O(\frac{(\ln k)^2}{k})$, *where* $d_4 = \min_{a_H \leq v \leq \theta_0} g_1(v) > 0$ *and* $0 < \tau_4 = \max_{a_H \leq v \leq \theta_0} \frac{g_1(v)}{\alpha^{(1)}(v)} < \infty$.

From (4.3) and Lemmas 4.1 - 4.4, we can establish an upper bound for the convergence rate of the regret Bayes risk $D_1(H, \delta_1^*)$. We summarize this main result as follows.

Theorem 4.2. *Suppose the prior distribution* H *is such that* (*a*) $\int_0^\infty \theta dH(\theta) < \infty$, *and* (*b*) $\int_0^\infty \frac{1}{\theta^m} dH(\theta) < \infty$. *Then the empirical Bayes simultaneous inspection procedure* $\underset{\sim}{\delta}^*$ *is asymptotically optimal,* $D_1(H, \delta_1^*) = O(\frac{(\ln k)^2}{k})$, *and* $\rho(H, \underset{\sim}{\delta}^*) = O(\frac{(\ln k)^2}{k})$.

5. A Lower Bound for $D_1(H, \delta_1^*)$

In the following, we establish a lower bound on the convergence rate of $D_1(H, \delta_1^*)$. More precisely, we provide a lower bound for $II_k + III_k$ where $II_k + III_k = \int_{a_H/2}^{a_H} P\{W_{1k}(v) \geq 0\}[-W(v)]\, dv$. Since $D_1(H, \delta_1^*) \geq II_k + III_k$, the lower bound of $II_k + III_k$ is also a lower bound of $D_1(H, \delta_1^*)$.

For each v in $[a_H/2, a_H], W(v) = \alpha(v) g_1(v) < 0$. From (6.6), $E_1[W_{1k}(v)] > W(v)$. From (6.1) - (6.2) and for sufficiently large k, it follows from Lemma 3, p. 47, of Lamperti (1966) that for all $\xi > 0$,

$$P\{W_{1k}(v) \geq 0\}$$

$$\begin{aligned}
&= P\left\{W_{1k}(v) - E_1[W_{1k}(v)] \geq -E_1[W_{1k}(v)]\right\} \\
&\geq P\left\{W_{1k}(v) - E_1[W_{1k}(v)] \geq -W(v)\right\} \\
&= P\left\{\frac{\sqrt{k-1}[W_{1k}(v) - E_1[W_{1k}(v)]]}{\sqrt{Var\left(Q_2(v)\right)}} \geq \frac{-\sqrt{k-1}W(v)}{Var\left(Q_2(v)\right)}\right\} \\
&\geq \exp\left\{-\frac{(k-1)W^2(v)(1+\xi)}{2Var\left(Q_2(v)\right)}\right\},
\end{aligned} \tag{5.1}$$

where $Q_2(v) = \frac{(v-\theta_0)}{h}\frac{u(v)I(V_2-v)}{u(V_2)}K\left(\frac{V_2-v}{h}\right)+(M-m)\frac{u(v)I(V_2-v)}{u(V_2)}$, see (6.1) - (6.2). Note that Var $(Q_2(v)) \geq E_H\left[\text{Var }(Q_2(v)\,|\theta)\right]$ and Var $(Q_2(v)\,|\theta) = E\left[Q_2^2(v)\,|\theta\right]$ $-\left(E\left[Q_2(v)\,|\theta\right]\right)^2$. Straightforward computations show that

$$\begin{aligned}
E\left[Q_2(v)\,|\theta\right] &= \frac{(v-\theta_0)\,u(v)\,e^{-v/\theta}}{\theta^m} \times [1-\exp(-\theta/h)] + \frac{(M-m)\,u(v)\,\theta}{\theta^m}e^{-v/\theta}; \\
E\left[Q_2^2(v)\,|\theta\right] &= \frac{(\theta_0-v)^2\,u^2(v)e^{-v/\theta}}{h\theta^m}\int_0^\infty \frac{1}{u(v+hy)}e^{-hy/\theta}K^2(y)\,dy \\
&\quad + \frac{h\,(M-m)^2\,u^2(v)e^{-v/\theta}}{\theta^m}\int_0^\infty \frac{1}{u(v+hy)}e^{-hy/\theta}dy \\
&\quad - \frac{2\,(M-m)\,(\theta_0-v)\,u^2(v)}{\theta^m}e^{-v/\theta}\int_0^\infty \frac{1}{u(v+hy)}e^{-hy/\theta}K(y)\,dy.
\end{aligned}$$

Hence, $E\left[Q_2^2(v)\,|\theta\right] - \left(E\left[Q_2(v)\,|\theta\right]\right)^2 = \frac{u^2(v)}{\theta^m}e^{-v/\theta}C(\theta,v,h)$, where

$$\begin{aligned}
C(\theta,v,h) =& \frac{(\theta_0-v)^2}{h}\int_0^\infty \frac{1}{u(v+hy)}e^{-hy/\theta}K^2(y)\,dy + h\int_0^\infty \frac{1}{u(v+hy)}e^{-hy/\theta}dy \\
&-2\,(M-m)\,(\theta_0-v)\int_0^\infty \frac{1}{u(v+hy)}e^{-hy/\theta}K(y)\,dy \\
&-\frac{(\theta_0-v)^2\,e^{-v/\theta}}{\theta^m}\left[1-e^{-\theta/h}\right]^2 - \frac{(M-m)^2\,\theta^2}{\theta^m}e^{-v/\theta} \\
&+\frac{2\,(\theta_0-v)\,(M-m)\,\theta e^{-v/\theta}}{\theta^m}\left[1-e^{-\theta/h}\right].
\end{aligned}$$

Let $0 < \theta_1 < \theta_2 < \infty$ be two finite points such that $H(\theta_1) < H(\theta_2)$. Note that $a_H/2 \leq v \leq a_H$. For each θ in $[\theta_1, \theta_2]$ and sufficiently large k,

$$\begin{aligned}
C(\theta,v,h) &\geq \frac{(\theta_0-v)^2}{2h}\int_0^\infty \frac{1}{u(v+hy)}e^{-hy/\theta}K^2(y)\,dy, \\
&\geq \frac{(\theta_0-v)^2}{2h}\int_0^\infty \frac{1}{u(v+hy)}e^{-hy/\theta_1}K^2(y)\,dy,
\end{aligned}$$

and therefore, Var $(Q_2(v)\,|\theta) \geq \frac{u^2(v)}{\theta^m}e^{-v/\theta}\frac{(\theta_0-v)^2}{2h}\int_0^\infty \frac{1}{u(v+hy)}e^{-hy/\theta_1}K^2(y)\,dy$.

Hence,

$$\begin{aligned}
&\text{Var}\left(Q_2\left(v\right)\right)\\
&\geq E_H\left[Var\left(Q_2\left(v\right)|\theta\right)\right]\\
&\geq \frac{\left(\theta_0-v\right)^2 u^2\left(v\right)}{2}\int_{\theta_1}^{\theta_2}\frac{e^{-v/\theta}}{\theta^m}dH\left(\theta\right)\times\frac{1}{h}\int_0^\infty\frac{1}{u\left(v+hy\right)}e^{-hy/\theta_1}K^2\left(y\right)dy\\
&\geq \frac{\left(\theta_0-a_H\right)^2 u^2\left(a_H/2\right)}{2}\int_{\theta_1}^{\theta_2}\frac{e^{-a_H/\theta}}{\theta^m}dH\left(\theta\right)\times\frac{1}{h}\int_0^\infty\frac{1}{u(v+hy)}e^{-hy/\theta_1}K^2\left(y\right)dy\\
&=\frac{C^*e\left(h\right)}{h},
\end{aligned}\tag{5.2}$$

where $C^*=\frac{(\theta_0-a_H)^2u^2(a_H/2)}{2}\int_{\theta_1}^{\theta_2}\frac{e^{-a_H/\theta}}{\theta^m}dH\left(\theta\right)>0$, and

$$e\left(h\right)=\int_0^\infty\frac{1}{u\left(v+hy\right)}e^{-hy/\theta_1}K^2\left(y\right)dy\rightarrow e\left(0\right)\equiv\int_0^\infty\frac{1}{u\left(v\right)}K^2\left(y\right)dy>0 \text{ as } k\rightarrow\infty.$$

Let $b_1=\max_{a_H/2\leq v\leq a_H} g_1^2(v)$ and $b_2=\min_{a_H/2\leq v\leq a_H}\left[\frac{g_1(v)}{\alpha^{(1)}(v)}\right]$. Then $b_1<\infty$ and $b_2>0$. Combining (5.1) and (5.2) and plugging the inequality into II_k+III_k, since $\alpha(a_H)=0$, we obtain

$$\begin{aligned}
&D_1\left(H,\delta_1^*\right)\\
&\geq\int_{a_H/2}^{a_H}P\left\{W_{1k}\left(v\right)\geq 0\right\}\left[-W\left(v\right)\right]dv\\
&\geq\int_{a_H/2}^{a_H}\exp\left\{-\frac{\left(k-1\right)h\left(1+\xi\right)W^2\left(v\right)}{2C^*e\left(h\right)}\right\}\left[-W\left(v\right)\right]dv\\
&=\int_{a_H/2}^{a_H}\exp\left\{-\frac{\left(k-1\right)h\left(k,v\right)\left(1+\xi\right)g_1^2\left(v\right)\alpha^2\left(v\right)}{2C^*e\left(h\right)}\right\}\left[-\alpha\left(v\right)\right]g_1\left(v\right)dv\\
&\geq\int_{a_H/2}^{a_H}\exp\left\{-\frac{\left(k-1\right)h\left(k,a_H\right)\left(1+\xi\right)b_1\alpha^2\left(v\right)}{2C^*e\left(h\right)}\right\}\left[-\alpha\left(v\right)\right]\alpha^{(1)}\left(v\right)\left[\frac{g_1\left(v\right)}{\alpha^{(1)}\left(v\right)}\right]dv\\
&\geq b_2\int_{a_H/2}^{a_H}\exp\left\{-\frac{\left(k-1\right)h\left(k,a_H\right)\left(1+\xi\right)b_1\alpha^2\left(v\right)}{2C^*e\left(h\right)}\right\}\left[-\alpha\left(v\right)\right]\alpha^{(1)}\left(v\right)dv\\
&\geq\frac{b_2C^*e\left(h\right)}{\left(k-1\right)h\left(k,a_H\right)\left(1+\xi\right)b_1}.
\end{aligned}$$

Hence, we have the following theorem.

Theorem 5.1. *Suppose that the distribution H is such that $H\left(\theta_1\right)<H\left(\theta_2\right)$ for some $0<\theta_1<\theta_2<\infty$. Then for any $\xi>0$, the following hold.*

(a) $D_1(H,\delta_1^*)\geq\frac{b_2C^*e(h)}{(k-1)h(k,a_H)(1+\xi)b_1}$;

(b) $\rho(H,\underset{\sim}{\delta}^*)\geq\frac{b_2C^*e(h)}{(k-1)h(k,a_H)(1+\xi)b_1R_1(H,\delta_1^H)}$.

Theorem 5.1 provides a lower bound of order $O((\ln k)^2/k)$ for the convergence rate of $\rho(H, \underset{\sim}{\delta}^*)$ and Theorem 4.2 gives an upper bound of the same order. Hence, the simultaneous inspection procedure $\underset{\sim}{\delta}^*$ is asymptotically optimal at a convergence rate of order $O((\ln k)^2/k)$.

6. Auxiliary Results

Let $C_H(m,v) = \int_0^\infty \frac{1}{\theta^m} e^{-v/\theta} dH(\theta)$. In this section, the analysis is made for sufficiently large k and under the assumption that $C_H(m,0) < \infty$. To investigate the asymptotic behavior of the four terms given in (4.4)-(4.7), we first study certain properties related to $W_{1k}(v)$.

Note that

$$W_{1k}(v) = \frac{1}{k-1} \sum_{j=2}^{k} Q_j(v), \tag{6.1}$$

where

$$Q_j(v) = \frac{(v-\theta_0)}{h} \frac{u(v) I(V_j - v)}{u(V_j)} K(\frac{V_j - v}{h}) + (M-m) \frac{u(v) I(V_j - v)}{u(V_j)}. \tag{6.2}$$

Since $0 \le \frac{u(v)I(V_j-v)}{u(V_j)} \le 1$ and $|K(t)| \le 1$, for sufficiently large k,

$$|Q_j(v)| \le \frac{2\theta_0}{h} \text{ and } |Q_j(v) - E_1[Q_j(v)]| \le \frac{4\theta_0}{h}. \tag{6.3}$$

Also, from (3.3),

$$\begin{aligned} \text{Var}(Q_j(v)) &\le 2\text{Var}\left(\frac{(v-\theta_0)}{h} \frac{u(v) I(V_j - v)}{u(V_j)} K(\frac{V_j - v}{h}) \right) \\ &\quad + 2\text{Var}\left((M-m) \frac{u(v) I(V_j - v)}{u(V_j)}\right) \\ &\le \frac{\theta_0^2 g_i(v)}{h} + 2M^2 \psi_1(v). \end{aligned} \tag{6.4}$$

From (3.6), $E_1[W_{1k}(v)] = W(v) + (\theta_0 - v) B_H(v,k)$, where

$$\begin{aligned} 0 < B_H(v,k) &= \int_0^\infty u(v) \exp(-v/\theta) \exp(-\theta/h)/\theta^m dH(\theta) \\ &\le \frac{u(v)}{k} \int_0^\infty \theta^{-m} dH(\theta) = \frac{u(v)}{k} C_H(m,0). \end{aligned} \tag{6.5}$$

Therefore,

$$E_1[W_{1k}(v)] > W(v) \text{ for all } v. \tag{6.6}$$

Since $\alpha(v)$ is strictly increasing in v and by the definition of $a_H(k)$, for $0 < v < a_H(k)$,

$$\alpha(v) \leq \alpha(a_H(k)) = -\ln^2 k/k. \tag{6.7}$$

Thus, for sufficiently large k and $0 < v < a_H(k)$, from (6.5) and (6.7),

$$\begin{aligned}\alpha(v) + (\theta_0 - v)\frac{B_H(v,k)}{u(v)\,C_H(m,v)} &\leq \frac{\alpha(v)}{2} + \frac{\alpha(v)}{2} + \frac{(\theta_0 - v)\,C_H(m,0)}{kC_H(m,v)} \\ &\leq \frac{\alpha(v)}{2} - \frac{\ln^2 k}{k} + \frac{(\theta_0 - v)\,C_H(m,0)}{kC_H(m,v)} \leq \frac{\alpha(v)}{2}.\end{aligned}$$

Therefore, for $0 < v < a_H(k)$, we obtain

$$\begin{aligned}E_1[W_{1k}(v)] &= g_1(v)\,[\alpha(v) + (\theta_0 - v)\frac{B_H(v,k)}{u(v)\,C_H(m,v)}] \leq g_1(v)\,\alpha(v)/2 \\ &= W(v)/2 < 0.\end{aligned} \tag{6.8}$$

Let $\beta(v) = \psi_1(v)/g_1(v)$. Then $\alpha(v) = v - \theta_0 + \beta(v)$. Note that both $\beta(v)$ and $\alpha(v)$ are nondecreasing and differentiable in v. Hence, $\alpha^{(1)}(v) = 1 + \beta^{(1)}(v) \geq 1$ for all v.

Proof of Lemma 4.1. For $0 < v < a_H/2$,

$$\frac{v^{m-1}}{\Gamma(m)}C_H(m, a_H/2) \leq g_1(v) \leq \frac{v^{m-1}}{\Gamma(m)}C_H(m,0) \text{ and } -\theta_0 < \alpha(v) < 0. \tag{6.9}$$

From (6.1) - (6.4), (6.8) - (6.9) and the Bernstein inequality. (see Shorack and Wellner (1986, p.855)), for $0 < v < a_H/2$,

$$\begin{aligned}P\{W_{1k}(v) \geq 0\} &\leq P\left\{W_{1k}(v) - E_1[W_{1k}(v)] \geq -\frac{1}{2}W(v)\right\} \\ &\leq \exp\left\{-\frac{(k-1)\left[\frac{1}{2}W(v)\right]^2/2}{Var(Q_2(v)) + \frac{4\theta_0}{3h} \times \left|\frac{W(v)}{2}\right|}\right\} \\ &= \exp\left\{-\frac{3}{8}\frac{(k-1)\,hg_1(v)\,\alpha^2(v)}{3\theta_0^2 + 6hM^2\beta(v) + 2\theta_0\,|\alpha(v)|}\right\} \\ &\leq \exp\left\{-\frac{3}{8}\frac{(k-1)\frac{v}{(\ln k)^2}\frac{v^{m-1}}{\Gamma(m)}C_H\left(m,\frac{a_H}{2}\right)\alpha^2\left(\frac{a_H}{2}\right)}{3\theta_0^2 + \frac{6a_H}{2(\ln k)^2}M^2\beta\left(\frac{a_H}{2}\right) + 2\theta_0^2}\right\} \\ &= \exp\{-C_1(k)v^m\},\end{aligned} \tag{6.10}$$

where $C_1(k) = \frac{3(k-1)C_H\left(m,\frac{a_H}{2}\right)\alpha^2\left(\frac{a_H}{2}\right)}{8\Gamma(m)\left[5\theta_0^2(\ln k)^2 + 3a_H M^2\beta\left(\frac{a_H}{2}\right)\right]}$.

Plugging (6.10) into I_k and by (6.9), we obtain

$$I_k \le \int_0^{\frac{a_H}{2}} \exp\{-C_1(k)v^m\} \frac{\theta_0}{\Gamma(m)} C_H(m,0)\, v^{m-1} dv \le \frac{\theta_0 C_H(m,0)}{m\Gamma(m)\, C_1(k)}$$
$$= \frac{8\theta_0 C_H(m,0)\left[5\theta_0^2(\ln k)^2 + 3a_H M^2 \beta\left(\frac{a_H}{2}\right)\right]}{3m(k-1)\, C_H\left(m, \frac{a_H}{2}\right) \alpha^2\left(\frac{a_H}{2}\right)}.$$

Proof of Lemma 4.2. For $\frac{a_H}{2} < v < a_H(k)$, following a discussion analogous to (6.10), we obtain

$$P\{W_{1k}(v) \ge 0\} \le \exp\left\{-\frac{3}{8} \times \frac{(k-1)\, h g_1(v)\, \alpha^2(v)}{5\theta_0^2 + 6hM^2\beta(v)}\right\}.$$
$$\le \exp\left\{-\frac{3}{8} \times \frac{(k-1)\, h\left(k, \frac{a_H}{2}\right) d_2 \alpha^2(v)}{5\theta_0^2 + 6h(k, a_H)\, M^2 \beta(a_H)}\right\} = \exp\left\{-C_2(k)\, \alpha^2(v)\right\}, \tag{6.11}$$

where $C_2(k) = \frac{3(k-1)h(k,\frac{a_H}{2})d_2}{8\left[5\theta_0^2 + 6h(k,a_H)M^2\beta(a_H)\right]},$ and $d_2 = \min_{a_H/2 \le v \le a_H} g_1(v) \ge \frac{u(a_H/2)}{\Gamma(m)} C_H(m, a_H) > 0$. Let $\tau_2 = \max_{a_H/2 \le v \le a_H} \frac{g_1(v)}{\alpha^{(1)}(v)}$. Note that $\tau_2 \le \max_{a_H/2 \le v \le a_H} g_1(v) \le \frac{u(a_H)}{\Gamma(m)} C_H(m, a_H/2) < \infty$, since $\alpha^{(1)}(v) \ge 1$. Plugging (6.11) and the preceding inequality into II_k, we obtain

$$II_k \le \int_{a_H/2}^{a_H} \exp\left\{-C_2(k)\, \alpha^2(v)\right\} [\alpha(v)]\, \alpha^{(1)}(v) \left[\frac{g_1(v)}{\alpha^{(1)}(v)}\right] dv.$$
$$\le \tau_2 \int_{a_H/2}^{a_H} \exp\left\{-C_2(k)\, \alpha^2(v)\right\} [-\alpha(v)]\, \alpha^{(1)}(v)\, dv \le \frac{\tau_2}{2C_2(k)}$$
$$= \frac{4\tau_2\left[5\theta_0^2 + 6h(k, a_H)\, M^2\beta(a_H)\right]}{3(k-1)h\left(k, \frac{a_H}{2}\right) d_2}.$$

Proof of Lemma 4.3. According to the definitions of $a_H(k)$ and a_H, and by the increasing property of the function $\alpha(v)$ in v, for $a_H(k) \le v \le a_H, -\frac{(\ln k)^2}{k} = \alpha(a_H(k)) \le \alpha(v) \le \alpha(a_H) = 0$. Hence,

$$III_k \le \int_{a_H(k)}^{a_H} [-\alpha(v)] g_1(v) dv \le \left(\frac{(\ln k)^2}{k}\right).$$

Proof of Lemma 4.4. For $a_H < v < \theta_0, W(v) > 0$. From (6.6) and following a discussion analogous to (6.10), we have

$$P\{W_{1k}(v) < 0\} \le P\{W_{1k}(v) - E_1[W_{1k}(v)] < -W(v)\}$$

$$\leq \exp\left\{-\frac{3}{2}\times\frac{(k-1)hg_1(v)\,\alpha^2(v)}{3\theta_0^2+6hM^2\beta(v)+4\theta_0\alpha(v)}\right\}$$
$$\leq \exp\left\{-\frac{3}{2}\times\frac{(k-1)h(k,a_H)\,d_4\alpha^2(v)}{3\theta_0^2+6h(k,\theta_0)\,M^2\beta(\theta_0)+4\theta_0\alpha(\theta_0)}\right\}$$
$$=\exp\left\{-C_4(k)\,\alpha^2(v)\right\}, \tag{6.12}$$

where $C_4(k)=\frac{3(k-1)h(k,a_H)d_4}{2\left[3\theta_0^2+6h(k,\theta_0)M^2\beta(\theta_0)+4\theta_0\alpha(\theta_0)\right]}$, and $d_4\geq\frac{u(a_H)}{\Gamma(m)}C_H(m,\theta_0)>0$. Let $\tau_4=\max_{a_H\leq v\leq\theta_0}\frac{g_1(v)}{\alpha^{(1)}(v)}$. Note that $\tau_4\leq\frac{u(\theta_0)}{\Gamma(m)}C_H(m,a_H)<\infty$. Plugging (6.12) and the preceding inequality into IV_k, we obtain

$$IV_k\leq\int_{a_H}^{\theta_0}\exp\{-C_4(k)\alpha^2(v)\}\alpha(v)\alpha^{(1)}(v)[\frac{g_1(v)}{\alpha^{(1)}(v)}]dv$$
$$\leq\tau_4\int_{a_H}^{\theta_0}\exp\{-C_4(k)\alpha^2(v)\}\alpha(v)\alpha^{(1)}(v)dv$$
$$\leq\frac{\tau_4[3\theta_0^2+6h(k,\theta_0)M^2\beta(\theta_0)+4\theta_0\alpha(\theta_0)]}{3(k-1)h(k,a_H)d_4}.$$

References

Balakrishnan, N. and Basu, A. P. (eds.) (1995). *The Exponential Distribution: Theory Methods and Applications*. Gordon and Breach Science Publishers, Amsterdam.

Balakrishnan, N. and Ma, Y. (1996). Empirical Bayes rules for selecting the most and least probable multivariate hypergeometric event. *Statist. Probab. Lett.* **27**, 181-188.

Balakrishnan, N. and Ma, Y. (1997). On empirical Bayes rules for selecting the best hypergeometric population. *Sequential Anal.* **16**, 269-286.

Bechhofer, R. E., Santner, T. J. and Goldsman, D. M. (1995). *Design and Analysis of Experiments for Statistical Selection, Screening, and Multiple Comparisons.* John Wiley, New York.

Gupta, S. S. and Hsiao, P. (1981). On Γ-minimax, minimax, and Bayes procedures for selecting populations close to a control. *Sankhya B* **43**, 291-318.

Gupta, S. S. and Miescke, K. J. (1985). Minimax multiple t-test for comparing k normal populations with a control. *J. Statist. Plann. Inference* **12**, 161-169.

Gupta, S. S. and Panchapakesan, S. (1988). Selection and ranking procedures in reliability models. In *Handbook of Statistics* **7** (Edited by P. R. Krishnaiah and C. R. Rao), 131-156. Elsevier Science Publishers, B. V.

Huang, W. T. (1975). Bayes approach to a problem of partitioning k normal populations. *Bull. Inst. Math. Acad. Sinica* **3**, 87-97.

Huang, W. T. and Lai, Y. T. (1999). Empirical Bayes procedures for selecting the best population with multiple criteria. *Ann. Inst. Statist. Math.* **51**, 281-299.

Johnson, N. L., Kotz, S. and Balakrishnan, N. (1994). *Continuous Univariate Distributions*, Vol. 1. 2nd edition. John Wiley, New York.

Lam, Y. (1994). Bayesian variable sampling plans for the exponential distribution with type-I censoring. *Ann. Statist.* **22**, 696-711.

Lam, Y. and Choy, S. T. B. (1995). Bayesian variable sampling plans for the exponential distribution with uniformly distributed random consoring. *J. Statist. Plann. Inference* **47**, 277-293.

Lamperti, J. (1966). *Probability.* W. A. Benjamin, New York.

Liang, T. (1997a). Simultaneously selecting normal populations close to a control. *J. Statist. Plann. Inference* **61**, 297-316.

Liang, T. (1997b). On hierarchical Bayesian estimation and selection for multivariate hypergeometric distributions. In *Advances in Statistical Decision Theory and Applications* (Edited by S. Panchapakesan and N. Balakrishnan). Birkhauser, Boston.

Mee, R. W., Shah, A. K. and Lefante, J. J. (1987). Comparing k independent sample means with a known standard. *J. Quality Tech.* **19**, 75-81.

Miescke, K. J. (1981). Γ minimax selection procedures in simultaneous testing problems. *Ann. Statist.* **9**, 215-220.

Pensky, M. and Singh, R. S. (1995). Empirical Bayes estimation of reliability characteristics for an exponential family. Technical Report, Department of Statistics, Purdue University, West Lafayette, IN.

Robbins, H. (1964). The empirical Bayes approach to statistical decision problems. *Ann. Math. Statist.* **35**, 1-20.

Shorack, G. R. and Wellner, J. A. (1986). *Empirical Process with Applications to Statistics.* John Wiley, New York.

Wetherill, G. B. (1977). *Sampling Inspection and Quality Control.* 2nd edtion. Chapman and Hall, London.

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