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ON THE CONSTRUCTION OF G_{rm}-OPTIMAL DESIGNS

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Abstract: We consider the problem of finding an efficient design for estimating the model parameters and the mean response surface simultaneously when the true degree of the polynomial model is unknown. A precision constraint is imposed on one of these estimates and the optimal design is found using the theory of canonical moments. Robustness properties of the optimal designs to model assumptions and their performances relative to similar designs are studied.

Key words and phrases: Canonical moment, continuous design, D-optimality, G-optimality, robust design criteria.

1. Introduction

Suppose an experiment is run and N independent observations of the response y are measured according to the model

$$y_i = f^T(x_i)\beta + e_i, \quad x_i \in [0,1] \text{ and } i = 1,2,\ldots,N. \tag{1.1}$$

Here $f^T(x) = (1, x, x^2, \ldots, x^m)$, β is the vector of unknown parameters and the e_i's are unobservable random errors with mean zero and constant variance. The problem of interest here is how to find an efficient design for estimating the model parameters and the mean response surface simultaneously when the true degree of the model (i.e. m) is unknown.

Some pioneering work addressing design issues when there is model uncertainty are Atwood (1971), Stigler (1971) and Läuter (1974). Both Atwood and Stigler's work are seminal; they formalized the concepts of robust design and found optimal designs for some simple models. Läuter (1974) proposed algorithms for generating D-optimal designs when it is possible to postulate the true model is in a class of models. Subsequent works include Studden (1982), Lee (1988), and Rosenberger and Pukelsheim (1993), where efficient designs are found when the mean regression function is not known precisely.

Stigler (1971) proposed two robust design criteria when there is model uncertainty. They were generalized by Studden (1982) as D_{rm} and G_{rm}-optimality. These designs are optimal for estimating the model parameters and the mean response surface respectively when the assumed model is a polynomial of degree r but, at the same time, also ensure a certain level of precision for estimating if additional terms $x^{r+1}, \ldots, x^m$ are needed. Studden (1982) used the theory of canonical moments and found closed form descriptions for D_{rm}-optimal designs when $m > r$ and $r = 1, 2$. Stigler (1971) and Studden (1982) noted that the D_{rm}-optimal designs are 'somewhat simpler to calculate than the G_{rm}-optimal designs'. Except in the simplest case, when $m = 2$ and $r = 1$, no other G_{rm}-optimal designs were found. One of the difficulties is that the G-optimality criterion is not differentiable and standard algorithms cannot be used to find the optimal design.

In this paper, we construct G_{rm}-optimal designs and derive explicit analytic formulae in terms of canonical moments, for any $m > r$ and $r = 1, 2$. In addition, we show that D_{1m} and G_{1m}-optimal designs are equivalent for any $m > 1$. Robustness properties of G_{rm}-optimal designs to model assumptions are studied, and their performances relative to D_{rm}-optimal designs and the constrained designs in Pukelsheim and Rosenberger (1993) are compared.

2. The D_{rm} and G_{rm}-optimality Criteria

Let ξ denote a design with n_j observations at the point $x_j \varepsilon [0, 1]$, $j = 1, \ldots, J$, subject to $\sum n_j = N$. We will treat ξ as a probability measure which puts mass ξ_j at x_j without insisting that each $N\xi_j$ is an integer, subject to the constraint that they sum to 1. Such designs are called continuous designs and, they are easier to study and generate than the traditional discrete designs (Kiefer (1959)). The information matrix of ξ is

$$M_m(\xi) = \int_0^1 f(x) f(x)^T \xi(dx)$$

and the covariance matrix of the LSE (least squares estimate) of β is proportional to $M_m(\xi)^{-1}$, if $M_m(\xi)$ is non-singular. Designs are non-singular if their information matrices are non-singular.

Various optimality criteria have been proposed, and many are formulated as convex functions of $M_m(\xi)$. For instance, a design is D-optimal if it maximizes $|M_m(\xi)|$ over the set Ξ containing all designs on $[0, 1]$. Another criterion is G-optimality, which seeks to minimize over Ξ, the maximum of the standardized variance function $d_m(x, \xi) = f^T(x) M_m(\xi)^{-1} f(x)$. Thus, D-optimal designs are useful for parameter estimation and G-optimal designs are useful for estimating the response surface. Kiefer and Wolfowitz (1960) showed that D and G-optimal designs are equivalent when the model is homoscedastic.

Let $f^T(x) = (f_1^T(x), f_2^T(x))$ where $f_1^T(x) = (1, x, \ldots, r^r)$, $f_2^T(x) = (x^{r+1}, \ldots, x^m)$ and $\beta^T = (\beta_1^T, \beta_2^T)$. The model of interest becomes $Ey = f_1^T(x)\beta_1 + f_2^T(x)\beta_2$. For any arbitrary design ξ, its information matrix can be partitioned as

$$M_m(\xi) = \begin{pmatrix} M_{11}(\xi) & M_{12}(\xi) \\ M_{21}(\xi) & M_{22}(\xi) \end{pmatrix},$$

where $M_{11}(\xi)$ has size $r+1$ and $M_{22}(\xi)$ has size $m-r$. The covariance matrix of the LSE of β_1 is proportional to $M_{11}(\xi)^{-1}$ if the polynomial model of degree r is fitted. In this case, the standardized variance function of the design ξ at the point x is given by $d_r(x,\xi) = f_1^T(x)M_{11}(\xi)^{-1}f_1(x)$. On the other hand, if the polynomial model of degree m is fitted, the covariance matrix of the LSE of β_2 is proportional to $M_{22.1}(\xi)^{-1}$, where

$$M_{22.1}(\xi) = M_{22}(\xi) - M_{21}(\xi)M_{11}(\xi)^{-1}M_{12}(\xi).$$

Let $\bar{d}_r(\xi) = \max_{0\leq x\leq 1} d_r(x,\xi)$ whenever $M_{11}(\xi)^{-1}$ exists.

Definition 2.1. (The D_{rm}-problem) A design is called D_{rm}-optimal if it

$$\text{maximizes } |M_{11}(\xi)| \text{ subject to } |M_{22.1}(\xi)| \geq \rho^{m-r} \max_{\eta\in\Xi} |M_{22.1}(\eta)|.$$

Definition 2.2. (The G_{rm}-problem) A design is called G_{rm}-optimal if it

$$\text{minimizes } \bar{d}_r(\xi) \text{ subject to } |M_{22.1}(\xi)| \geq \rho^{m-r} \max_{\eta\in\Xi} |M_{22.1}(\eta)|.$$

These definitions are motivated by design efficiencies: Recall that if the D-optimal design for model (1.1) is ξ^*, then the D and G-efficiency of a non-singular design ξ is $\{|M_m(\xi)|/|M_m(\xi^*)|\}^{1/(m+1)}$ and $(m+1)/\bar{d}_m(\xi)$ respectively. The reciprocal of these numbers represent how many times the design ξ has to be replicated for it to do as well as the optimal design. Clearly, designs with high efficiencies are sought. It follows that the inequalities in the above definitions ensure that the D_{rm} and G_{rm}-optimal design has a guaranteed D-efficiency of at least ρ for estimating β_2, while still being as close to optimal as possible under the other criterion. If $\rho = 0(1)$, this corresponds to the case when estimating $\beta_1(\beta_2)$ is the only goal: intermediate values of ρ represent a compromised goal of balancing model uncertainty and precision of the various estimates. We will call the set of all information matrices that satisfy the inequality constraint in the above definitions as the constrained set.

3. Canonical Moments

We now define and review properties of canonical moments useful for finding G_{rm}-optimal designs. For an arbitrary design ξ on $[0,1]$, let c_k denote the kth ordinary moment of ξ, $k = 0,1,\ldots$ Given a set of moments $c_0, c_1, \ldots, c_{i-1}$, let c_i^+ be the maximum value of the ith moment over the set of designs having the given moments $c_0, c_1, \ldots, c_{i-1}$. Similarly, let c_i^- be the corresponding minimum. The canonical moments are defined by

$$p_i = (c_i - c_i^-)/(c_i^+ - c_i^-) \quad i = 1,2,\ldots$$

Note that $0 \le p_i \le 1$ and if $c_i^- = c_i^+$, p_i is left undefined and the sequence is terminated. There are many interesting properties of canonical moments; for example, if $p_{2m} = 1$, the design is supported at $(m+1)$points (Dette and Roeder (1996)). Other pertinent facts about canonical moments include the following.

Property 3.1. (Skibinsky (1986)) Every probability measure on $[0, 1]$ is uniquely determined by its canonical moment sequence.

Property 3.2. (Skibinsky (1969), Lau (1983)) The canonical moments are invariant under linear transformations, and all odd canonical moments of a design on $[0, 1]$ are equal to $1/2$ if and only if the design is symmetric about $x = 1/2$.

Property 3.3. (Lau (1983)) For the polynomial regression model of degree m, $|M_m(\xi)| = \prod_{i=1}^{m}(\zeta_{2i-1}\zeta_{2i})^{m+1-i}$ where $\zeta_i = p_i q_{i-1}$, $q_i = 1 - p_i$, $i = 1, 2, \ldots$ and $q_0 = 1$.

Property 3.3 is useful because it facilitates the calculation of the optimal design; each p_i varies independently over the space $[0, 1]$ so the determinant can be maximized by maximizing each p_i one at a time. Once the canonical moments are found, the optimal design is recovered using standard technique (see, for example, the appendix in Dette and Roeder (1996)).

4. G_{rm}-optimal Designs

In general it is quite laborious to find G_{rm}-optimal designs. The following lemmas help reduce the complexity of the problem.

Lemma 1. *There exists a symmetric G_{rm}-optimal design about the midpoint of the design space* $[0, 1]$.

Proof. We observe that the constrained set is convex because $\log |M_{22.1}(\xi)|$ is concave on the set $\{M_m(\xi)|\xi \in \Xi$ and $|M_{11}(\xi)| > 0\}$ (Pázman (1986), p. 104) and the logarithm function is an increasing function. This implies that the set of all G_{rm}-optimal designs is convex and consequently, there exists a symmetric G_{rm}-optimal design since $\bar{d}_r(\xi)$ is strictly convex on $\{M_{11}(\xi)|\xi \in \Xi$ and $|M_{11}(\xi)| > 0\}$ (Pázman (1986), p. 86).

Lemma 2. *If ξ is a symmetric non-singular design on* $[0, 1]$, *then the standardized variance function $d_m(x, \xi)$ is symmetric about $x = 1/2$.*

Proof. This is immediate if the design is transformed linearly onto $[-1, 1]$, observing that the resulting variance function is symmetric with respect to 0.

Lemma 3. *The information matrix of a G_{rm}-optimal design is on the boundary of the constrained set. In other words, if ξ^* is a G_{rm}-optimal design, then*

$$|M_{22.1}(\xi^*)| = \rho^{m-r} \max_{\eta \in \Xi} |M_{22.1}(\eta)|.$$

Proof. Since $\bar{d}_r(\xi)$ is strictly convex on $\{M_{11}(\xi)|\xi \in \Xi$ and $|M_{11}(\xi)| > 0\}$ (Pázman (1986), p. 86), it is also strictly convex on $\{M_m(\xi)|\xi \in \Xi$ and $|M_{11}(\xi)| > 0\}$. If $M_m(\xi^*)$ is an interior point of the constrained set, then clearly $\bar{d}_r(\xi^*)$ is the global minimum and hence ξ^* is a unconstrained G-optimal design for the problem when $\rho = 0$. Since $f_1(x)$ is a polynomial of degree r, ξ^* is supported at $r+1$ points. This implies $|M_{22.1}(\xi^*)| = |M_m(\xi^*)|/|M_{11}(\xi^*)| = 0$ because $m > r$ by assumption. This is impossible and so $M_m(\xi^*)$ is on the boundary of the constrained set.

We now consider the problem of finding G_{1m} and G_{2m}-optimal designs (though the procedure can be generalized to find G_{rm}-designs when $m > r > 2$). Theorem 4.1 assumes a simple linear model but some protection is needed for the terms $\beta_2, \beta_3, \ldots, \beta_m$. The result extends the equivalence result found by Stigler (1971) where he showed G_{12} and D_{12}-optimal designs coincide.

Theorem 4.1. *For any ρ and m, the G_{1m} and D_{1m}-optimal designs are equivalent.*

Proof. By Lemma 1 and Property 3.2, we only need to consider designs ξ, which are symmetric and non-singular. Since $c_1 = p_1$, $c_2 = p_1(p_1 + q_1 p_2)$, we have

$$M_{11}(\xi)^{-1} = \frac{1}{p_2}\begin{pmatrix} 1+p_2 & -2 \\ -2 & 4 \end{pmatrix}$$

and the standardized variance function is $d_1(x,\xi) = (4x^2 - 4x + 1 + p_2)/p_2$.

By definition, the constrained set for the G_{1m} problem is the same constrained set as the D_{1m}-problem. Since $\bar{d}_1(\xi) = d_1(0,\xi) = 1 + 1/p_2$, the G_{1m}-optimal design minimizes $1 + 1/p_2$, or maximizes p_2, among all designs whose information matrices are in the constrained set. But Studden (1982) showed the D_{1m}-optimal design maximizes p_2 among designs from the same set; consequently their sequences of canonical moments coincide and by Property 3.1, the G_{1m} and D_{1m}-optimal designs are equivalent. Formulas for D_{1m}-optimal designs are given in Studden (1982).

We now state and prove the results for G_{2m}-optimal design.

Theorem 4.2. *Let ρ, m be given , let $R(p) = \{1 + \sqrt{1 - \rho/[4p(1-p)]}\}/2$ and let p_2 be the root bounded between $(1 \pm \sqrt{1-\rho})/2$ of the following equation:*

$$\frac{d}{dp}\{\frac{1}{p} + \frac{1-p}{pR(p)}\} = 0 \quad if \quad \rho > 0.96, \tag{4.0}$$

or

$$2p - 1 - (1-p)R(p) = 0 \quad if \quad \rho \le 0.96. \tag{4.1}$$

The G_{2m}-optimal design has canonical moments given by $p_{2i-1} = 1/2$, $i = 1, 2, \ldots, m$, p_2 (above), $p_4 = R(p_2)$, $p_{2i} = (m-i+1)/(2m-2i+1)$, $i = 3, \ldots, m-1$ and $p_{2m} = 1$.

Proof. We prove the result for $m = 3$ first. The first diagonal block of the information matrix of any design for the G_{23}-problem is 3×3 and can be expressed in terms of canonical moments. Replacing all odd canonical moments by 1/2, a direct calculation shows the standardized variance function is

$$\begin{aligned} d_2(x,\xi) = \{&(p_2 - 2p_2^2 + p_2^3 + p_2q_2p_4 + p_2^2q_2p_4) + 4(2p_2^2 - 2p_2 - p_2q_2p_4)x \\ &+4(6p_2 - 2p_2^2 + p_2q_2p_4)x^2 - 32p_2x^3 + 16p_2x^4\}/p_2^2q_2p_4. \end{aligned}$$

This function depends only on the second and fourth canonical moments of ξ and it is instructive to write $d_2(x,\xi)$ as $d_2(x,p_2,p_4)$ in what is to follow. By property 3.3, we have $|M_{22.1}(\xi)| = (p_1q_1)(p_2q_2)(p_3q_3)(p_4q_4)(p_5q_5)p_6$. Thus, when we replace the odd moments by 1/2 and set $p_6 = 1$, the problem is reduced to

$$\text{minimize } \bar{d}_2(\xi) \quad \text{subject to } p_2q_2p_4q_4 = \rho/16 \tag{4.2}$$

by Lemma 3. The nature of the constraint implies that for each p_2 satisfying

$$L = \{1 - \sqrt{1-\rho}\}/2 \le p_2 \le \{1 + \sqrt{1-\rho}\}/2 = U \tag{4.3}$$

we must have $p_4 = R(p_2)$ or $r(p_2)$ where

$$R(p_2) = \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{\rho}{4p_2q_2}} \text{ and } r(p_2) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{\rho}{4p_2q_2}}.$$

It follows that for each p_2 satisfying (4.3), it suffices to compare $\bar{d}_2(p_2, R(p_2))$ and $\bar{d}_2(p_2, r(p_2))$. Because $d_2(x,p_2,p_4)$ is a 4th order polynomial with positive coefficient of x^4 and symmetric about $x = 0.5$ (Lemma 2), this function attains its maximum at 0 and 1, and/or at $x = 0.5$; thus $d_2(x,p_2,R(p_2))$ and $d_2(x,p_2,r(p_2))$ are maximized at $x = 0$ and 1 and/or 0.5. A direct calculation shows $d_2(0,p_2,p_4) = 1+1/p_2+q_2/(p_2p_4)$ and $d_2(0.5,p_2,p_4) = 1+p_2/(q_2p_4)$. Since $R(p_2) > r(p_2)$, we have $d_2(0,p_2,R(p_2)) < d_2(0,p_2,r(p_2))$ and $d_2(0.5,p_2,R(p_2)) < d_2(0.5,p_2,r(p_2))$. Hence, $\bar{d}_2(p_2,R(p_2)) < \bar{d}_2(p_2,r(p_2))$ for all p_2 satisfying (4.3).

To find the $G_{2,3}$-optimal design, we search for p_2 in (4.3) to minimize $\bar{d}_2(p_2, R(p_2))$ or equivalently to minimize max $\{d_2(0,p_2,R(p_2)),\ d_2(0.5,p_2,R(p_2))\}$.

Claim. If $\rho \ge 0.96$, $\max\{d_2(0,p_2,R(p_2)),\ d_2(0.5,p_2,R(p_2))\} = d_2(0,p_2,R(p_2))$ for all p_2 satisfying (4.3).

Proof. Let $g(p_2) = d_2(0.5,p_2,R(p_2)) - d_2(0,p_2,R(p_2)) = 2p_2 - 1 - (1-p_2)R(p_2)$, ignoring an unimportant positive constant. For $L \le p_2 \le 1/2$, observe that

$g(p_2) \leq 0 - (1-p_2)R(p_2) \leq 0$ and so $\bar{d}_2(p_2, R(p_2)) = d_2(0, p_2, R(p_2))$. If $1/2 \leq p_2 \leq U$, $R(p_2)$ is a decreasing function and so $g(p_2)$ is an increasing function. This implies $g(U) = 2U - 1 - (1-U)R(U) = 2U - 1 - (1/2)(1-U) = (5U-3)/2 \leq 0$ if $U \leq 0.6$, or equivalently if $\rho \geq 0.96$. Thus, $\bar{d}_2(p_2, R(p_2)) = d_2(0, p_2, R(p_2))$ for all p_2 satisfying (4.3) as claimed.

The optimal p_2 is the one which minimizes $d_2(0, p_2, R(p_2))$ in the interval $[L, U]$. It can be verified that this value is the root of the equation in (4.0); its existence is guaranteed since

$$\lim_{p\to L} \frac{d}{dp}\{1/p + (1-p)/(pR(p))\} = -\infty \text{ and } \lim_{p\to U} \frac{d}{dp}\{1/p + (1-p)/(pR(p))\} = \infty.$$

On the other hand, if $\rho < 0.96$, we have $L \leq 0.4$, $U \geq 0.6$, $g(L) \leq 0$ and $g(U) \geq 0$. This implies the optimal value of p_2 is found by solving $g(p) = 0$ given in (4.1). Again, the existence of a root between U and L is assured since $g(U)g(L) = (5U-3)(5L-3)/4 < 0$. This completes the proof when $m = 3$.

To obtain $G_{2,m}$-optimal designs when $m > 3$, a direct calculation yields $|M_{22.1}(\xi)| = (p_1q_1p_2q_2p_3q_3p_4q_4p_5q_5)^{m-2}p_6^{m-2}q_6^{m-3}(p_7q_7)^{m-3}p_8^{m-3}q_8^{m-4}\cdots p_{2m}$. If we replace all odd moments by $1/2$ and set $p_{2k} = (m-k+1)/(2m-2k+1)$, $k = 3, \ldots, m-1$, $p_{2m} = 1$, the problem is reduced exactly to (4.2). Thus, the $G_{2,m}$-optimal design has the same p_2 and p_4 as G_{23} has, with $p_{2k} = (m-k+1)/(2m-2k+1)$, $k = 3, \ldots, m$. This completes the proof of the theorem.

As an example, suppose $\rho = 0.8$. The G_{23}-optimal design has canonical moments $p_1 = p_3 = p_5 = 1/2$, $p_2 = 0.6282$, $p_4 = 0.6896$ and $p_6 = 1$. Table 1 shows some G_{23}-optimal designs. All designs are supported at 0, $1-t$, t and 1 with mass w, $1/2 - w$, $1/2 - w$ and w respectively. The corresponding numbers for the D_{23}-optimal design are given in Studden (1982) and are displayed in parentheses.

Table 1. $G_{23}(D_{23})$-optimal designs for various values of ρ.

ρ	p_2	p_4	t	w
0.1	0.6634 (0.6633)	0.9712 (0.9712)	0.5691 (0.5691)	0.3284 (0.3284)
0.2	0.6600 (0.6595)	0.9408 (0.9408)	0.5988 (0.5988)	0.3231 (0.3229)
0.3	0.6562 (0.6551)	0.9085 (0.9087)	0.6225 (0.6223)	0.3171 (0.3166)
0.4	0.6520 (0.6499)	0.8739 (0.8743)	0.6434 (0.6429)	0.3104 (0.3094)
0.5	0.6474 (0.6436)	0.8363 (0.8373)	0.6628 (0.6618)	0.3028 (0.3010)
0.6	0.6422 (0.6357)	0.7946 (0.7968)	0.6816 (0.6797)	0.2939 (0.2908)
0.7	0.6360 (0.6254)	0.7470 (0.7515)	0.7006 (0.6971)	0.2831 (0.2782)
0.8	0.6282 (0.6109)	0.6896 (0.6992)	0.7208 (0.7143)	0.2691 (0.2616)
0.9	0.6168 (0.5872)	0.6096 (0.6339)	0.7454 (0.7318)	0.2476 (0.2371)
0.98	0.5675 (0.5450)	0.5212 (0.5548)	0.7606 (0.7463)	0.2031 (0.1996)

Table 1 shows that the second canonical moments of the D_{23}- and G_{23}-optimal design are different; thus by Property 3.1, G_{2m} and D_{2m}-optimal designs are no longer equivalent.

5. Discussion

To see if G_{rm}-optimal designs are robust to model misspecification, we consider the case where $r = 2$ and the true model is of degree 1, 2 and 3. Let $\xi_{G,2m}$ denote the G_{2m}-optimal design. From Property 3.3, the D-efficiencies of $\xi_{G,rm}$ for the kth degree polynomial can be expressed in terms of canonical moments. When $k = 1, 2$ and 3, these expression are respectively given by, for all $m > 2$,

$$e_1^D(\xi_{G,2m}) = p_2^{1/2}, \quad e_2^D(\xi_{G,2m}) = 3(0.25p_2^2q_2p_4)^{1/3},$$
$$e_3^D(\xi_{G,2m}) = \{(m-2)/(2m-5)\}^{1/4}e_3^D(\xi_{G,23})$$

and

$$e_3^D(\xi_{G,23}) = 0.5(5^5p_2^3q_2^2p_4^2q_4)^{1/4}.$$

A similar argument shows that the G-efficiencies of $\xi_{G,rm}$ for the simple and quadratic models are independent of m if $m > 2$, and are $e_1^G(\xi_{G,2m}) = 2p_2/(1+p_2)$ and $e_2^G(\xi_{G,2m}) = 3p_2^2q_2p_4/(p_2 - 2p_2^2 + p_2^3 + p_2q_2p_4 + p_2^2q_2p_4)$, respectively. Table 2 shows the D- and G-efficiencies of the G_{23}-optimal designs for various ρ under different model assumptions. Note that the fourth column is specific to the G_{23}-optimal designs only; the entries in the other columns apply to G_{2m}-optimal designs for any $m > 2$. Unlike the other columns, the entries in the fourth column increase as ρ increases until ρ is roughly 0.8 and it decreases after that. The rationale for this observation is that for very high values of ρ, one essentially estimates only the cubic coefficient in the cubic model and therefore the efficiency for estimating all the parameters declines. This pattern is also reflected in Table 3 of Studden (1982) for D_{rm}-optimal designs.

Table 2. Selected D and G-efficiencies of G_{rm}-optimal designs.

ρ	$e_1^D(\xi_{G,2m})$	$e_2^D(\xi_{G,2m})$	$e_3^D(\xi_{G,23})$	$e_1^G(\xi_{G,2m})$	$e_2^G(\xi_{G,2m})$
0.1	0.8145	0.9903	0.6474	0.7977	0.9902
0.2	0.8124	0.9798	0.7637	0.7951	0.9795
0.3	0.8101	0.9683	0.8377	0.7924	0.9675
0.4	0.8075	0.9556	0.8914	0.7894	0.9541
0.5	0.8046	0.9414	0.9320	0.7860	0.9387
0.6	0.8014	0.9250	0.9627	0.7821	0.9207
0.7	0.7975	0.9055	0.9846	0.7775	0.8986
0.8	0.7926	0.8806	0.9970	0.7716	0.8695
0.9	0.7854	0.8434	0.9940	0.7630	0.8241

Note that for the quadratic model, all G_{2m}-optimal designs have very high D and G-efficiencies for small ρ, and the efficiencies decrease rather slowly as ρ increases. For the cubic model, these efficiencies grow rather fast as ρ increases. In any case, all the efficiencies remain quite stable if the linear model is assumed.

We also evaluate the D and G-efficiency of $\xi_{G,2m}$ relative to the D_{2m}-optimal design $\xi_{D,rm}$ when the assumed polynomial model is of degree k. For any two designs ξ_1 and ξ_2, these relative D-efficiencies are defined by $\{|M_k(\xi_1)|/|M_k(\xi_2)|\}^{1/(k+1)}$ and $\bar{d}(\xi_2)/\bar{d}(\xi_1)$. Since the canonical moments of $\xi_{G,2m}$ and $\xi_{D,2m}$ differ only in p_2 and p_4, the above for these designs are independent of m for $k = 1$ and 2. Our calculation showed these efficiencies range from 0.99 to 1.05 when $0.1 \leq \rho \leq 0.9$, suggesting that the G_{2m} and D_{2m}-optimal designs have almost identical performances if we have simple, quadratic or cubic models. This means the D_{2m} and the G_{2m}-optimal designs are roughly optimal for both parameter estimation and response surface estimation, and at the same time offer some protection against model miss-specification for low order polynomial models.

Finally, we compare our designs with the constrained optimal designs obtained by Pukelsheim and Rosenberger (1993). We follow their notation and let Θ_3 denote the 'cubic' coefficient, let $\Theta_{(A)}$ denote the vector of parameters in the quadratic model and let $\Theta_{(B)}^T = (\Theta_{(A)}^T, \Theta_3)$. They reported the efficiencies of two D-optimal designs for $\Theta_{(A)}$, (a) and (b) below, which guarantee a 50% efficiency for estimating Θ_3. In our case, the G_{23}-optimal design with $\rho = 0.5$ is symmetrically supported on ± 1 and ± 0.3256 with mass 0.3208 at 1 and 0.1972 at 0.3256 respectively. Table 3, row (c), shows this design performs similarly to theirs and has the same numerical efficiencies with the D_{23}-optimal design.

Table 3. Comparing some properties of G_{rm}-optimal designs with those obtained by Pukelsheim and Rosenberger (1993).

		Efficiencies		
	design description	Θ_3	$\Theta_{(B)}$	$\Theta_{(A)}$
(a)	D-optimal for $\Theta_{(A)}$, 50% efficient for Θ_3 on $[-1, 1]$	0.50	0.93	0.94
(b)	D-optimal for $\Theta_{(A)}$, 50% for Θ_3 on ± 1, $\pm 1/2$, 0	0.50	0.92	0.93
(c)	G-optimal for $\Theta_{(A)}$, 50% for Θ_3 on $[-1, 1]$	0.50	0.93	0.94

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