Statistica Sinica **6**(1996), 245-257

BOUNDS AND ESTIMATORS OF A BASIC CONSTANT IN EXTREME VALUE THEORY OF GAUSSIAN PROCESSES

Qi-Man Shao

National University of Singapore

Abstract. Let H_α be the constant in the extremal theorem for Gaussian processes. In this note a lower and upper bound as well as two statistical estimators of H_α are given. Simulation results are also discussed. As a by-product of the proof, a more precise bound on the small ball probability of fractional Brownian motion is obtained.

Key words and phrases: Gaussian process, extremal theorem, small ball probability.

1. Introduction

Let $\{\xi(t), t \geq 0\}$ be a stationary Gaussian process with mean zero, variance one and covariance $r(\tau)$ satisfying

$$r(\tau) = 1 - C|\tau|^\alpha + o(|\tau|^\alpha) \quad \text{as} \quad \tau \to 0, \tag{1.1}$$

here, and in the sequel, α is a constant, $0 < \alpha \leq 2$, and C is a positive constant. Let $\{Y(t), t \geq 0\}$ be a non-stationary Gaussian process with mean $-|t|^\alpha$ and covariance $|s|^\alpha + |t|^\alpha - |t-s|^\alpha$. Let

$$H_\alpha = \lim_{N\to\infty} N^{-1} \int_0^\infty e^x \, P\Big(\sup_{0\leq t\leq N} Y(t) > x\Big)\, dx. \tag{1.2}$$

It is well-known (see Pickands (1969a,b), Berman (1971), Qualls and Watanabe (1972) and Leadbetter, Lindgren and Rootzen (1983)) that for each fixed $h > 0$ such that $\sup_{\varepsilon\leq t\leq h} r(t) < 1$ for all $\varepsilon > 0$,

$$\lim_{u\to\infty} \frac{1}{u^{\frac{2}{\alpha}-1}\phi(u)} P\left(\sup_{0\leq t\leq h} \xi(t) > u\right) = h\, C^{1/\alpha}\, H_\alpha, \tag{1.3}$$

where $\phi(u)$ is the standard normal density function and H_α is defined as in (1.2).

Based on (1.3), one can prove that (cf. Leadbetter, Lindgren and Rootzen (1983), p.237) if $r(t)\ln t \to 0$ as $t \to \infty$, then

$$P\left(a_T\Big(\sup_{0\leq t\leq T} \xi(t) - b_T\Big) \leq x\right) \longrightarrow \exp\left(-e^{-x}\right)$$

as $T \to \infty$, where $a_T = (2\ln T)^{1/2}$ and

$$b_T = (2\ln T)^{1/2} + \frac{1}{2}(\ln T)^{-1/2}\left(\frac{2-\alpha}{2\alpha}\ln\ln T + \ln\left(C^{1/\alpha}H_\alpha(2\pi)^{-1/2}2^{(2-\alpha)/2\alpha}\right)\right).$$

Clearly, H_α is a very important parameter in the above extremal theorem. H_α appears also in the Erdős-Révész type law of the iterated logarithm (Shao (1992)). Unfortunately, the exact value of H_α is unknown except for two special cases, $H_1 = 1$ and $H_2 = 1/\sqrt{\pi}$. The main aim of this note is to give a lower and upper bound of H_α.

Theorem 1. *We have*

$$5.2^{-1/\alpha}\,0.625 \le H_\alpha \le (\alpha\, e/\sqrt{\pi})^{2/\alpha} \quad \text{if} \quad 1 \le \alpha \le 2 \tag{1.4}$$

and

$$\begin{aligned}(\alpha/4)^{1/\alpha}\left(1 - e^{-1/\alpha}(1+1/\alpha)\right) &\le H_\alpha \\ \le \left(\sqrt{\alpha}\left(0.77\sqrt{\alpha} + 2.41(8.8 - \alpha\,\ln(0.4+2.5/\alpha))^{1/2}\right)\right)^{2/\alpha}&\end{aligned} \tag{1.5}$$

if $0 < \alpha < 1$. *In particular, we have*

$$0.12 \le H_\alpha \le 3.1 \quad \text{if} \quad 1 \le \alpha \le 2, \tag{1.6}$$

$$\lim_{\alpha\to 0}\frac{\alpha\,\ln H_\alpha}{\ln\alpha} = 1. \tag{1.7}$$

We will prove this theorem in the next section. In Section 3 we propose two estimators of H_α and present some simulation results. As a by-product of the proof of Theorem 1, we will establish a more precise bound on the small ball probability of fractional Bownian motion.

2. Proofs

One may note that the value of H_α depends only on α. Therefore, to estimate H_α, one can choose a special Gaussian process $\{\xi(t),\, t \ge 0\}$ provided that (1.1) is satisfied.

Throughout this section let $0 < \alpha < 2$, $\{X(t),\, t \ge 0\}$ be a fractional Brownian motion of order α (cf., Mandelbrot and Van Ness (1968)), i.e., a centered Gaussian process with stationary increments and variance $EX^2(t) = t^\alpha$. Set

$$\xi(t) = X(e^t)/e^{\alpha t/2}, \qquad M(h) = \sup_{0\le t\le h} X(t), \qquad M = M(1), \quad t,\ h \ge 0. \tag{2.1}$$

It is easy to show that $E\xi(t) = 0, \quad E\xi^2(t) = 1,$

$$E\xi(t)\xi(s) = \frac{1}{2}\left(e^{\alpha(t-s)/2} + e^{\alpha(s-t)/2} - |e^{(t-s)/2} - e^{(s-t)/2}|^{\alpha}\right)$$

and

$$r(t) = 1 - \frac{1}{2}|t|^\alpha + o(|t|^\alpha) \quad \text{as} \quad t \to 0.$$

Hence, $\{\xi(t),\, t \geq 0\}$ is a stationary Gaussian process with correlation function $r(t)$ satisfying (1.1) with $C = 1/2$.

We start with the following lemmas, some of which are of independent interest.

Lemma 1. *If U and V are centered Gaussian processes on a parameter set T such that $EU^2(t) = EV^2(t)$ and*

$$EU(t)U(s) \geq EV(t)V(s) \text{ for all } s, t \in T,$$

then for all real x

$$P\left(\sup_{t\in T} U(t) > x\right) \leq P\left(\sup_{t\in T} V(t) > x\right).$$

This is the well-known Slepian (1962) lemma.

Lemma 2. *If U is a centered Gaussian process on a parameter set T, then for all $x > 0$*

$$P\left(\sup_{t\in T} U(t) > x + \operatorname{med}\sup_{t\in T} U(t)\right) \leq \Psi(x).$$

Here and in the sequel, $\Psi(x) = (2\pi)^{-\frac{1}{2}}\int_x^\infty e^{-t^2/2}\,dt$ and med *denotes the median.*

This elegant result was due to Borell (1975).

Lemma 3. *Let $\{X(t),\, t \geq 0\}$ be a fractional Brownian motion of order α, $0 < \alpha < 2$. Then*

$$\limsup_{h\to 0} h^{-\alpha/2} \operatorname{med} \sup_{0\leq t\leq h} X(e^t) \leq E \sup_{0\leq t\leq 1} X(t). \tag{2.2}$$

Proof. Let η be a standard normal random variable independent of $\{X(t),\, t \geq 0\}$. Put

$$\eta(t) = X(e^t - 1) + \left(e^{t\alpha} - (e^t - 1)^\alpha\right)^{1/2} \eta, \quad t \geq 0.$$

Clearly, we have $E\eta^2(t) = EX^2(e^t)$ and for any $0 \leq s \leq t \leq 1$

$$\begin{aligned}
&E\eta(t)\eta(s) \\
&= EX(e^t-1)X(e^s-1) + (e^{t\alpha} - (e^t-1)^\alpha)^{1/2}(e^{s\alpha} - (e^s-1)^\alpha)^{1/2} \\
&= \frac{1}{2}\Big((e^t-1)^\alpha + (e^s-1)^\alpha - (e^t-e^s)^\alpha + 2(e^{t\alpha} - (e^t-1)^\alpha)^{1/2}(e^{s\alpha} - (e^s-1)^\alpha)^{1/2}\Big) \\
&\leq \frac{1}{2}\left(e^{t\alpha} + e^{s\alpha} - (e^t - e^s)^\alpha\right) \\
&= EX(e^t)X(e^s).
\end{aligned}$$

Hence, by the Slepian lemma, we have, for any given $h > 0$, $\tau > E\sup_{0\le t\le 1} X(t)$

$$P\left(\sup_{0\le t\le h} X(e^t) > \tau h^{\alpha/2}\right) \le P\left(\sup_{0\le t\le h} \eta(t) > \tau h^{\alpha/2}\right)$$
$$= P\left(\sup_{0\le t\le h} X(e^t-1) + (e^{t\alpha}-(e^t-1)^\alpha)^{1/2}\,\eta > \tau h^{\alpha/2}\right).$$

Define

$$a_1 = a_1(\alpha,h) = I\{\alpha<1\} + (e^{h\alpha}-(e^h-1)^\alpha)I\{\alpha\ge 1\},$$
$$a_2 = a_2(\alpha,h) = I\{\alpha\ge 1\} + (e^{h\alpha}-(e^h-1)^\alpha)I\{\alpha< 1\},$$

where I is the indicator function. Clearly,

$$a_1 \ge a_2, \qquad \lim_{h\to 0} a_1 = \lim_{h\to 0} a_2 = 1. \tag{2.3}$$

Noting that $e^{t\alpha}-(e^t-1)^\alpha$ is an increasing function if $\alpha > 1$ and non-increasing if $0<\alpha\le 1$, we get

$$P\Big(\sup_{0\le t\le h} X(e^t-1) + (e^{t\alpha}-(e^t-1)^\alpha)^{1/2}\,\eta > \tau h^{\alpha/2}\Big)$$
$$\le P\Big(\eta\ge 0, \sup_{0\le t\le h} X(e^t-1)+\eta\, a_1 > \tau h^{\alpha/2}\Big) + P\Big(\eta<0, \sup_{0\le t\le h} X(e^t-1)+\eta\, a_2 > \tau h^{\alpha/2}\Big)$$
$$= P\Big(\eta\ge 0, (e^h-1)^{\alpha/2}M+\eta\, a_1 > \tau h^{\alpha/2}\Big) + P\Big(\eta<0, (e^h-1)^{\alpha/2}M+\eta\, a_2 > \tau h^{\alpha/2}\Big)$$
$$= I_1(h,\tau) + I_2(h,\tau),$$

where, in the above equality we used the fact that $\sup_{0\le t\le 1} X(a\,t)$ and $a^{\alpha/2}\sup_{0\le t\le 1} X(t)$ have the same distribution for any given $a>0$, and M is defined as in (2.1). Note that

$$I_1(h,\tau) \le P(\eta a_1 \ge \tau h^{\alpha/2}) + P\left(0\le \eta\le \tau h^{\alpha/2}/a_1, (e^h-1)^{\alpha/2}M > \tau h^{\alpha/2} - \eta\, a_1\right)$$
$$= \frac{1}{2} - P\left(0\le\eta\le \tau h^{\alpha/2}/a_1\right)$$
$$+ \frac{1}{\sqrt{2\pi}}\int_0^{\tau h^{\alpha/2}/a_1} e^{-x^2/2} P\left(M > (\tau h^{\alpha/2} - x a_1)(e^h-1)^{-\alpha/2}\right)\,dx$$
$$\le \frac{1}{2} - P\left(0\le\eta\le\tau h^{\alpha/2}/a_1\right) + \frac{(e^h-1)^{\alpha/2}}{a_1\sqrt{2\pi}}\int_0^{\tau h^{\alpha/2}(e^h-1)^{-\alpha/2}} P(M>x)\,dx$$

and

$$I_2(h,\tau) \le \frac{(e^h-1)^{\alpha/2}}{a_2\sqrt{2\pi}}\int_{\tau h^{\alpha/2}(e^h-1)^{-\alpha/2}}^{\infty} P(M>x)\,dx.$$

Therefore, by (2.3)

$$\begin{aligned} I_1(h,\tau)+I_2(h,\tau) &\le \frac{1}{2}-P\left(0\le \eta\le \tau h^{\alpha/2}/a_1\right)+\frac{(e^h-1)^{\alpha/2}}{a_2\sqrt{2\pi}}\int_0^\infty P\left(M>x\right)dx \\ &= \frac{1}{2}-P\left(0\le\eta\le\tau h^{\alpha/2}/a_1\right)+\frac{(e^h-1)^{\alpha/2}}{a_2\sqrt{2\pi}}E(M). \end{aligned}$$

It is easy to show that, as $h\to 0$,

$$P\left(0\le \eta\le \tau h^{\alpha/2}/a_1\right)\sim \frac{\tau\, h^{\alpha/2}}{\sqrt{2\pi}} \quad\text{and}\quad \frac{(e^h-1)^{\alpha/2}}{a_2\sqrt{2\pi}}E(M)\sim \frac{h^{\alpha/2}\,E(M)}{\sqrt{2\pi}}.$$

Putting the above inequalities together, for any given $\tau > E(M)$ we have

$$I_1(h,\tau)+I_2(h,\tau)\le \frac{1}{2}$$

and hence

$$\operatorname{med}\sup_{0\le t\le h} X(e^t)\le \tau h^{\alpha/2}$$

provided that h is sufficiently small, as desired.

Lemma 4. *Let H_α be defined as in (1.2) and $\{X(t),\, t\ge 0\}$ a fractional Brownian motion of order α. Then, we have*

$$H_\alpha \le \left(\frac{e\,\alpha}{\sqrt{2}}E\sup_{0\le t\le 1} X(t)\right)^{2/\alpha}. \tag{2.4}$$

Proof. Let $\xi(t)$ be defined as in (2.1). Put

$$\tau = 2/(\alpha\, E\sup_{0\le t\le 1} X(t)),\quad h=h(u)=(\tau/u)^{2/\alpha},\ u\ge 1. \tag{2.5}$$

Using the Borell inequality, we obtain, for $u > \operatorname{med}\sup_{0\le t\le 1}\xi(t)$,

$$\begin{aligned} &P\Big(\sup_{0\le t\le 1}\xi(t)>u\Big) \\ &\le (1+1/h)P\Big(\sup_{0\le t\le h}\xi(t)>u\Big)\le (1+1/h)\Psi\Big(u-\operatorname{med}\sup_{0\le t\le h}\xi(t)\Big). \end{aligned}$$

From (1.3), (2.2) and (2.5) it follows that

$$
\begin{aligned}
H_\alpha &= \lim_{u\to\infty} \frac{2^{1/\alpha}}{u^{\frac{2}{\alpha}-1}\phi(u)} P\left(\sup_{0\le t\le 1} \xi(t) > u\right) \\
&\le \lim_{u\to\infty} \frac{2^{1/\alpha}}{u^{\frac{2}{\alpha}-1}\phi(u)} (1+1/h)\Psi\left(u - \operatorname{med}\sup_{0\le t\le h} \xi(t)\right) \\
&\le \frac{2^{1/\alpha}}{\tau^{2/\alpha}} \exp\left(u \operatorname{med}\sup_{0\le t\le h} \xi(t)\right) \\
&\le \frac{2^{1/\alpha}}{\tau^{2/\alpha}} \exp\left(u\, h^{\alpha/2}\, h^{-\alpha/2} \operatorname{med}\sup_{0\le t\le h} X(e^t)\right) \\
&\le \frac{2^{1/\alpha}}{\tau^{2/\alpha}} \exp\left(\tau\, E \sup_{0\le t\le 1} X(t)\right) \\
&= \left(\frac{e\,\alpha}{\sqrt{2}} E \sup_{0\le t\le 1} X(t)\right)^{2/\alpha}.
\end{aligned}
$$

This proves (2.4).

Lemma 5. *If* $1 \le \alpha < 2$, *then*

$$E \sup_{0\le t\le 1} X(t) \le (2/\pi)^{1/2}. \tag{2.6}$$

Proof. Let $\{W(t),\, t \ge 0\}$ be a Brownian motion. Note that for $0 \le s,\, t \le 1$

$$E(X(t)-X(s))^2 = |t-s|^\alpha \le |t-s| = E(W(t)-W(s))^2.$$

Hence, applying the Sudakov-Fernique inequality (cf. Adler (1990)), we have

$$E \sup_{0\le t\le 1} X(t) \le E \sup_{0\le t\le 1} W(t) = (2/\pi)^{1/2}.$$

Lemma 6. *If* $0 < \alpha < 1$, *then*

$$E \sup_{0\le t\le 1} X(t) \le \frac{1}{(2\pi)^{1/2}} + \frac{e^{2.2}+1}{e^{2.2}-1}\left(\frac{8.8}{\alpha} - \ln\left(\frac{2.5}{\alpha}+0.4\right)\right)^{1/2}. \tag{2.7}$$

Proof. From the Borell inequality it follows immediately that

$$E \sup_{0\le t\le 1} X(t) \le (2\pi)^{-\frac{1}{2}} + \operatorname{med}\sup_{0\le t\le 1} X(t). \tag{2.8}$$

So, it suffices to show that

$$\operatorname{med}\sup_{0\le t\le 1} X(t) \le \frac{e^{2.2}+1}{e^{2.2}-1}\left(\frac{8.8}{\alpha} - \ln\left(\frac{2.5}{\alpha}+0.4\right)\right)^{1/2}. \tag{2.9}$$

Set

$$c = 2.2, \qquad h = e^{-2c/\alpha}, \qquad m = \text{med} \sup_{0\le t\le 1} X(t),$$
$$\theta = \ln(0.4 + 2.5/\alpha), \qquad a = (4\,c/\alpha \,-\, \theta)^{1/2}.$$

Using the Borell inequality again, we have

$$\begin{aligned}
&P\Big(\sup_{0\le t\le 1} X(t) > a + h^{\alpha/2}(a+m)\Big)\\
&\le P\Big(\max_{1\le i\le 1/h} X(i\,h) > a\Big) + P\Big(\max_{0\le i\le 1/h}\ \sup_{ih\le t\le (i+1)h} (X(t) - X(ih)) > h^{\alpha/2}(a+m)\Big)\\
&\le \frac{\Psi(a)}{h} + (1+\frac{1}{h})P\Big(\sup_{0\le t\le h} X(t) > h^{\alpha/2}(a+m)\Big)\\
&= \frac{\Psi(a)}{h} + (1+\frac{1}{h})P\Big(\sup_{0\le t\le 1} X(t) > a+m\Big)\\
&\le \frac{\Psi(a)}{h} + (1+\frac{1}{h})\Psi(a) \le (1+\frac{2}{h})\exp(-\frac{a^2}{2})/(a\sqrt{2\pi})\\
&= (2+h)\exp(\frac{\theta}{2})/(a\sqrt{2\pi}) \le 2.013\ \exp(\frac{\theta}{2})/(a\sqrt{2\pi}) \le 1/2,
\end{aligned}$$

where the last inequality was obtained by elementary argument. Therefore,

$$m \le a + h^{\alpha/2}(a+m)$$

and

$$m \le a(1+h^{\alpha/2})/(1-h^{\alpha/2}) = a(e^c+1)/(e^c-1).$$

This proves (2.9).

Lemma 7. *If $\{Y(t),\, t\ge 0\}$ is the non-stationary Gaussian process defined as in Section 1, then for any integer N and positive a*

$$E \sup_{0\le t\le a\,N} e^{Y(t)} \ge N\left(1 - \sum_{k=1}^{\infty} \exp\left(-\frac{(k\,a)^\alpha}{4}\right)\right). \tag{2.10}$$

Proof. Obviously, we have

$$\begin{aligned}
E \sup_{0\le t\le a\,N} e^{Y(t)} &\ge E \sup_{1\le i\le N} e^{Y(ia))}\\
&\ge E\left(\sum_{i=1}^{N} e^{Y(ia)} - \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} e^{\frac{Y(ia)+Y(ja)}{2}}\right)\\
&= \sum_{i=1}^{N} Ee^{Y(ia)} - \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} Ee^{\frac{Y(ia)+Y(ja)}{2}}.
\end{aligned} \tag{2.11}$$

Recalling that $EY(t) = -t^\alpha$ and $\mathrm{Var}Y(t) = 2t^\alpha$, we have for any $j > i \geq 1$

$$Ee^{Y(ia)} = 1, \qquad E\left(\frac{Y(ia)+Y(ja)}{2}\right) = -\frac{a^\alpha}{2}(i^\alpha + j^\alpha),$$

$$\mathrm{Var}\left(\frac{Y(ia)+Y(ja)}{2}\right) = a^\alpha(i^\alpha + j^\alpha - \frac{1}{2}(j-i)^\alpha)$$

and hence

$$Ee^{\frac{Y(ia)+Y(ja)}{2}} = \exp\left(-\frac{(j-i)^\alpha a^\alpha}{4}\right). \tag{2.12}$$

(2.10) now follows from (2.11) and (2.12).

We are now ready to prove Theorem 1.

Proof of Theorem 1. The right hand side of (1.4) and (1.5) follows from Lemmas 4, 5 and 6. We prove below the left hand side. It is easy to see from (1.2) that

$$H_\alpha = \lim_{T\to\infty} T^{-1} E\exp(\sup_{0\leq t\leq T} Y(t)) = \lim_{T\to\infty} T^{-1} E \sup_{0\leq t\leq T} e^{Y(t)}. \tag{2.13}$$

We next divide the proof into two cases.
CASE I: $1 \leq \alpha \leq 2$. Put $a = 5.2^{1/\alpha}$ in Lemma 7. By (2.10) and (2.13), we get

$$\begin{aligned} H_\alpha &\geq a^{-1}\left(1 - \sum_{k=1}^\infty \exp(-(k\,a)^\alpha/4)\right) \\ &\geq a^{-1}\left(1 - \sum_{k=1}^\infty \exp(-k\,a^\alpha/4)\right) \\ &= a^{-1}\left(1 - 1/(e^{a^\alpha/4} - 1)\right) \\ &\geq 5.2^{-1/\alpha}\, 0.625 \end{aligned}$$

This proves the left hand side of (1.4).
CASE II: $0 < \alpha < 1$. Let $a^\alpha = 4/\alpha$ in Lemma 7. Note that

$$\begin{aligned} \sum_{k=1}^\infty \exp(-(k\,a)^\alpha/4) &= \sum_{k=1}^\infty \exp(-k^\alpha/\alpha) \\ &\leq \exp(-1/\alpha) + \int_1^\infty \exp(-x^\alpha/\alpha)\,dx. \end{aligned} \tag{2.14}$$

An elementary argument yields

$$\int_1^\infty \exp(-x^\alpha/\alpha)\,dx \leq (1/\alpha)e^{-1/\alpha}. \tag{2.15}$$

From (2.10), (2.13), (2.14) and (2.15) it follows that

$$\begin{aligned} H_\alpha &\geq a^{-1}\left(1-\sum_{k=1}^{\infty}\exp(-(k\,a)^\alpha/4)\right) \\ &\geq a^{-1}\left(1-e^{-1/\alpha}(1+1/\alpha)\right)=(\alpha/4)^{1/\alpha}\left(1-e^{-1/\alpha}(1+1/\alpha)\right), \end{aligned}$$

as desired.

3. Estimators of H_α

Since we are unable to obtain the precise numerical value of H_α, it would certainly be interesting to introduce some estimators of H_α. A natural way is by using the empirical process. Let $\{X_i(t), t\geq 0\}_{i=1}^{\infty}$ be i.i.d. fractional Brownian motion of order α. Set

$$\xi_i(t)=X_i(e^t)/e^{\alpha t/2},\quad 0\leq t\leq 1.$$

Let $\{n_k, k\geq 1\}$ be a sequence of integers and $\{a_k, k\geq 1\}$ a sequence of real numbers with $n_k\geq 2^{a_k^2}$ and $a_k\geq k$ for every $k\geq 1$. Put

$$D_{k,1}=\sum_{i=1}^{n_k} I\{\sup_{0\leq t\leq 1}\xi_i(t)>a_k\}/\left(n_k(a_k/\sqrt{2})^{2/\alpha}\,\phi(a_k)/a_k\right),\quad k=1,2,\ldots \tag{3.1}$$

On the other hand, noting that by (1.3)

$$H_\alpha=2^{1/\alpha}\lim_{T\to\infty}T^{-2/\alpha}e^{-T^2/2}E\exp\left(T\sup_{0\leq t\leq 1}\xi(t)\right), \tag{3.2}$$

we present another estimator of H_α.

$$D_{k,2}=2^{1/\alpha}(4/3a_k)^{2/\alpha}\exp(-9a_k^2/32)\sum_{i=1}^{n_k}\exp\left((3\,a_k/4)\sup_{0\leq t\leq 1}\xi_i(t)\right)/n_k. \tag{3.3}$$

Theorem 2. *We have*

$$D_{k,1}\longrightarrow H_\alpha\qquad a.s. \tag{3.4}$$

and

$$D_{k,2}\longrightarrow H_\alpha\qquad a.s. \tag{3.5}$$

as $k\to\infty$.

Proof. Using the Chebyshev inequality and (1.3), one has

$$\begin{aligned} &P\left(\Big|\sum_{i=1}^{n_k}\Big(I\{\sup_{0\leq t\leq 1}\xi_i(t)>a_k\}-P\Big(\sup_{0\leq t\leq 1}\xi(t)>a_k\Big)\Big)\Big|>n_k\phi(a_k)\right) \\ &\leq (n_k\phi(a_k))^{-2}P\left(\sup_{0\leq t\leq 1}\xi_1(t)>a_k\right)\leq (n_k\phi(a_k))^{-2}a_k^{2/\alpha}\phi(a_k)\leq \exp(-0.1\,k^2) \end{aligned} \tag{3.6}$$

provided that k is sufficiently large. (3.4) now follows from (1.3), (3.6) and the Borel-Cantelli lemma.

We next prove (3.5). By (3.2) and the Chebyshev inequality,

$$\begin{aligned}
&P\Big(\Big|\sum_{i=1}^{n_k}\Big(\exp\Big((3\,a_k\,/4)\sup_{0\le t\le 1}\xi_i(t)\Big)-E\exp\Big((3\,a_k\,/4)\sup_{0\le t\le 1}\xi_i(t)\Big)\Big)\Big| \\
&\qquad > n_k\exp(9\,a_k^2\,/32)\Big) \\
&\le n_k^{-1}\exp(-9a_k^2/16)E\exp\Big((3\,a_k\,/2)\sup_{0\le t\le 1}\xi(t)\Big) \\
&\le n_k^{-1}\exp(-9\,a_k^2\,/16)a_k^{2/\alpha}\exp(9\,a_k^2\,/8)\le e^{-0.1\,k} \qquad (3.7)
\end{aligned}$$

for k sufficiently large. This proves (3.5), by (3.2), (3.7) and the Borel-Cantelli lemma.

Simulations on $D_{k,1}$, $D_{k,2}$, $E\sup_{0\le t\le 1}X(t)$ as well as the upper bound of H_α in (2.7) are given in the following table with $n_1=20000$, $a_1=3$.

Table. Simulation on $D_{k,1}$ and upper bound of H_α

α	$D_{1,1}$	$E\sup_{0\le t\le 1}X(t)$	Upper bound of H_α
0.1	1.16E-5	1.94	2.6E-9
0.2	1.54E-2	1.76	1.98E-2
0.3	0.14	1.59	0.55
0.4	0.35	1.43	1.64
0.5	0.52	1.28	2.34
0.6	0.56	1.16	2.61
0.7	0.72	1.04	2.63
0.8	0.85	0.94	2.53
0.9	0.85	0.86	2.40
1.0	0.87	0.80	2.36
1.1	0.88	0.75	2.29
1.2	0.85	0.67	2.04
1.3	0.86	0.63	2.05
1.4	0.83	0.57	1.85

Remark 1. According to the simulation, we conjecture that H_α is increasing on $(0,\ 1]$.

Remark 2. From the simulation on $E\sup_{0\le t\le 1}X(t)$, we should have a much better estimate for $E\sup_{0\le t\le 1}X(t)$ than (2.10).

4. An Application to the Small Ball Probability of Fractional Brownian Motion

Let $0 < \alpha \le 1$, $\{X(t), t \ge 0\}$ be a fractional Brownian motion of order α. In Shao (1993) we proved

$$\begin{aligned} &\exp\left(-2\left(1+3\,e(2\pi/\alpha)^{1/2}\right)^{2/\alpha} x^{-2/\alpha}\right) \\ &\le P\Big(\sup_{0\le t\le 1}|X(t)| \le x\Big) \le 2\exp\left(-0.17\,x^{-2/\alpha}\right) \end{aligned} \tag{4.1}$$

for every $0 < x < 1$. Along the same lines of the proof of (4.1) (see (2.5) of Shao (1993)), one can also obtain

$$\begin{aligned} &\exp\left(-11\left(1+(1/\alpha)^{1/2}\right)^{2/\alpha} x^{-2/\alpha}\right) \\ &\le P\Big(\sup_{0\le t\le 1}|X(t)| \le x\Big) \le 2\exp\left(-0.17\,x^{-2/\alpha}\right). \end{aligned} \tag{4.2}$$

The above so-called small ball probability is a key step in establishing a Chung type law of the iterated logarithm and has attracted a lot of attention recently (see Shao (1993), Monrad and Rootzén (1992) and Kuelbs and Li (1992)). It should be mentioned that the precise constant in the small ball problem is unknown except for the case of $\alpha = 1$. As a by-product of the proof of Theorem 1, we are now able to achieve a more precise upper bound.

Theorem 3. *Let $0 < \alpha < 1$, and $\{X(t), t \ge 0\}$ be a fractional Brownian motion of order α. Then*

$$\begin{aligned} &\exp\left(-11\left(1+(1/\alpha)^{1/2}\right)^{2/\alpha} x^{-2/\alpha}\right) \le P\left(\sup_{0\le t\le 1}|X(t)| \le x\right) \\ &\le 3\exp\left(-\left(4\,e\sqrt{\alpha}\right)^{-2/\alpha} x^{-2/\alpha}\,\ln 2\right). \end{aligned} \tag{4.3}$$

From Theorem 3 and similar to the proof of Theorem 3.3 in Monrad and Rootzén (1992), we have, immediately,

Theorem 4. *Under the condition of Theorem* 3,

$$\frac{(\ln 2)^{\alpha/2}}{4\,e\sqrt{\alpha}} \le \liminf_{t\to 0}\Big(\frac{\log\log 1/t}{t}\Big)^{\alpha/2} \sup_{0\le s\le t}|X(s)| \le 11^{\alpha/2}(1+\frac{1}{\sqrt{\alpha}}) \quad a.s. \tag{4.4}$$

Remark 5. It is known that (cf. Monrad and Rootzén (1992)) there exists c_α such that

$$\liminf_{t\to 0}\left(\frac{\log\log 1/t}{t}\right)^{\alpha/2}\sup_{0\le s\le t}|X(s)| = c_\alpha \quad \text{a.s.}$$

In terms of (4.4), we have

$$1/(4e) \le \liminf_{\alpha\downarrow 0}\sqrt{\alpha}\,c_\alpha \le \limsup_{\alpha\downarrow 0}\sqrt{\alpha}\,c_\alpha \le 1.$$

Proof of Theorem 3. When $0.2 \le \alpha < 1$, (4.3) is an immediate consequence of (4.2). So, we only need to deal with the case $0 < \alpha < 0.2$. Let $m = \operatorname{med}\sup_{0\le t\le 1} X(t)$. By Lemma 4, (2.8) and (1.5),

$$m \ge \frac{1}{e\sqrt{\alpha}}\left(1 - e^{-1/\alpha}(1 + 1/\alpha)\right)^{\alpha/2} - (2\pi)^{-1/2}$$

and hence

$$m \ge 1/(2\,e\sqrt{\alpha}) \tag{4.5}$$

for every $0 < \alpha \le 0.2$. Put

$$h = (m/(2\,x))^{2/\alpha}.$$

Applying the Slepian lemma and (4.5),

$$\begin{aligned}
P\left(\sup_{0\le t\le 1}|X(t)| \le x\right) &\le P\left(\max_{0\le i\le[1/h]-1}\sup_{0<t<h}|X(t+ih) - X(ih)| \le 2\,x\right)\\
&\le P\left(\max_{0\le i\le[1/h]-1}\sup_{0<t<h}(X(t+ih) - X(ih)) \le 2\,x\right)\\
&\le \prod_{0\le i\le[1/h]-1} P\left(\sup_{0<t<h}(X(t+ih) - X(ih)) \le 2\,x\right)\\
&= \left(P\left(\sup_{0<t<1} X(t) \le m\right)\right)^{[1/h]}\\
&= \exp(-[1/h]\,\ln 2)\\
&\le 3\,\exp\left(-\left(\frac{1}{4\,e\sqrt{\alpha}}\right)^{2/\alpha} x^{-2/\alpha}\,\ln 2\right),
\end{aligned}$$

as desired.

Acknowledgments

The author is grateful to an associate editor and a referee for helpful comments on an earlier draft of this manuscript. Thanks are also due to Xuming He for his help in preparing the simulation.

References

Adler, R. J. (1990). An Introduction to Continuity, Extrema, and Related Topics for General Gaussian Processes. IMS Lecture Notes-Monograph Series **12**.

Berman, S. M. (1971). Maxima and high level excursions of stationary Gaussian processes. Trans. Amer. Math. Soc. **160**, 65-85.

Borell, C. (1975). The Brunn-Minkowski inequality in Gauss space. Invent. Math. **30**, 205-221.

Kuelbs, J. and Li, W. V. (1992). Metric entropy and the small ball problem for Gaussian measures. J. Funct. Anal. **116**, 133-157.

Leadbetter, M. R., Lindgren, G. and Rootzen, H. (1983). Extremes and Related Properties of Random Sequences and Processes. Springer-Verlag, New York.

Mandelbrot, B. B. and Ness, J. W. V. (1968). Fractional Brownian motions, fractional noises and applications. Siam Review **10**, 422-437.

Monrad, D. and Rootzén, H. (1992). Small values of fractional Brownian motion and locally nondeterministic Gaussian processes. Manuscript.

Pickands, J. III. (1969a). Upcrossing probabilities for stationary Gaussian processes. Trans. Amer. Math. Soc. **145**, 51-73.

Pickands, J. III. (1969b). Asymptotic properties of the maximum in a stationary Gaussian process. Trans. Amer. Math. Soc. **145**, 75-86.

Qualls, C. and Watanabe, H. (1972). Asymptotic properties of Gaussian processes. Ann. Math. Statist. **43**, 580-596.

Shao, Q. M. (1992). An Erdős - Révész type law of the iterated logarithm for stationary Gaussian processes. Probab. Theory Related Fields **94**, 119-133.

Shao, Q. M. (1993). A note on small ball probability of Gaussian process with stationary increments. J. Theory Probab. **6**, 595-602.

Slepian, D. (1962). The one-sided barrier problem for Gaussian noise. Bell System Tech. J. **41**, 463-501.

Department of Mathematics, National University of Singapore, Singapore 0511, Singapore.

(Received April 1993; accepted October 1994)