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MAX CHART: COMBINING X-BAR CHART AND S CHART

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Abstract: Control chart techniques have been widely used in industries to monitor a process in quality improvement. Whenever we deal with variables data, we usually employ a combination of X-bar chart and R chart (or S chart) to monitor both the center and the spread of the process. In this paper, we propose a simple alternative, that is, we design a single chart to monitor both the center and the spread for variables data. When compared with the combination of X-bar chart and S chart, the proposed chart is shown to be just as effective. An example is given to show how to use this new chart.

Key words and phrases: Average run length, control chart, variables data.

1. Introduction

Ever since Shewhart introduced control charts, it has become a common practice for practitioners to use various control charts to monitor different processes. When we deal with variables data, the control chart technique usually employs a chart (such as an X-bar chart) to monitor the process center and a chart (such as an R or an S chart) to monitor the process spread. Efforts have been made to use a single control chart to monitor both the process center and the process spread at the same time (see, for example, Chan, Cheng and Spiring (1990), Domangue and Patch (1991), Chao and Cheng (1996), and the references therein). A major difficulty when designing a single chart is to keep the chart simple, and to be able to indicate clearly whether the process mean is out of control, or the process variability is out of control, or both when an out-of-control signal is observed. The charts proposed in Domangue and Patch (1991) are sensitive to changes in the mean and/or the variability, but cannot indicate which change has actually occurred; the chart proposed in Chan, Cheng and Spiring (1990) can indicate which change has actually occurred, but it requires plotting *two* types of quantities separately in a chart; therefore, it is not simple; and the semicircle chart in Chao and Cheng (1996) is essentially a 2-dimensional chart which also loses track of the time sequence of the plotted points. In this paper, we propose a new statistic which can measure shifts in the center and/or the spread for variables data. We investigate the sampling behavior of the proposed statistic and provide a procedure for constructing a new single chart which, to a large degree, satisfies the criteria discussed above.

2. The Statistic

Let X denote a certain characteristic of a process, let μ denote the process mean, and let σ denote the process standard deviation. Let X_{ij}, $i = 1, 2, 3, \ldots$, and $j = 1, \ldots, n_i$, be measurements of X arranged in groups of size n_i with i indexing the group number. We suppose that for each i, $X_{i1}, \ldots, X_{in_i}$ is a random sample from a normal distribution with mean $\mu + a\sigma$ and standard deviation $b\sigma$, where $a = 0$ and $b = 1$ indicate that the process is in control, otherwise, the process has changed or drifted. Let $\bar{X}_i = (X_{i1} + \cdots + X_{in_i})/n_i$ be the ith sample mean, and let $S_i^2 = \sum_{j=1}^{n_i}(X_{ij} - \bar{X}_i)^2/(n_i - 1)$ be the ith sample variance. The X-bar chart and the S chart are directly based on $\bar{X}_i$ and S_i. For our purpose, we define

$$U_i = \frac{(\bar{X}_i - \mu)}{\sigma/\sqrt{n_i}} \tag{1}$$

$$V_i = \Phi^{-1}\Big\{H\Big(\frac{(n_i - 1)S_i^2}{\sigma^2}; n_i - 1\Big)\Big\}, \tag{2}$$

where $\Phi(z) = P(Z \leq z)$ for $Z \sim N(0, 1)$, the standard normal distribution, $\Phi^{-1}(\cdot)$ is the inverse function of $\Phi(\cdot)$, and $H(w; \nu) = P(W \leq w \mid \nu)$ for $W \sim \chi^2_\nu$, the chi-squared distribution with ν degrees of freedom.

It is known that U_i and V_i are independent because $\bar{X}_i$ and S_i are, and when $a = 0$ and $b = 1$, we have both $U_i \sim N(0, 1)$ and $V_i \sim N(0, 1)$. The advantage of transforming $\bar{X}_i$ to U_i and S_i to V_i is twofold: (i) the distributions of U_i and V_i are both independent of the sample size n_i when $a = 0$ and $b = 1$, therefore we can handle the case of variable sample size easily; (ii) both U_i and V_i have the same distribution so that we can construct a single chart to monitor both the process center and the process spread. Specifically, we define a statistic $M(n_i)$ by

$$M(n_i) = \max\{|U_i|, |V_i|\}. \tag{3}$$

The statistic $M(n_i)$ will be large when the process center is drifted away from μ and/or when the process variability is increased or decreased. On the other hand, the statistic $M(n_i)$ will be small when the process center and process variability stay close to their respective targets.

3. The Distribution of $M(n_i)$

Let $\chi^2_{\gamma,\nu}$ satisfy $P(\chi^2_\nu \leq \chi^2_{\gamma,\nu}) = \gamma$, where $\gamma \in (0, 1)$. The distribution of $M(n_i)$ is found, for any $y > 0$, to be

$$\begin{aligned} F(y; n_i, a, b) &= P(M(n_i) \leq y) = P(|U_i| \leq y,\ |V_i| \leq y) \\ &= \Big\{\Phi\Big(\frac{y}{b} - \frac{a}{b}\sqrt{n_i}\Big) - \Phi\Big(-\frac{y}{b} - \frac{a}{b}\sqrt{n_i}\Big)\Big\} \\ &\quad \times \Big\{H\Big(\frac{\chi^2_{\Phi(y),n_i-1}}{b^2}; n_i - 1\Big) - H\Big(\frac{\chi^2_{\Phi(-y),n_i-1}}{b^2}; n_i - 1\Big)\Big\}. \end{aligned} \tag{4}$$

4. The *Max* Chart

We use the statistic $M(n_i)$ to construct a new control chart. Because $M(n_i)$ is the maximum of two statistics, we name this new chart a *Max* chart. Let $a = 0$ and $b = 1$ in equation (4). We obtain

$$F(y; n_i, 0, 1) = \Big\{\Phi(y) - \Phi(-y)\Big\}^2 = P(\chi_1^2 \le y^2)^2. \tag{5}$$

Therefore, for $F(y; n_i, 0, 1) = 1 - \alpha$ to hold, we must have $y = \{\chi^2_{\sqrt{1-\alpha},1}\}^{1/2}$. The center line (CL) and the upper control limits (UCL) of the *Max* chart are then easily determined for various values of Type I Error probability α; the results are given in Table 1.

Table 1. Center line (CL) and upper control limits (UCL) of the *Max* chart for various values of type I error probability α.

α	0.5000	α	0.0054	0.0027	0.00135
CL	1.0518	UCL	2.9996	3.2049	3.3994

The following procedure can be used to set up a *Max* chart:

Case 1. Both μ and σ are known

1. For each sample, compute U_i, V_i and $M(n_i)$.
2. Find the center line CL and the upper control limit UCL from Table 1 for the desired α, and set up a chart with CL and UCL marked.
3. When $M(n_i) \le UCL$, plot a dot against i. When $M(n_i) > UCL$, check both $|U_i|$ and $|V_i|$ against UCL. If $|U_i|$ alone is greater than UCL, plot "m+" against i when $U_i > 0$ to indicate the process center is up, and plot "m−" against i when $U_i < 0$ to indicate the process center is down. If $|V_i|$ alone is greater than UCL, plot "v+" against i when $V_i > 0$ to indicate the process variability is up, and plot "v−" against i when $V_i < 0$ to indicate the process variability is down. If both $|U_i|$ and $|V_i|$ are greater than UCL, plot "++", "+−", "−+", or "−−" according to $U_i > 0$ and $V_i > 0$, $U_i > 0$ and $V_i < 0$, $U_i < 0$ and $V_i > 0$, or $U_i < 0$ and $V_i < 0$, with similar interpretations.
4. Examine the cause(s) for each out of control point.

Case 2. At least one of μ and σ is unknown

1. Estimate the unknown process parameter(s). For example, if μ is unknown, use the grand average $\bar{\bar{x}}$ of the (preliminary) data to estimate μ; if σ is unknown, use $\bar{R}/d_2$ or $\bar{S}/c_4$ to estimate σ, where $\bar{R} = (R_1 + \cdots + R_m)/m$ is the average of the sample ranges, $\bar{S} = (S_1 + \cdots + S_m)/m$ is the average of the sample standard deviations, and $d_2 = d_2(\bar{n})$ and $c_4 = c_4(\bar{n})$ are known constants with $\bar{n} = [(n_1 + \cdots + n_m)/m]$, where $[y]$ denotes the largest integer smaller than or equal to y (see, for example, Duncan (1986) and Montgomery (1996)).

2. Follow the steps described in Case 1.

5. Comparison With Other Charts

Of the commonly used control charts, such as the X-bar, R, S, CUSUM, and EWMA, few are *designed* to monitor both process center and process spread at the same time. Some of the above mentioned charts can detect changes in process center and/or spread, but they usually cannot indicate which is which. Therefore, it is impossible to compare the Max chart with the existing charts on a completely equal footing.

Table 2. The Average Run Length (ARL) of the Max chart and the chart based on a combination of the X-bar chart and the S chart.

		Max Chart				Combined $\bar{X}$ and S Chart			
		a				a			
n	b	0.0	0.5	1.0	2.0	0.0	0.5	1.0	2.0
	0.25	13.2	13.2	13.2	1.9	13.2	13.2	13.2	1.9
	0.50	95.0	94.7	30.2	2.0	95.1	94.8	30.3	2.0
4	1.00	185.2	39.3	6.2	2.0	185.4	39.3	6.2	2.0
	1.50	8.6	6.2	3.2	1.9	8.6	6.2	3.2	1.9
	2.00	2.7	2.5	2.0	1.6	2.7	2.5	2.0	1.6
	0.25	4.8	4.8	4.8	1.1	4.8	4.8	4.8	1.1
	0.50	51.3	51.1	12.3	1.3	51.4	51.2	12.3	1.3
5	1.00	185.2	30.7	4.5	1.6	185.4	30.7	4.5	1.6
	1.50	7.3	5.2	2.7	1.6	7.3	5.2	2.7	1.6
	2.00	2.3	2.1	1.7	1.4	2.3	2.1	1.7	1.4
	0.25	1.5	1.5	1.5	1.0	1.5	1.5	1.5	1.0
	0.50	18.4	18.2	3.6	1.0	18.4	18.3	3.6	1.0
7	1.00	185.2	20.2	2.8	1.2	185.4	20.3	2.8	1.2
	1.50	5.6	3.9	2.0	1.3	5.6	3.9	2.0	1.3
	2.00	1.8	1.7	1.4	1.2	1.8	1.7	1.4	1.2
	0.25	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
	0.50	6.1	6.1	1.5	1.0	6.1	6.1	1.5	1.0
10	1.00	185.2	12.4	1.8	1.0	185.4	12.4	1.8	1.0
	1.50	4.0	2.9	1.6	1.1	4.0	2.9	1.6	1.1
	2.00	1.4	1.3	1.2	1.1	1.4	1.3	1.2	1.1

On the other hand, since our goal is to provide a simple alternative to the use of combinations of the X-bar chart with the S or R chart, we proceed to compare the Max chart with a combination of the X-bar chart and the S chart. We follow the tradition to use the 3-sigma X-bar chart, which has a Type I Error probability 0.0027 when the process is in control. For the S chart, we use the version with probability control limits, where a probability 0.00135 is assigned to each tail so that the Type I Error probability is also 0.0027 when the process is in control. With this combination of the X-bar chart and the S chart, the comparable Max chart should have a Type I Error probability equal to $1-(1-0.0027)^2 = 0.0053927 \approx 0.0054$. For various changes in center alone, in spread alone, and in both center and spread, we have calculated the average run length (ARL) for the Max chart and the (X-bar, S) chart combination; some representative results are given in Table 2.

It is not surprising to see from Table 2 that the performance of the Max chart is nearly the same as that of the (X-bar, S) chart combination. However, the Max chart has effectively combined the X-bar chart and the S chart into one single chart.

6. An Example

DeVor, Chang and Sutherland (1992), page 165, Table 6.1 contains a data set consisting of the measurements of the inside diameter of the cylinder bores in an engine block. The measurements are made to 1/10,000 of an inch. Samples of size $n = 5$ are taken roughly every half hour, and the first 35 samples are given in Table 3. The actual measurements are of the form 3.5205, 3.5202, 3.5204, and so on. The entries given in Table 3 provide the last three digits in the measurements.

Table 3. Cylinder diameter data.

Sample i	X_{i1}	X_{i2}	X_{i3}	X_{i4}	X_{i5}	Sample i	X_{i1}	X_{i2}	X_{i3}	X_{i4}	X_{i5}
1	205	202	204	207	205	19	207	206	194	197	201
2	202	196	201	198	202	20	200	204	198	199	199
3	201	202	199	197	196	21	203	200	204	199	200
4	205	203	196	201	197	22	196	203	197	201	194
5	199	196	201	200	195	23	197	199	203	200	196
6	203	198	192	217	196	24	201	197	196	199	207
7	202	202	198	203	202	25	204	196	201	199	197
8	197	196	196	200	204	26	206	206	199	200	203
9	199	200	204	196	202	27	204	203	199	199	197
10	202	196	204	195	197	28	199	201	201	194	200
11	205	204	202	208	205	29	201	196	197	204	200
12	200	201	199	200	201	30	203	206	201	196	201
13	205	196	201	197	198	31	203	197	199	197	201
14	202	199	200	198	200	32	197	194	199	200	199
15	200	200	201	205	201	33	200	201	200	197	200
16	201	187	209	202	200	34	199	199	201	201	201
17	202	202	204	198	203	35	200	204	197	197	199
18	201	198	204	201	201						

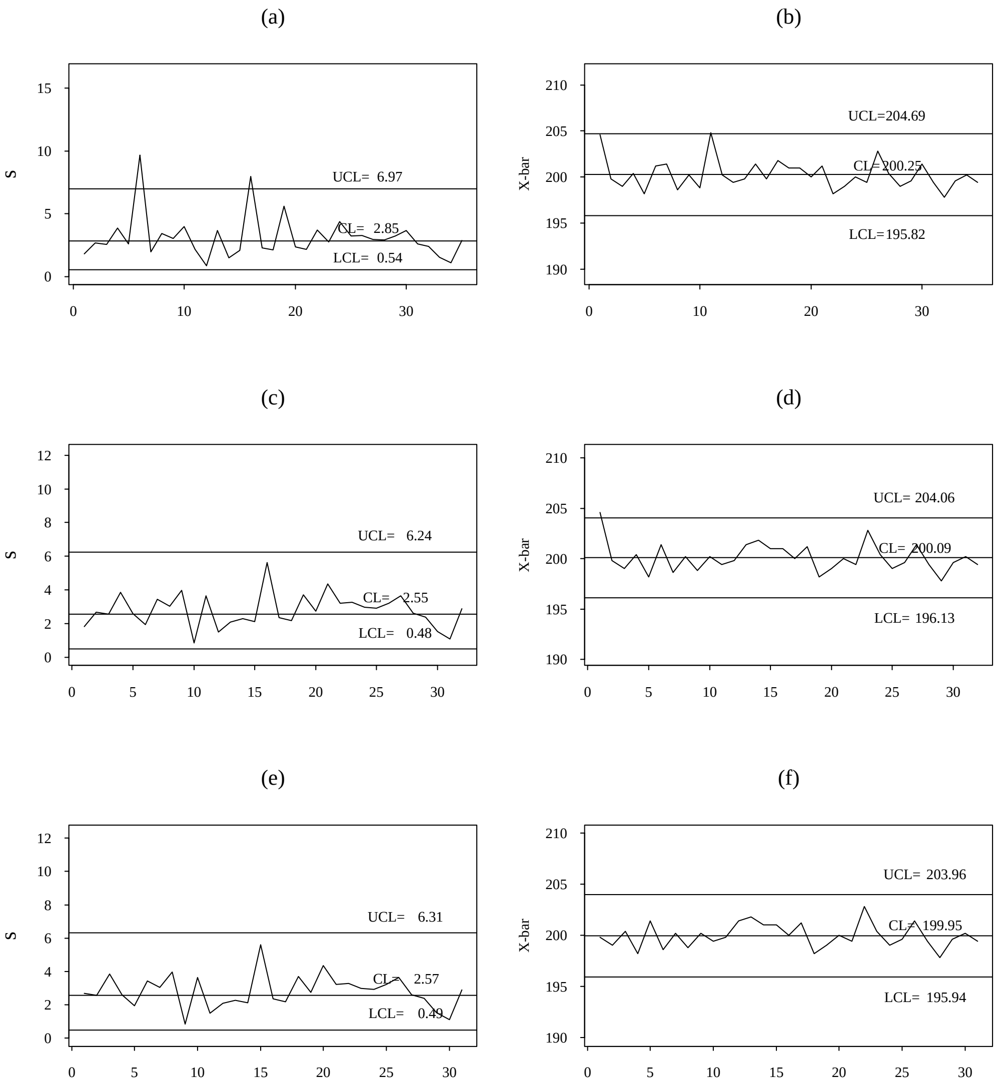

Figure 1. The S charts and the X-bar charts for the cylinder diameter data.

Since μ and σ are unknown, we estimate μ by $\bar{\bar{x}}= 200.25$ and estimate σ by $\bar{S}/c_4 = 3.31$. Plots (a) and (b) in Figure 1 are the S chart and the X-bar chart for the data in Table 3, each with a Type I Error probability 0.0027. There are two points (samples 6 and 16) exceeding the upper control limit in the S chart, and one point (sample 11) exceeding the upper control limit in the X-bar chart. According to DeVor, Chang and Sutherland (1992), samples 6 and 16 corresponded to the time when the regular operator was absent, and a relief operator, who was less experienced, was in charge of the production line, and sample 11 occurred at 1:00 P.M., corresponding roughly to the startup of the production line directly after the lunch hour, when the machine was down for tool changing. When samples 6, 11 and 16 are removed from the data, we have $\bar{\bar{x}}= 200.09$ and $\bar{S}/c_4 = 2.96$, and the related S chart and X-bar chart are shown as plots (c) and (d) in Figure 1. This time, there is one point (sample 1 in the original data set) exceeding the upper control limit in the X-bar chart. An investigation reveals that this sample occurred at 8:00 A.M., corresponding roughly to the startup of the production line in the morning, when the machine was cold. Once the machine warmed up (about 10 minutes), the problem seemed to disappear. When sample 1 is removed from the data, we re-estimate μ by $\bar{\bar{x}}= 199.95$ and re-estimate σ by $\bar{S}/c_4 = 2.99$. The new S chart and X-bar chart are given as plots (e) and (f) in Figure 1. There is no point falling outside the control limits.

Now we use the Max chart to monitor the cylinder production process. Based on the estimates $\bar{\bar{x}}= 200.25$ and $\bar{S}/c_4 = 3.31$, our first Max chart with a Type I Error probability 0.0054 is shown as plot (a) in Figure 2. There are three points (samples 6, 11, and 16) exceeding the upper control limit, where samples 6 and 16 are related to the process variability and sample 11 is related to the process mean. When these three samples are removed, we obtain $\bar{\bar{x}}= 200.09$ and $\bar{S}/c_4 = 2.96$, and our second Max chart is given as plot (b) in Figure 2. This time, there is one point (sample 1 in the original data set) exceeding the control limit, and this sample is related to the process mean. When this sample is removed, we estimate μ by $\bar{\bar{x}}= 199.95$ and estimate σ by $\bar{S}/c_4 = 2.99$, and our third Max chart is displayed as plot (c) in Figure 2. As expected, there is no point falling outside the control limit.

7. Discussion

The Max chart is essentially equivalent to a combination of the X-bar chart and the S chart. The main advantage of using the Max chart is that one can monitor both the process center and the process spread by looking at one chart.

To implement the Max chart, it is best to use a computer to speed up the computation and graphing.

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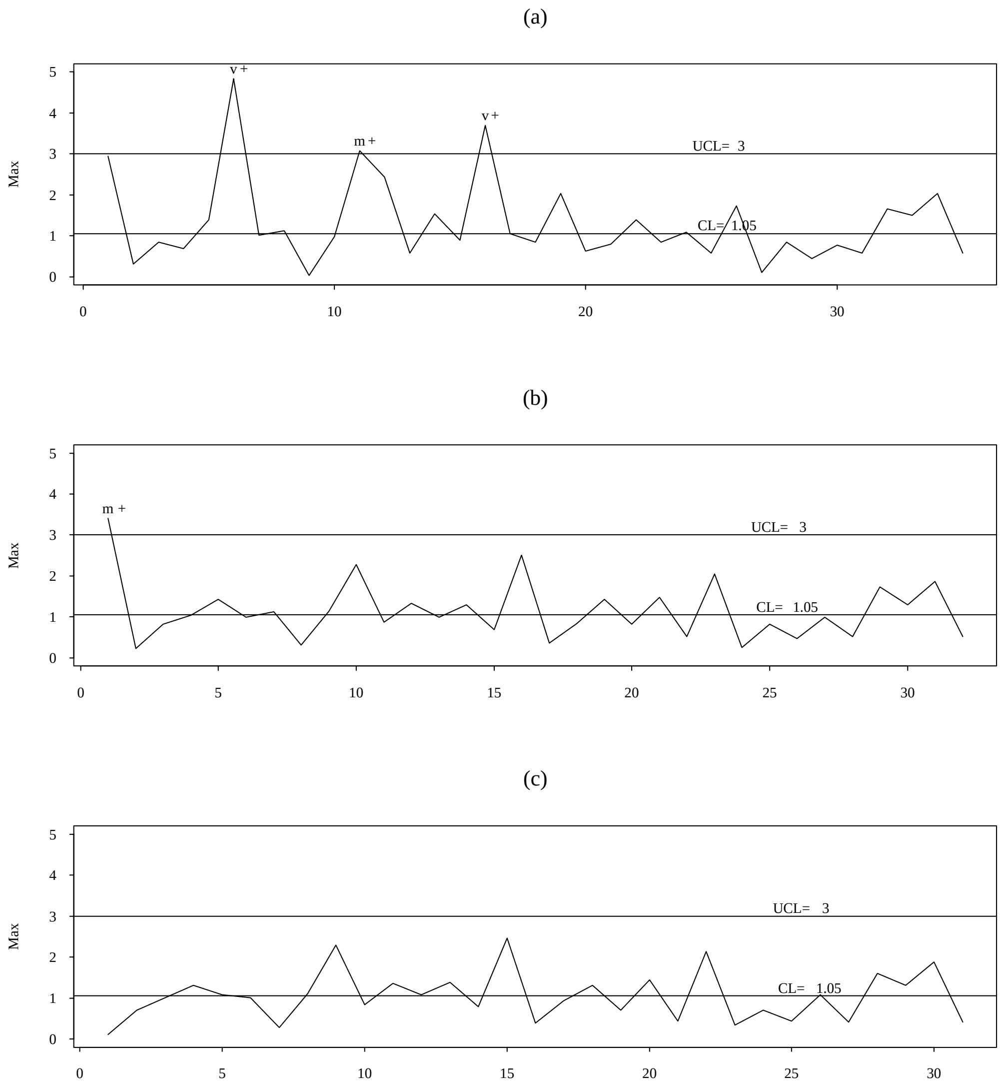

Figure 2. The Max charts for the cylinder diameter data.

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