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FISHER INFORMATION IN ORDERED RANDOMLY CENSORED DATA WITH APPLICATIONS TO CHARACTERIZATION PROBLEMS

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Abstract: The proportional hazards model of Koziol-Green is often considered in survival analysis. If the lifetime and censoring random variables are independent, the Koziol-Green model implies that the variable indicating whether the observation is censored or not does not contain Fisher information about the parameters of the underlying lifetime distribution. The Koziol-Green model, however, is not uniquely characterized by this result on the lack of Fisher information in the censoring indicator. Given the ordered randomly censored lifetimes with corresponding indicators, we obtain a necessary and sufficient condition, weaker than the Koziol-Green model, which ensures that a set of any number of censoring indicators does not contain Fisher information about the parameters of the lifetime distribution. Under this weaker condition, the results are applied to characterize the Weibull distribution within the class of scale parameter families of lifetime distributions and the factorization of the hazard function in terms of the Fisher information in randomly censored data.

Key words and phrases: Factorization, Fisher information, hazard function, Koziol-Green model, order statistics, time-dependent, Weibull family.

1. Introduction

Let X and Y be independent absolutely continuous random variables on the sample space E with distribution functions F_θ and G_θ and density functions $f_\theta(x)$ and $g_\theta(y)$, respectively, where $E = (a, b) \subset R^1$ (a and b can be infinite), and $\theta = (\theta_1 \cdots \theta_m) \in \Theta \subset R^m$. Denote $Z = \min(X, Y)$ and $\delta = \mathrm{I}(X \le Y)$, where $\mathrm{I}(A)$ is the indicator function of A. In survival or lifetime analysis, X and Y are lifetime and censoring random variables, respectively. In the random censorship setting, one only observes the pair (Z, δ). The distribution and density functions of Z are given by $H_\theta(z) = 1 - \bar{F}_\theta(z)\bar{G}_\theta(z)$ and $h_\theta(z) = f_\theta(z)\bar{G}_\theta(z) + g_\theta(z)\bar{F}_\theta(z)$, respectively, where $\bar{F}_\theta = 1 - F_\theta$ and $\bar{G}_\theta = 1 - G_\theta$. The full likelihood function for a single pair (Z, δ) is given by $L(z, \delta) = \{f_\theta(z)\bar{G}_\theta(z)\}^\delta\{g_\theta(z)\bar{F}_\theta(z)\}^{1-\delta}$. Under regularity conditions, the Fisher information matrix about θ contained in a single pair (Z, δ), based on the full likelihood function, is $I^{Z,\delta}(\theta) = E_\theta[\{\partial/\partial\theta \log L(Z,\delta)\}\{\partial/\partial\theta^T \log L(Z,\delta)\}]$. Notice that the censoring distribution G_θ may not depend on the parameter θ and, when this is so, the likelihood function can be simplified for the calculation of Fisher information. Different expressions for $I^{Z,\delta}(\theta)$ have been obtained by Prakasa Rao (1995) and Zheng and Gastwirth (2001).

One important model of random censorship is the Koziol-Green model (KGM) of random censoring (e.g., Koziol and Green (1976); Csörgő and Horváth (1981); Chen, Hollander and Langberg (1982); Stute (1992); and Pawlitschko (1999)), in which the lifetime and censoring distributions satisfy

$$g_\theta(x)\bar{F}_\theta(x) = \beta f_\theta(x)\bar{G}_\theta(x), \tag{1}$$

for some positive constant β. Denote the hazard functions of X and Y by $\lambda_X = f/\bar{F}$ and $\lambda_Y = g/\bar{G}$, respectively. Then (1) is equivalent to $\lambda_Y = \beta\lambda_X$. It is well known that Z and δ are independent under the KGM when X and Y are independent (Chen et al. (1982)). This implies that (Z, δ) and Z contain the same Fisher information about θ, since the distribution of δ is independent of θ under the KGM when the lifetime and censoring random variables are independent. However, the reverse is not necessarily true. For a single pair (Z, δ), when the lifetime and censoring random variables are independent (assumed throughout), Zheng and Gastwirth (2001) showed that (Z, δ) and Z contain the same Fisher information about θ if and only if

$$g_\theta(x)\bar{F}_\theta(x) = \beta(x) f_\theta(x)\bar{G}_\theta(x), \tag{2}$$

for some positive function $\beta(x)$, independent of $\theta \in \Theta$. Note that (2) is equivalent to $\lambda_Y = \beta(x)\lambda_X$, a weaker assumption than (1). It may be called the time-dependent Koziol-Green model of random censoring. When (2) holds, the hazard function of Z can be expressed as

$$\lambda_Z = \lambda_X + \lambda_Y = [1 + \beta(x)]\lambda_X = [1 + 1/\beta(x)]\lambda_Y, \tag{3}$$

which shows that H_θ and F_θ (H_θ and G_θ) satisfy the time-dependent KGM.

In this paper, we extend earlier results based on a single pair (Z, δ) (Zheng and Gastwirth (2001)). Given a randomly censored sample, $(Z_1, \delta_1), \ldots, (Z_n, \delta_n)$, of size n, one can rank Z_i, $i = 1, \ldots, n$, in ascending order as $Z_{(1)} \le \cdots \le Z_{(n)}$, the order statistics from the distribution $H(z)$ of $Z = \min(X, Y)$. Let $\delta_{[i]}$ be associated with $Z_{(i)}$ such that if $Z_{(i)} = Z_j$, then $\delta_{[i]} = \delta_j$ for $i = 1, \ldots, n$. Denote any k order statistics $\{Z_{(r_1)}, \ldots, Z_{(r_k)}\}$ by $D_{r_1\cdots r_k:n}(Z)$, and these order statistics with their associated censoring indicators $\{(Z_{(r_1)}, \delta_{[r_1]}), \ldots, (Z_{(r_k)}, \delta_{[r_k]})\}$ by $D_{r_1\cdots r_k:n}(Z, \delta)$, $1 \le r_1 < \cdots < r_k \le n$. We show that, for any $1 \le r_1 < \cdots <$

$r_k \le n$ and any $1 \le k < n$, $D_{r_1\cdots r_k:n}(Z,\delta)$ and $D_{r_1\cdots r_k:n}(Z)$ contain the same Fisher information about θ if and only if (2) holds for some $\beta(x) > 0$.

The calculation of the Fisher information (FI) contained in one or several order statistics is more complex than in the case of a single random variable or random sample because of dependence between the order statistics. The situation where there is no censoring was considered by Gastwirth (1965), Mehrotra, Johnson and Bhattacharyya (1979), Park (1996), and Zheng and Gastwirth (2000). Here we apply and extend the results in Zheng and Gastwirth (2000) to obtain the FI about θ contained in $D_{r_1\cdots r_k:n}(Z,\delta)$. Furthermore, using this result under (2), we characterize the Weibull distribution within the class of scale parameter families of lifetime distributions and the factorization of the hazard function.

2. The Likelihood Function for Ordered Randomly Censored Data

The joint likelihood function for $D_{1\cdots n:n}(Z,\delta)$ is

$$L_{1\cdots n}(z_{(1)},\delta_{[1]};\cdots;z_{(n)},\delta_{[n]}) = n!\prod_{i=1}^{n} L(z_{(i)},\delta_{[i]}),\quad z_{(1)} < \cdots < z_{(n)}, \tag{4}$$

where $L(Z,\delta)$ is defined in Section 1. The likelihood function of $D_{r_1\cdots r_k:n}(Z,\delta)$, $1 \le r_1 < \cdots < r_k \le n$, can be obtained by integrating over the other variables in (4). Thus, the joint likelihood of $D_{r_1\cdots r_k:n}(Z,\delta)$, denoted as $L_{r_1\cdots r_k}(Z,\delta)$, can be written as

$$\begin{aligned} L_{r_1\cdots r_k}(Z,\delta) &= L_{r_1\cdots r_k}(z_{(r_1)},\delta_{[r_1]};\cdots;z_{(r_k)},\delta_{[r_k]}) \\ &= n!\prod_{i=1}^{k+1}\frac{[H(z_{(r_i)})-H(z_{(r_{i-1})})]^{r_i-r_{i-1}-1}}{(r_i-r_{i-1}-1)!}\prod_{i=1}^{k} L(z_{(r_i)},\delta_{[r_i]}), \end{aligned} \tag{5}$$

where $z_{(r_1)} < \cdots < z_{(r_k)}$, $r_0 = 0$, $r_{k+1} = n+1$, $z_{(r_0)} = a$, and $z_{(r_{k+1})} = b$. The joint likelihood function of $D_{r_1\cdots r_k:n}(Z)$, denoted as $L_{r_1\cdots r_k}(Z)$, can be obtained directly from (5) if we replace $L(z_{(r_i)},\delta_{[r_i]})$ by $h(z_{(r_i)})$.

3. Fisher Information in Randomly Censored Data

3.1. Regularity conditions and definition of Fisher information

The regularity conditions for the FI in a single random variable appear in Rao (1973, p.329). Aboeleneen and Nagaraja (2002) showed they suffice to define the FI in order statistics. Similarly the regularity conditions of Prakasa Rao (1995) for the FI in the pair (Z,δ) are also sufficient for the FI in $D_{r_1\cdots r_k:n}(Z,\delta)$.

Let $S = (S_1\cdots S_m)^T$, where $S_i = \partial \log L_{r_1\cdots r_k}(Z,\delta)/\partial\theta_i$. Under regularity conditions, the FI matrix about θ contained in $D_{r_1\cdots r_k:n}(Z,\delta)$ is $I^{Z,\delta}_{r_1\cdots r_k:n}(\theta) = E_\theta(SS^T)$. Similarly, we have the FI matrix for $D_{r_1\cdots r_k:n}(Z)$, denoted by $I^{Z}_{r_1\cdots r_k:n}(\theta)$.

3.2. Calculation of Fisher information

We are interested in calculating $I^{Z,\delta}_{r_1\cdots r_k:n}(\theta)$. The techniques employed to show Theorem 2.1 in Zheng and Gastwirth (2000) for $I^{Z}_{r_1\cdots r_k:n}(\theta)$ can be used to obtain $I^{Z,\delta}_{r_1\cdots r_k:n}(\theta)$.

Lemma 3.1. *Assume regularity conditions hold. Denote*

$$\begin{aligned}\Delta_{1,s-1} &= I^{Z,\delta}_{1\cdots n:n}(\theta) - I^{Z,\delta}_{s\cdots n:n}(\theta),\\ \Delta_{s+1,t-1} &= I^{Z,\delta}_{1\cdots n:n}(\theta) - I^{Z,\delta}_{1\cdots s\,t\cdots n:n}(\theta),\\ \Delta_{t+1,n} &= I^{Z,\delta}_{1\cdots n:n}(\theta) - I^{Z,\delta}_{1\cdots t:n}(\theta),\end{aligned} \tag{6}$$

where $1 \le s \le n$, $s \le t-1$ *and* $1 \le t \le n$, *respectively. Then*
(a) *for* $s < j_1 < j_2 < \cdots < j_k \le n$, $\Delta_{1,s-1} = I^{Z,\delta}_{1\cdots s\,j_1\,j_2\cdots j_k:n}(\theta) - I^{Z,\delta}_{s\,j_1\,j_2\cdots j_k:n}(\theta)$;
(b) *for* $1 \le i_1 < i_2 < \cdots < i_l < s < t < j_1 < j_2 < \cdots < j_k \le n$,

$$\Delta_{s+1,t-1} = I^{Z,\delta}_{i_1\,i_2\cdots i_l\,s\cdots t\,j_1\,j_2\cdots j_k:n}(\theta) - I^{Z,\delta}_{i_1\,i_2\cdots i_l\,s\,t\,j_1\,j_2\cdots j_k:n}(\theta);$$

(c) *for* $1 \le i_1 < i_2 < \cdots < i_l < t$, $\Delta_{t+1,n} = I^{Z,\delta}_{i_1\,i_2\cdots i_l\,t\cdots n:n}(\theta) - I^{Z,\delta}_{i_1\,i_2\cdots i_l\,t:n}(\theta)$.

To illustrate how Lemma 3.1 can be used to obtain $I^{Z,\delta}_{r\cdots s\,u\cdots v:n}(\theta)$, $1 \le r \le s < u \le v \le n$, first from (b), we have $I^{Z,\delta}_{r\cdots s\,u\cdots v:n}(\theta) = I^{Z,\delta}_{r\cdots v:n}(\theta) - \Delta_{s+1,u-1}$. Then from (a), $I^{Z,\delta}_{r\cdots v:n}(\theta) = I^{Z,\delta}_{1\cdots v:n}(\theta) - \Delta_{1,r-1}$. Finally from (c), $I^{Z,\delta}_{1\cdots v:n}(\theta) = I^{Z,\delta}_{1\cdots n:n}(\theta) - \Delta_{v+1,n}$. Hence we obtain $I^{Z,\delta}_{r\cdots s\,u\cdots v:n}(\theta) = I^{Z,\delta}_{1\cdots n:n}(\theta) - \Delta_{1,r-1} - \Delta_{s+1,u-1} - \Delta_{v+1,n}$. This leads to the following result.

Lemma 3.2. *Under regularity conditions, for* $1 \le r_1 < \cdots < r_k \le n$,

$$I^{Z,\delta}_{r_1\cdots r_k:n}(\theta) = I^{Z,\delta}_{1\cdots n:n}(\theta) - \sum_{i=0}^{k} \Delta_{r_i+1,r_{i+1}-1},$$

where $\Delta_{1,0} = \Delta_{n+1,n} = 0$, $r_0 = 0$ *and* $r_{k+1} = n+1$, *and* Δ_{1,r_1-1}, $\Delta_{r_i+1,r_{i+1}-1}$ *and* $\Delta_{r_k+1,n}$ *are given by* (6).

Remark 3.1. In the uncensored case, i.e., $G(x) = 0$ for all x, Lemmas 3.1 and 3.2 reduce to known results for FI in order statistics (Zheng and Gastwirth (2000)).

Lemmas 3.1 and 3.2 show that the FI in multiply randomly censored data is a linear combination of the FI in (i) complete randomly censored data ($I^{Z,\delta}_{1\cdots n:n}(\theta)$), (ii) the left portion ($I^{Z,\delta}_{1\cdots u:n}(\theta)$), (iii) the right portion ($I^{Z,\delta}_{v\cdots n:n}(\theta)$), and (iv) the two-tail portion ($I^{Z,\delta}_{1\cdots u\,v\cdots n:n}(\theta)$) of the randomly censored data ($u < v$). The FI in each of the last three cases is given in the following result, proved in Appendix.

Theorem 3.1. *Under regularity conditions, let* U_z *be a binomial random variable with parameters* $N = n-1$ *and* $p = H(z) = 1-\bar{F}(z)\bar{G}(z)$, *and* λ_X (λ_Y) *be hazard functions for* X (Y). *For* $1 \le u \le v \le n$ *we have*

$$I^{Z,\delta}_{u\cdots v:n}(\theta) = I^{Z}_{u\cdots v:n}(\theta) + I_{u\cdots v:n}(\theta), \tag{7}$$

$$I^{Z,\delta}_{1\cdots u\, v\cdots n:n}(\theta) = I^{Z}_{1\cdots u\, v\cdots n:n}(\theta) + I_{1\cdots u\, v\cdots n:n}(\theta), \tag{8}$$

where $I_{u\cdots v:n}(\theta) = n\int_E K(z)Pr(u-1 \le U_z \le v-1)dz$, $I_{1\cdots u\, v\cdots n:n}(\theta) = n\int_E K(z)$ $Pr(U_z \le u-1$ *or* $U_z \ge v-1)dz$, *and where* $K(z)$ *is an* $m \times m$ *matrix with the* (i,j)*th element given by*

$$K_{ij}(z) = \frac{(f\bar{G})(g\bar{F})}{f\bar{G}+g\bar{F}}\left[\frac{\partial}{\partial\theta_i}\log(\frac{\lambda_X}{\lambda_Y})\right]\left[\frac{\partial}{\partial\theta_j}\log(\frac{\lambda_X}{\lambda_Y})\right]. \tag{9}$$

Remark 3.2. From Theorem 3.1, $I^{Z,\delta}_{u\cdots v:n} \ge I^{Z}_{u\cdots v:n}$ and $I^{Z,\delta}_{1\cdots u\, v\cdots n:n}(\theta) \ge I^{Z}_{1\cdots u\, v\cdots n:n}(\theta)$ (here and below $A \ge B$ means $A-B$ is a semi-positive definite matrix), where equalities hold if and only if $(\partial/\partial\theta_i)\log(\lambda_X(z)/\lambda_Y(z)) = 0$ for all z and $i = 1,\ldots,m$, i.e., the distributions of X and Y satisfy (2). In the uncensored case, expressions for $I^{Z}_{u\cdots v:n}(\theta)$ and $I^{Z}_{1\cdots u\, v\cdots n:n}(\theta)$ were derived by Park (1996) and Zheng and Gastwirth (2000).

Remark 3.3. In Theorem 3.1, let $u = 1$ and $v = n$. From (7), we obtain $I^{Z,\delta}_{1\cdots n:n}(\theta) = I^{Z}_{1\cdots n:n}(\theta) + n\int_E K(z)dz$, for complete randomly censored data (Zheng and Gastwirth (2001)).

3.3. Asymptotic Fisher information

In this section, we consider a single parameter θ and $D_{1\cdots r:n}(Z,\delta)$, where $Z_{(r)}$ approaches the pth percentile of $H(z)$ as $n \to \infty$ and $r/n \to p \in (0,1)$. Define

$$I^{Z,\delta}_{[0,p]}(\theta) = \lim_{n\to\infty}\frac{1}{n}I^{Z,\delta}_{1\cdots r:n}(\theta) \text{ and } I^{Z}_{[0,p]}(\theta) = \lim_{n\to\infty}\frac{1}{n}I^{Z}_{1\cdots r:n}(\theta).$$

Set $I^{Z,\delta}_{[0,0]}(\theta) = 0$ and $I^{Z,\delta}_{[0,1]}(\theta) = I^{Z,\delta}(\theta)$, where $I^{Z,\delta}(\theta)$ is defined in Section 1. Denote by λ_p the pth percentile of $H(z)$. Then, from Zheng (2001),

$$I^{Z}_{[0,p]}(\theta) = \int_a^{\lambda_p}\left[\frac{\partial}{\partial\theta}\log\lambda_Z(z)\right]^2 h(z)dz,$$

where λ_Z is the hazard function of Z. By the dominated convergence theorem,

$$\lim_{n\to\infty} I_{1\cdots r:n}(\theta)/n = \lim_{n\to\infty}\int_E K(z)\Pr(U_z \le r-1)dz = \int_a^{\lambda_p} K(z)dz.$$

Hence, for any $p \in (0,1)$,

$$I^{Z,\delta}_{[0,p]}(\theta) = I^{Z}_{[0,p]}(\theta) + \int_a^{\lambda_p} K(z)dz. \tag{10}$$

4. Main Result

Theorem 4.1. *Assume regularity conditions hold. For any* $1 \le r_1 < \cdots < r_k \le n$ *and* $1 \le k < n$, $I^{Z,\delta}_{r_1\cdots r_k:n}(\theta) = I^{Z}_{r_1\cdots r_k:n}(\theta)$ *holds if and only if* $g_\theta(x)\bar{F}_\theta(x) = \beta(x)f_\theta(x)\bar{G}_\theta(x)$ *for some* $\beta(x) > 0$.

Proof. First, assume no random censoring occurs ($G(x) = 0$, for all $x \in E$). From Lemmas 3.1 and 3.2,

$$\begin{aligned} I^{Z}_{r_1\cdots r_k:n}(\theta) &= I^{Z}_{1\cdots n:n}(\theta) - \Big[I^{Z}_{1\cdots n:n}(\theta) - I^{Z}_{r_1\cdots n:n}(\theta)\Big] \\ &\quad - \sum_{i=1}^{k-1}[I^{Z}_{1\cdots n:n}(\theta) - I^{Z}_{1\cdots r_i\, r_{i+1}\cdots n:n}(\theta)] - [I^{Z}_{1\cdots n:n}(\theta) - I^{Z}_{1\cdots r_k:n}(\theta)]. \end{aligned}$$

Similarly, with random censoring,

$$\begin{aligned} I^{Z,\delta}_{r_1\cdots r_k:n}(\theta) &= I^{Z,\delta}_{1\cdots n:n}(\theta) - [I^{Z,\delta}_{1\cdots n:n}(\theta) - I^{Z,\delta}_{r_1\cdots n:n}(\theta)] \\ &\quad - \sum_{i=1}^{k-1}[I^{Z,\delta}_{1\cdots n:n}(\theta) - I^{Z,\delta}_{1\cdots r_i\, r_{i+1}\cdots n:n}(\theta)] - [I^{Z,\delta}_{1\cdots n:n}(\theta) - I^{Z,\delta}_{1\cdots r_k:n}(\theta)]. \end{aligned}$$

Then from Theorem 3.1 and the two expressions above, we have

$$\begin{aligned} I^{Z,\delta}_{r_1\cdots r_k:n}(\theta) &= I^{Z}_{r_1\cdots r_k:n}(\theta) + I_{1\cdots n:n}(\theta) - [I_{1\cdots n:n}(\theta) - I_{r_1\cdots n:n}(\theta)] \\ &\quad - \sum_{i=1}^{k-1}[I_{1\cdots n:n}(\theta) - I_{1\cdots r_i\, r_{i+1}\cdots n:n}(\theta)] - [I_{1\cdots n:n}(\theta) - I_{1\cdots r_k:n}(\theta)] \\ &= I^{Z}_{r_1\cdots r_k:n}(\theta) + n\int_E K(z)\sum_{i=1}^{k}\Pr(U_z = r_i - 1)\,dz. \end{aligned}$$

Hence $I^{Z,\delta}_{r_1\cdots r_k:n}(\theta) \ge I^{Z}_{r_1\cdots r_k:n}(\theta)$ and equality holds if and only if $K(z) = 0$, i.e., (2) holds.

5. Applications to Characterization

5.1. The Weibull family

Zheng (2001) characterized the Weibull family for type II censored order statistics. He showed that the hazard function $\lambda_X(x,\theta) = u(x)v(\theta)$ for some positive functions u and v if and only if, for any $1 \le r \le n$ and all $n \ge 1$, (i) $I^{X}_{1\cdots r:n}(\theta) = (r/n)I^{X}_{1\cdots n:n}(\theta) = rI^{X}(\theta)$, or (ii) $I^{X}_{[0,p]}(\theta) = pI^{X}_{[0,1]}(\theta)$ for any

$0 < p < 1$, where $I^X_{1\cdots r:n}(\theta)$ and $I^X_{[0,p]}(\theta)$ can be defined analogously to $I^Z_{1\cdots r:n}(\theta)$ and $I^Z_{[0,p]}(\theta)$, respectively. When θ is a scale parameter, (i) or (ii) holds if and only if $F_\theta(x)$ is the Weibull distribution. Generalizing Type II censoring to random censoring, we have the following.

Theorem 5.1. *Assume regularity conditions and* (2) *hold for the lifetime distribution F_θ and the censoring distribution G_θ, where θ is a scale parameter. For every $1 \le r \le n$ and all $n \ge 1$,*

$$I^{Z,\delta}_{1\cdots r:n}(\theta) = \frac{r}{n} I^{Z,\delta}_{1\cdots n:n}(\theta) = r\, I^{Z,\delta}(\theta), \tag{11}$$

holds if and only if $F_\theta(x) = 1 - \exp\{-(x/\theta)^\alpha\}$, where α is a positive constant.

Proof. Let $F_\theta(x)$ be the Weibull distribution. When (2) holds, from Theorem 4.1, $I^{Z,\delta}_{1\cdots r:n}(\theta) = I^Z_{1\cdots r:n}(\theta)$ for every $1 \le r \le n$ and any $n \ge 1$. Since X follows the Weibull distribution, from (3), the hazard function of $Z = \min(X, Y)$ can be factored as $\lambda_Z(x, \theta) = u(x)v(\theta)$ for some functions u and v. Thus, from Zheng (2001), $I^Z_{1\cdots r:n}(\theta) = rI^Z(\theta) = rI^{Z,\delta}(\theta)$. Hence $I^{Z,\delta}_{1\cdots r:n}(\theta) = rI^{Z,\delta}(\theta)$, which implies (11). On the other hand, if (11) holds for every $1 \le r \le n$ and any $n \ge 1$, then $I^{Z,\delta}_{[0,p]}(\theta) = pI^{Z,\delta}_{[0,1]}$ for any $p \in (0,1)$. Since (2) holds, implying $K(z) = 0$, from (10) we have $I^Z_{[0,p]}(\theta) = I^{Z,\delta}_{[0,p]}(\theta) = pI^{Z,\delta}_{[0,1]}(\theta) = pI^Z_{[0,1]}$ for any $p \in (0, 1)$. Hence, from Zheng (2001), the hazard function of Z can be factored as $\lambda_Z(x, \theta) = u(x)v(\theta)$ for some functions u and v. When (2) holds, from (3) we have $\lambda_X = u(x)v(\theta)/(\beta(x) + 1)$, i.e., the hazard function of X can be factored. Thus, from Zheng (2001), X follows the Weibull distribution.

Note that in the weaker KGM setting (2), (11) implies that the percentage of the FI in the first pth portion of ordered randomly censored data is exactly $100p\%$. This has relevance to life testing when it is cost-effective not to wait for the last failures to occur (Kimball, Burnett and Doherty (1957); Gastwirth and Wang (1987); Li, Tiwari and Wells (1996); and Chakraborti and vander Laan (1997)), as one can increase the number of items on test to increase the information yielded by the experiment.

5.2. The factorization of the hazard function

As the factorization of the hazard function is used in Theorem 5.1, we give a characterization of the factorization of the hazard function by the FI in a fraction of randomly censored data in terms of the loss of FI under the condition (2). Let F_θ be the lifetime distribution. For any positive function of x, $\beta(x)$, define a censoring distribution G_θ such that (2) holds for $F_\theta(x)$ and $G_\theta(x)$ with respect to $\beta(x)$. Define a collection of these censoring distributions with respect to $F_\theta(x)$ as

$$\mathcal{C} = \{G_\theta : (2) \text{ holds for } F_\theta \text{ and } G_\theta \text{ with respect to some } \beta(x) > 0\}.$$

Thus, for any $\beta(x) > 0$, there exists $G_\theta \in \mathcal{C}$ such that F_θ and G_θ satisfy (2). Let $\mathcal{C}^*$ be a subset of $\mathcal{C}$ such that, for every $G_\theta \in \mathcal{C}^*$, the corresponding $\beta(x) = \beta$ is a positive constant.

Theorem 5.2. *Assume regularity conditions hold. Let X be a lifetime random variable from $F_\theta(x)$. There exist functions u and v such that the hazard function of X satisfies $\lambda_X = u(x)v(\theta)$ if and only if, for any $1 \le r \le n$, all $n \ge 1$, and every $G_\theta \in \mathcal{C}$, $I^{Z,\delta}_{1\cdots r:n}(\theta) = I^X_{1\cdots r:n}(\theta)$.*

Proof. If $\lambda_X = u(x)v(\theta)$, from Zheng (2001), $I^X_{1\cdots r:n}(\theta) = (r/n)I^X_{1\cdots n:n}(\theta)$. Since λ_X can be factored, from (2), λ_Z can also be factored. Thus by Zheng (2001), $I^Z_{1\cdots r:n}(\theta) = (r/n)I^Z_{1\cdots n:n}(\theta)$. From Efron and Johnstone (1990), $I^Z_{1\cdots n:n}(\theta) = nE[(\partial/\partial\theta)\log\lambda_Z]^2 = nE[(\partial/\partial\theta)\log\lambda_X]^2 = I^X_{1\cdots n:n}(\theta)$. Thus from the three expressions above and Theorem 4.1, $I^X_{1\cdots r:n}(\theta) = I^Z_{1\cdots r:n}(\theta) = I^{Z,\delta}_{1\cdots r:n}(\theta)$.

On the other hand, for any $1 \le r \le n$ and any $n \ge 1$, and any $G_\theta \in \mathcal{C}$, $I^{Z,\delta}_{1\cdots r:n}(\theta) = I^X_{1\cdots r:n}(\theta)$. Hence, we have $I^{Z,\delta}_{1\cdots r:n}(\theta) = I^X_{1\cdots r:n}(\theta)$ when we restrict to any $G_\theta \in \mathcal{C}^*$, which implies that $I^{Z,\delta}_{[0,p]}(\theta) = I^X_{[0,p]}(\theta)$ for any $p \in (0,1)$ and any $G_\theta \in \mathcal{C}^*$. Since $K(z)$ in (10) is zero, we have $I^Z_{[0,p]}(\theta) = I^X_{[0,p]}(\theta)$ for any $0 < p < 1$ and any $G_\theta \in \mathcal{C}^*$ (thus for any constant $\beta > 0$). Taking the derivative with respect to p on both sides of $I^Z_{[0,p]}(\theta) = I^X_{[0,p]}(\theta)$ and using $\partial\log\lambda_Z/\partial\theta = \partial\log\lambda_X/\partial\theta$, we have

$$\left[\frac{\partial}{\partial\theta}\log\lambda_X(x;\theta)\right]^2_{x=\lambda_p^Z} = \left[\frac{\partial}{\partial\theta}\log\lambda_X(x;\theta)\right]^2_{x=\lambda_p^X}, \tag{12}$$

for any $0 < p < 1$ and any $\beta > 0$, where $\lambda_p^X = F^{-1}(p)$ and $\lambda_p^Z = F^{-1}(1-(1-p)^{1/(1+\beta)})$. We need to show that $\partial\log\lambda_X/\partial\theta$ is only a function of θ if it is not a constant. Notice that $\lambda_p^X > \lambda_p^Z$ for $\beta > 0$. For any two real numbers $x_1 \in E$ and $x_2 \in E$ such that $x_1 > x_2$, define $p_1 = F(x_1)$ and $\beta^* = \log(1-p_1)/\log(1-F(x_2)) - 1 > 0$ such that $F^{-1}(1-(1-p_1)^{1/(1+\beta^*)}) = x_2$. Hence from (12), for any fixed θ, we obtain

$$\left[\frac{\partial}{\partial\theta}\log\lambda_X(x;\theta)\right]^2_{x=x_2} = \left[\frac{\partial}{\partial\theta}\log\lambda_X(x;\theta)\right]^2_{x=x_1},$$

i.e., $(\partial/\partial\theta)\log\lambda_X$ is only a function of θ. Thus, λ_X can be factored.

Theorem 5.2 shows that, under the time-dependent KGM (2), if the hazard function can be factorized, then the first pth fraction of randomly censored data contains the same FI as that of the uncensored data. On the other hand, under the time-dependent KGM, if the first pth fraction of randomly censored data contains the same FI as that of the uncensored data, then the hazard function must factor. For example, assume the lifetime distribution is the Weibull with any censoring random variable such that (2) holds. If we are interested in inference about the scale parameter of the Weibull distribution based on the smallest $r < n$ randomly censored observations, then there is no loss of FI (efficiency) about the scale parameter in the first r randomly censored data relative to the inference based on the smallest r ordered data without random censoring.

Under KGM, there is no loss of FI in any randomly censored data. When KGM does not hold, randomly censored data may contain less FI. For example, let the lifetime X and censoring variable Y follow the exponential (with the scale parameter θ) and Gamma (with the shape parameter β and the scale parameter equal to 1) distributions, respectively. When $\theta = 1$, numerical results show that $I^{Z,\delta}_{[0,p]}(\theta)/I^{X}_{[0,p]}(\theta) = 0.956$, 0.841, 0.767 if $\beta = 2.0$, and 0.221, 0.331, 0.378 if $\beta = 0.7$, for $p = 0.1$, 0.5, 0.9, respectively.

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Appendix: Proof of Theorem 3.1

We only prove a special case of (7) where $u = 1$, $1 \le v \le n$, and θ is a single parameter. The proof of the general case can be done similarly. Let $h_i = h(z_{(i)})$ and $H_i = H(z_{(i)})$ with similar notations for f_i, F_i, g_i and G_i. Also denote the cumulative distribution functions of $(Z_{(i)}, Z_{(j)})$ and $Z_{(i)}$ by $H_{ij:n}$ and $H_{i:n}$, respectively. Then

$$I^{Z,\delta}_{1\cdots r:n}(\theta) = E\left[\sum_{i=1}^{r} \frac{\partial}{\partial\theta}\log L(Z_{(i)}, \delta_{[i]})\right]^2 + (n-r)^2 E\left[\frac{\partial}{\partial\theta}\log \bar{H}_r\right]^2$$
$$+2(n-r)\sum_{i=1}^{r} E\left[\frac{\partial}{\partial\theta}\log L(Z_{(i)}, \delta_{[i]})\frac{\partial}{\partial\theta}\log \bar{H}_r\right],$$

and $I^{Z}_{1\cdots r;n}(\theta)$ is the same if we replace $L(Z_{(i)}, \delta_{[i]})$ by h_i.

For $1 \le i \le r$, it can be shown that

$$E\left[\frac{\partial}{\partial\theta}\log L(Z_{(i)}, \delta_{[i]})\frac{\partial}{\partial\theta}\log \bar{H}_r\right] = E\left[\frac{\partial}{\partial\theta}\log h_i \frac{\partial}{\partial\theta}\log \bar{H}_r\right],$$

$$E\left[\sum_{i=1}^{r} \frac{\partial}{\partial\theta}\log L(Z_{(i)}, \delta_{[i]})\right]^2$$
$$= \sum_{i=1}^{r}\int_E \left\{(f_i\bar{G}_i)[\frac{\partial}{\partial\theta}\log(f_i\bar{G}_i)]^2 + (g_i\bar{F}_i)[\frac{\partial}{\partial\theta}\log(g_i\bar{F}_i)]^2\right\} dH_{i:n} \tag{13}$$
$$+\sum_{i\neq j} E\left[\frac{\partial}{\partial\theta}\log L(Z_{(i)}, \delta_{[i]})\frac{\partial}{\partial\theta}\log L(Z_{(j)}, \delta_{[j]})\right], \tag{14}$$

$$E\left[\sum_{i=1}^{r}\frac{\partial}{\partial\theta}\log h_i\right]^2$$
$$=\sum_{i=1}^{r}\int_E\frac{1}{h_i}\left[(f_i\bar{G}_i)\frac{\partial}{\partial\theta}\log(f_i\bar{G}_i)+(g_i\bar{F}_i)\frac{\partial}{\partial\theta}\log(g_i\bar{F}_i)\right]dH_{i:n} \tag{15}$$
$$+\sum_{i\neq j}E\left[\frac{\partial}{\partial\theta}\log h_i\frac{\partial}{\partial\theta}\log h_j\right]. \tag{16}$$

It can be shown that (14) equals to (16). Thus, with $z_{(i)} = z$, we have

$$I^{Z,\delta}_{1\cdots r:n}(\theta)-I^{Z}_{1\cdots r:n}(\theta)=(13)-(15)=\sum_{i=1}^{r}\int_E\frac{(f_i\bar{G}_i)(g_i\bar{F}_i)}{f_i\bar{G}_i+g_i\bar{F}_i}\left[\frac{\partial}{\partial\theta}\log(\frac{f_i\bar{G}_i}{g_i\bar{F}_i})\right]^2dH_{i:n}$$
$$=n\sum_{i=1}^{r}\int_E K(z)\frac{(n-1)!}{(i-1)!(n-i)!}H^{i-1}(z)\bar{H}^{n-i}(z)dz.$$

This completes the proof.

References

Aboeleneen, Z. A. and Nagaraja, H. N. (2002). Fisher information in an order statistic and its concomitant. *Ann. Inst. Statist. Math.* **54**, 667-680.

Chakraborti, S. and vander Laan, P. (1997). An overview of precedence-type tests for censored data. *Biometrical J.* **39**, 99-116.

Chen, Y. Y., Hollander, M. and Langberg, N. A. (1982). Small-Sample Results for the Kaplan-Meier Estimator. *J. Amer. Statist. Assoc.* **77**, 141-144.

Csörgő, S. and Horváth, L. (1981). On the Koziol-Green model for random censorship. *Biometrika* **68**, 391-401.

Efron, B. and Johnstone, I. (1990). Fisher information in terms of the hazard rate. *Ann. Statist.* **18**, 38-62.

Gastwirth, J. L. (1965). Asymptotic most powerful rank tests for the two-sample problem with censored data. *Ann. Math. Statist.* **36**, 1243-1247.

Gastwirth, J. L. and Wang, J. L. (1988). Control Percentile Test Procedures for Censored Data. *J. Statist. Plann. Inference* **18**, 267-276.

Kimball, A. W., Burnett, W. T. Jr. and Doherty, D. G. (1957). Chemical protection against ionizing radiation. I. Sampling methods for screening compounds in radiation protection studies with mice. *Radiation Research* **7**, 1-12.

Koziol, J. A. and Green, S. B. (1976). A Cramér-von Mises statistic for randomly censored data. *Biometrika* **63**, 465-474.

Li, G. Tiwari, R. C. and Wells, M. T. (1996). Quantile comparison functions in two-sample problems, with application to comparisons of diagnostic markers. *J. Amer. Statist. Assoc.* **91**, 689-698.

Mehrotra, K. G., Johnson, R. A. and Bhattacharyya, G. K. (1979). Exact Fisher information for censored samples and the extended hazard rate functions. *Comm. Statist. Theory Methods* **15**, 1493-1510.

Park, S. (1996). Fisher information in order statistics. *J. Amer. Statist. Assoc.* **91**, 385-390.

Pawlitschko, J. (1999). A comparison of survival function estimators in the Koziol-Green model. *Statistics* **32**, 277-291.

Prakasa Rao, B. L. S. (1995). Remarks on Cramer-Rao type integral inequalities for randomly censored data. In *Analysis of Censored Data* **27** (Edited by H. L. Koul and J. V. Deshpande), IMS Lecture Notes - Monograph Series.

Rao, C. R. (1973). *Linear Statistical Inference and Its Applications*. 2nd edited. John Wiley, New York.

Stute, W. (1992). Strong consistency under the Koziol-Green model. *Statist. Probab. Lett.* **14**, 313-320.

Zheng, G. (2001). A characterization of the factorization of hazard function by the Fisher information under Type II censoring with application to the Weibull family. *Statist. Probab. Lett.* **52**, 249-253.

Zheng, G. and Gastwirth, J. L. (2000). Where is the Fisher information in an ordered sample? *Statist. Sinica* **10**, 1267-1280.

Zheng, G. and Gastwirth, J. L. (2001). On the Fisher information in randomly censored data. *Statist. Probab. Lett.* **52**, 421-426.

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