Supplement to "Hypothesis Testing for Network Data with Power Enhancement"

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A1 Technical Lemmas

Lemma S1 (Bonferroni Inequality)**.** *Let $A = \cup_{t=1}^{p} A_t$. For any $k < [p/2]$, we have*

$$\sum_{t=1}^{2k}(-1)^{t-1}E_t \leq \mathsf{P}(A) \leq \sum_{t=1}^{2k-1}(-1)^{t-1}E_t,$$

where $E_t = \sum_{1\leq i_1<\cdots<i_t\leq p}\mathsf{P}(A_{i_1}\cap\cdots\cap A_{i_t})$.

Lemma S2. *For any random vector $\boldsymbol{W} = (w_1,\ldots,w_b)$, with $\mathsf{E}(\boldsymbol{W}) = 0$, and $\boldsymbol{W} = \xi_1 + \cdots + \xi_n$, where $\{\xi_k = (\xi_{1,k},\ldots,\xi_{b,k}), k = 1,\ldots,n\}$ are independent random vectors and $|\xi_{i,k}| \leq \tau$, $1 \leq i \leq b$, we have, for any $y, \epsilon > 0$,*

$$\begin{aligned}\mathsf{P}(|\boldsymbol{W}| \geq y) &\leq \mathsf{P}(|\boldsymbol{N}| \geq y-\epsilon) + c_1 b^{5/2}\exp\left(-\frac{\epsilon}{c_2 b^3\tau}\right),\\ \mathsf{P}(|\boldsymbol{W}| \geq y) &\geq \mathsf{P}(|\boldsymbol{N}| \geq y+\epsilon) - c_1 b^{5/2}\exp\left(-\frac{\epsilon}{c_2 b^3\tau}\right),\end{aligned}$$

for some absolute constants $c_1, c_2 > 0$, where $|\cdot|$ is any vector norm, $\boldsymbol{N}$ is a normal random vector with $\mathsf{E}(\boldsymbol{N}) = 0$ and the same covariance matrix as $\boldsymbol{W}$.

Lemma S2 is based on Theorem 1 of Zaïtsev (1987), and its proof is omitted. Lemma 1 in Section 2.2 of the paper is proved by applying Lemma S2, similarly as done in the proof of Theorem 1 in Cai et al. (2013), and its proof is also omitted here.

A2 Proof of Theorem 1

We rearrange the indices of $\{S_{l,d,i,j}, 1 \leq i < j \leq p\}$ by $\{S_{l,d,i}, i = 1, \ldots, q\}$. Let $(U_{l,1}, ..., U_{l,q})^{\intercal}$ be a zero mean random vector, with the covariance matrix $\Sigma = (\sigma_{i,j})$ and the diagonal $\{\sigma_{i,i}\}_{i=1}^{q} = 1$, which satisfies the moment conditions (C1) or (C2) and the regularity conditions (A1) and (A2), $l = 1, \ldots, n$. Note that,

$$|V_{d,i,j}/\mathsf{Var}(S_{l,d,i,j}) - 1| = O_p\{(\log q/n)^{1/2}\}, \quad 1 \leq i < j \leq p, d = 1, 2,$$

Then under the event that $\{|V_{d,i,j}/\mathsf{Var}(S_{l,d,i,j}) - 1| = O\{(\log q/n)^{1/2})\}$, to prove the theorem, it suffices to show that, for any $x \in R$, as $p \to \infty$,

$$\mathsf{P}\left\{\max_{1\leq i\leq q}\left(n^{-1/2}\sum_{l=1}^{n} U_{l,i}\right)^2 - 2\log q + \log\log q \leq x\right\} \to \exp\{-\pi^{-1/2}\exp(-x/2)\}.$$

Let $\hat{U}_{l,i} = U_{l,i}I\{|U_{l,i}| \leq \tau_n\} - \mathsf{E}(U_{l,i}I\{|U_{l,i}| \leq \tau_n\})$, $l = 1, \ldots, n$, where $\tau_n = \eta^{-1/2}2\sqrt{\log(q+n)}$ if (C1) holds, and $\tau_n = \sqrt{n}/(\log q)^8$ if (C2) holds. Let $W_i = \sum_{l=1}^{n} U_{l,i}/\sqrt{n}$ and $\hat{W}_i = \sum_{l=1}^{n} \hat{U}_{l,i}/\sqrt{n}$. If (C1) holds, then we have,

$$\begin{aligned}
&\max_{1\leq i\leq q} n^{-1/2}\sum_{l=1}^{n}\mathsf{E}(|U_{l,i}|)I\left\{|U_{l,i}| \geq \eta^{-1/2}2\sqrt{\log(q+n)}\right\}\\
&\quad\leq Cn^{1/2}\max_{1\leq l\leq n}\max_{1\leq i\leq q}\mathsf{E}(|U_{l,i}|)I\left\{|U_{l,i}| \geq \eta^{-1/2}2\sqrt{\log(q+n)}\right\}\\
&\quad\leq Cn^{1/2}(q+n)^{-2}\max_{1\leq l\leq n}\max_{1\leq i\leq q}\mathsf{E}(|U_{l,i}|)\exp\left(\eta U_{l,i}^2/2\right)\\
&\quad\leq Cn^{1/2}(q+n)^{-2}.
\end{aligned}$$

If (C2) holds, then we have,

$$\begin{aligned}
&\max_{1\leq i\leq q} n^{-1/2}\sum_{l=1}^{n}\mathsf{E}(|U_{l,i}|)I\left\{|U_{l,i}| \geq \sqrt{n}/(\log q)^8\right\}\\
&\quad\leq Cn^{1/2}\max_{1\leq l\leq n_1+n_2}\max_{1\leq i\leq q}\mathsf{E}(|U_{l,i}|)I\left\{|U_{l,i}| \geq \sqrt{n}/(\log q)^8\right\}\\
&\quad\leq Cn^{-2\gamma_0-\epsilon/4}.
\end{aligned}$$

Therefore,

$$\mathsf{P}\left\{\max_{1\le i\le q}|W_i-\hat{W}_i|\ge(\log q)^{-1}\right\} \le \mathsf{P}\left(\max_{1\le i\le q}\max_{1\le l\le t}|U_{l,i}|\ge\tau_n\right) \le tq\max_{1\le i\le q}\mathsf{P}(|U_{1i}|\ge\tau_n)=O(q^{-1}+n^{-\epsilon/4}). \quad (S1)$$

Note that

$$\left|\max_{1\le i\le q}W_i^2-\max_{1\le i\le q}\hat{W}_i^2\right|\le 2\max_{1\le i\le q}|W_i|\max_{1\le i\le q}|W_i-\hat{W}_i|+\max_{1\le i\le p}|W_i-\hat{W}_i|^2. \quad (S2)$$

By (S1) and (S2), it suffices to prove that, for any $x\in R$, as $p\to\infty$,

$$\mathsf{P}\left(\max_{1\le i\le q}\hat{W}_i^2-2\log q+\log\log q\le x\right)\to\exp\left\{-\pi^{-1/2}\exp(-x/2)\right\}.$$

Let $x_q=2\log q-\log\log q+x$. It follows from Lemma S1 that, for any fixed $k\le[q/2]$,

$$\sum_{s=1}^{2k}(-1)^{s-1}\sum_{1\le i_1<\dots<i_n\le q}\mathsf{P}\left(|\hat{W}_{i_1}|\ge x_q,\dots,|W_{i_s}|\ge x_q\right)\le\mathsf{P}\left(\max_{1\le i\le q}|\hat{W}_i|\ge x_q\right) \le\sum_{s=1}^{2k-1}(-1)^{s-1}\sum_{1\le i_1<\dots<i_s\le q}\mathsf{P}\left(|\hat{W}_{i_1}|\ge x_q,\dots,|W_{i_s}|\ge x_q\right). \quad (S3)$$

Define $|\hat{\boldsymbol{W}}|_{\min}=\min_{1\le b\le s}|\hat{W}_{i_b}|$. Then by Lemma S2, we have,

$$\mathsf{P}\left(|\hat{\boldsymbol{W}}|_{\min}\ge x_q\right) \le \mathsf{P}\left\{|\boldsymbol{Z}|_{\min}\ge x_q-\epsilon_n(\log q)^{-1/2}\right\} +c_1s^{5/2}\exp\left\{-\frac{n^{1/2}\epsilon_n}{c_2s^3\tau_n(\log q)^{1/2}}\right\}, \quad (S4)$$

where $c_1>0$ and $c_2>0$ are absolute constants, $\epsilon_n\to0$ is to be specified later, and $\boldsymbol{Z}=(Z_{i_1},\dots,Z_{i_s})'$ is a s-dimensional normal vector with the same covariance structure as $\hat{\boldsymbol{W}}$. Because $\log p=o(n^{1/5})$, we can let $\epsilon_n\to0$ sufficiently slow, such that

$$c_1s^{5/2}\exp\left\{-\frac{n^{1/2}\epsilon_n}{c_2s^3\tau_n(\log q)^{1/2}}\right\}=O(q^{-M}) \quad (S5)$$

for any large $M > 0$. It then follows from (S3), (S4) and (S5) that

$$\mathsf{P}\left(\max_{1\le i\le q}|\hat{W}_i|\ge x_q\right)\le\sum_{s=1}^{2k-1}(-1)^{s-1}\sum_{1\le i_1<\ldots<i_s\le p}\mathsf{P}\left(|Z|_{\min}\ge x_q-\epsilon_n(\log q)^{-1/2}\right)+o(1).\text{(S6)}$$

Similarly, using Lemma S2 again, we get

$$\mathsf{P}\left(\max_{1\le i\le q}|\hat{W}_i|\ge x_q\right)\ge\sum_{s=1}^{2k}(-1)^{s-1}\sum_{1\le i_1<\ldots<i_s\le q}\mathsf{P}\left(|Z|_{\min}\ge x_q-\epsilon_n(\log q)^{-1/2}\right)-o(1).\text{(S7)}$$

By (S6), (S7) and the proof of Theorem 1 in Cai et al. (2014), our theorem is proved. □

A3 Proof of Theorem 2

We first show that, as $n_1, n_2, p \to \infty$,

$$\inf_{(s_1,s_2)\in U(2\sqrt{2})}\mathsf{P}(\Psi_\alpha=1)\to 1.$$

Let

$$M_n^1=\max_{1\le i\le j\le p}\frac{\left(\bar{S}_{1,i,j}-\bar{S}_{2,i,j}-s_{1,i,j}+s_{2,i,j}\right)^2}{V_{1,i,j}/n_1+V_{2,i,j}/n_2}.$$

By the self-normalized large deviation theorem for independent random variables (Jing et al., 2003, Theorem 1), we have that,

$$\max_{1\le i\le j\le p}\mathsf{P}\left\{\frac{\left(\bar{S}_{1,i,j}-\bar{S}_{2,i,j}-s_{1,i,j}+s_{2,i,j}\right)^2}{V_{1,i,j}/n_1+V_{2,i,j}/n_2}\ge x^2\right\}\le C\{1-\Phi(x)\},$$

uniformly for $0 \le x \le (8\log p)^{1/2}$. Thus we have that,

$$\mathsf{P}\left(M_n^1\le 2\log q-\frac{1}{2}\log\log q\right)\to 1$$

as $t, q \to \infty$. Note that,

$$\max_{1\le i\le j\le p}\frac{(s_{1,i,j}-s_{2,i,j})^2}{V_{1,i,j}/n_1+V_{2,i,j}/n_2}\le 2M_n^1+2M_n$$

and that

$$\max_{1\le i\le j\le p} \frac{(s_{1,i,j}-s_{2,i,j})^2}{\mathsf{Var}(S_{1,l,i,j})/n_1+\mathsf{Var}(S_{2,l,i,j})/n_2} \ge 8\log q.$$

By the fact that

$$|V_{d,i,j}/\mathsf{Var}(S_{l,d,i,j})-1| = O_p\left\{(\log q/n)^{1/2}\right\}, \quad 1\le i<j\le p, d=1,2,$$

we have that,

$$\mathsf{P}\left(M_n \ge q_\alpha + 2\log q - \log\log p\right) \to 1$$

as $n, q \to \infty$.

We next prove that, there exists a constant $c_0 > 0$ such that, for all sufficiently large n_d and p,

$$\inf_{(s_1,s_2)\in U(c_0)} \sup_{T_\alpha\in\mathcal{T}_\alpha} \mathsf{P}(T_\alpha = 1) \le 1-\beta,$$

Since $\mathcal{T}_\alpha$ contains all the α-level tests over the collection of distributions satisfying (C1) or (C2), it suffices to take $\mathcal{T}_\alpha$ as the set of α-level tests over Gaussian distributions. Then following the proof of Theorem 4 of Cai et al. (2014), our theorem is proved. □

A4 Proof of Theorem 3

Under the assumption that $|\mathcal{S}_\rho| \ge [1/\{\pi^{1/2}\alpha\}+\delta](\log q)^{1/2}$, we have that,

$$\sum_{(i,j)\in\mathcal{H}} I\left(|T_{i,j}| \ge \sqrt{2\log q}\right) \ge \left\{\frac{1}{\pi^{1/2}\alpha}+\delta\right\}\sqrt{\log q},$$

with probability tending to 1. Henceforth, with probability going to one, we have

$$\frac{q}{\sum_{(i,j)\in\mathcal{H}} I\left(|T_{i,j}| \ge \sqrt{2\log q}\right)} \le q\left\{\frac{1}{\pi^{1/2}\alpha}+\delta\right\}^{-1}(\log q)^{-1/2}.$$

Define $h_q = \sqrt{2\log q - 2\log\log q}$. Because $1-\Phi(h_q) \sim (\sqrt{2\pi}h_q)^{-1}\exp(-h_q^2/2)$, we have $\mathsf{P}(0 \le \hat{h} \le h_q) \to 1$ according to the definition of $\hat{h}$ in Algorithm 1. Namely, we have

$$\mathsf{P}\left(\hat{h} \text{ exists in } [0, h_q]\right) \to 1.$$

Thus it suffices to prove the theorem under the event that $\left\{\hat{h} \text{ exists in } [0, h_q]\right\}$.

Note that, by the definition of $\hat{h}$, for any $h < \hat{h}$, we have

$$\frac{G(h)q}{\max\left\{\sum_{(i,j)\in\mathcal{H}} I(|T_{i,j}| \ge h), 1\right\}} > \alpha.$$

Because $\max\left\{\sum_{(i,j)\in\mathcal{H}} I(|T_{i,j}| \ge \hat{h}), 1\right\} \le \max\left\{\sum_{(i,j)\in\mathcal{H}} I(|T_{i,j}| \ge h), 1\right\}$, we have that,

$$\frac{G(h)q}{\max\left\{\sum_{(i,j)\in\mathcal{H}} I(|T_{i,j}| \ge \hat{h}), 1\right\}} > \alpha.$$

Thus, by letting $h \to \hat{h}$, we have,

$$\frac{G(\hat{h})q}{\max\left\{\sum_{(i,j)\in\mathcal{H}} I(|T_{i,j}| \ge \hat{h}), 1\right\}} \ge \alpha.$$

On the other hand, based on the definition of $\hat{h}$, there exists a sequence $\{h_l\}$ with $h_l \ge \hat{h}$ and $h_l \to \hat{h}$, such that,

$$\frac{G(h_l)q}{\max\left\{\sum_{(i,j)\in\mathcal{H}} I(|T_{i,j}| \ge h_l), 1\right\}} \le \alpha.$$

Since $\max\left\{\sum_{(i,j)\in\mathcal{H}} I(|T_{i,j}| \ge h_l), 1\right\} \le \max\left\{\sum_{(i,j)\in\mathcal{H}} I(|T_{i,j}| \ge \hat{h}), 1\right\}$, it implies that,

$$\frac{G(h_l)q}{\max\left\{\sum_{(i,j)\in\mathcal{H}} I(|T_{i,j}| \ge \hat{h}), 1\right\}} \le \alpha.$$

Letting $h_l \to \hat{h}$, we have that,

$$\frac{G(\hat{h})q}{\max\left\{\sum_{(i,j)\in\mathcal{H}} I(|T_{i,j}| \ge \hat{h}), 1\right\}} \le \alpha.$$

Thus by focusing on the event $\left\{\hat{h} \text{ exists in } [0, h_q]\right\}$, we have that,

$$\frac{G(\hat{h})q}{\max\left\{\sum_{(i,j)\in\mathcal{H}} I(|T_{i,j}| \geq \hat{h}), 1\right\}} = \alpha.$$

Then it suffices to show that

$$\sup_{0\leq h\leq h_q}\left|\frac{\sum_{(i,j)\in\mathcal{H}_0} I(|T_{i,j}| \geq h) - |\mathcal{H}_0|G(h)}{qG(h)}\right| \to 0$$

in probability. Let $0 \leq h_0 < h_1 < \cdots < h_b = h_q$, such that $h_\iota - h_{\iota-1} = v_q$ for $1 \leq \iota \leq b-1$, and $h_b - h_{b-1} \leq v_q$, where $v_q = 1/\sqrt{\log q(\log_4 q)}$. Then we have $b \sim h_q/v_q$. For any h such that $h_{\iota-1} \leq h \leq h_\iota$, we have that

$$\begin{aligned}\frac{\sum_{(i,j)\in\mathcal{H}_0} I(|T_{i,j}| \geq h_\iota)}{|\mathcal{H}_0|G(h_\iota)}\frac{G(h_\iota)}{G(h_{\iota-1})} &\leq \frac{\sum_{(i,j)\in\mathcal{H}_0} I(|T_{i,j}| \geq h)}{|\mathcal{H}_0|G(h)} \\ &\leq \frac{\sum_{(i,j)\in\mathcal{H}_0} I(|T_{i,j}| \geq h_{\iota-1})}{|\mathcal{H}_0|G(h_{\iota-1})}\frac{G(h_{\iota-1})}{G(h_\iota)}.\end{aligned}$$

Thus it suffices to prove that

$$\max_{0\leq\iota\leq b}\left|\frac{\sum_{(i,j)\in\mathcal{H}_0}\{I(|T_{i,j}| \geq h_\iota) - G(h_\iota)\}}{|\mathcal{H}_0|G(h_\iota)}\right| \to 0$$

in probability. Note that

$$\begin{aligned}&\mathsf{P}\left(\max_{0\leq\iota\leq b}\left|\frac{\sum_{(i,j)\in\mathcal{H}_0}\{I(|T_{i,j}| \geq h_\iota) - G(h_\iota)\}}{|\mathcal{H}_0|G(h_\iota)}\right| \geq \epsilon\right) \\ &\leq \sum_{\iota=1}^{b}\mathsf{P}\left(\left|\frac{\sum_{(i,j)\in\mathcal{H}_0}\{I(|T_{i,j}| \geq h_\iota) - G(h_\iota)\}}{|\mathcal{H}_0|G(h_\iota)}\right| \geq \epsilon\right) \\ &\leq \frac{1}{v_q}\int_0^{h_q}\mathsf{P}\left(\left|\frac{\sum_{(i,j)\in\mathcal{H}_0}\{I(|T_{i,j}| \geq h) - G(h)\}}{|\mathcal{H}_0|G(h)}\right| \geq \epsilon\right)dt \\ &\quad + \sum_{\iota=b-1}^{b}\mathsf{P}\left(\left|\frac{\sum_{(i,j)\in\mathcal{H}_0}\{I(|T_{i,j}| \geq h_\iota) - G(h_\iota)\}}{|\mathcal{H}_0|G(h_\iota)}\right| \geq \epsilon\right).\end{aligned}$$

By the proof of Theorem 1, we have that

$$\mathsf{P}\left(|T_{i,j}| \geq h\right) = \{1 + o(1)\}G(h).$$

Thus it suffices to prove the following two statements are true for any $\epsilon > 0$:

$$\int_0^{h_q} \mathsf{P}\left(\left|\frac{\sum_{(i,j)\in\mathcal{H}_0}\{I(|T_{i,j}|\geq h)-\mathsf{P}(|T_{i,j}|\geq h)\}}{qG(h)}\right|\geq\epsilon\right)dh = o(v_q), \tag{S8}$$

and

$$\sup_{0\leq h\leq h_q}\mathsf{P}\left(\left|\frac{\sum_{(i,j)\in\mathcal{H}_0}\{I(|T_{i,j}|\geq h)-\mathsf{P}(|T_{i,j}|\geq h)\}}{qG(h)}\right|\geq\epsilon\right)=o(1). \tag{S9}$$

Next we prove (S8), while the proof of (S9) is similar and is thus omitted. Note that the variance can be calculated as follows

$$\begin{aligned} &\mathsf{E}\left[\frac{\sum_{(i,j)\in\mathcal{H}_0}\{I(|T_{i,j}|\geq h)-\mathsf{P}(|T_{i,j}|\geq h)\}}{qG(h)}\right]^2 \\ = \;& \frac{\sum_{(i,j),(i',j')\in\mathcal{H}_0}\{\mathsf{P}\left(|T_{i,j}|\geq h,|T_{i',j'}|\geq h\right)-\mathsf{P}\left(|T_{i,j}|\geq h\right)\mathsf{P}\left(|T_{i',j'}|\geq h\right)\}}{q^2G^2(h)}. \end{aligned}$$

We further split the pairs of indices in $\mathcal{H}_0$ into three subsets as below. We rearrange the indices of $\{(i,j), 1\leq i<j\leq p\}$ by $\{k, k=1,\ldots,q\}$, and denote by $k_{i,j}$ the corresponding index of (i,j) after rearranging:

$$\begin{aligned} \mathcal{H}_{01} &= \{(i,j),(i',j')\in\mathcal{H}_0,(i,j)=(i',j')\}, \\ \mathcal{H}_{02} &= \left\{(i,j),(i',j')\in\mathcal{H}_0,(i,j)\neq(i',j'),k_{i,j}\in\mathcal{A}_{k_{i',j'}}(\xi)\text{ or }k_{i',j'}\in\mathcal{A}_{k_{i,j}}(\xi)\right\}, \\ \mathcal{H}_{03} &= \{(i,j),(i',j')\in\mathcal{H}_0\}\setminus(\mathcal{H}_{01}\cup\mathcal{H}_{02}). \end{aligned}$$

For the subset $\mathcal{H}_{01}$, the cardinality is q_0, thus we have

$$\frac{\sum_{(i,j),(i',j')\in\mathcal{H}_{01}}\{\mathsf{P}(|T_{i,j}|\geq h,|T_{i',j'}|\geq h)-\mathsf{P}(|T_{i,j}|\geq h)\mathsf{P}(|T_{i',j'}|\geq h)\}}{q^2G^2(h)}\leq\frac{C}{qG(h)}. \tag{S10}$$

For the subset $\mathcal{H}_{02}$, recall that

$$\mathcal{A}_i(\xi)=\{j:\ \max\{|r_{1,i,j}|,|r_{2,i,j}|\}\geq(\log q)^{-2-\xi}\},$$

and $\max_{1\le i\le q}|\mathcal{A}_i(\xi)| = o(q^\nu)$ for $0 < \nu < (1-r)/(1+r)$. Thus we have $|\mathcal{H}_{02}| = O(q^{1+\nu})$. Note that, by Assumption (A2), we have that, uniformly for $(i,j),(i',j') \in \mathcal{H}_{02}$, $\mathsf{Corr}(T_{i,j}, T_{i',j'}) \le r' < 1$, with $r' < r + \epsilon < 1$, $0 < \epsilon < \frac{1-\nu}{1+\nu} - r$. Therefore, similar to (S4) in the proof of Theorem 1, under (C1) or (C2), by applying Lemma S2 and Lemma 6.2 in Liu (2013), we have that

$$\begin{aligned}&\frac{\sum_{(i,j),(i',j')\in\mathcal{H}_{02}}\{\mathsf{P}(|T_{i,j}|\ge h, |T_{i',j'}|\ge h) - \mathsf{P}(|T_{i,j}|\ge h)\mathsf{P}(|T_{i',j'}|\ge h)\}}{q^2G^2(h)}\\ \le\ & C\frac{q^{1+\nu}h^{-2}\exp\{-h^2/(1+r')\}}{q^2G^2(h)} \le \frac{C}{q^{1-\nu}\{G(h)\}^{2r'/(1+r')}}. \qquad \text{(S11)}\end{aligned}$$

For the subset $\mathcal{H}_{03}$, $T_{i,j}$ and $T_{i',j'}$ are weakly correlated with each other. Based on the conditions in the theorem, by applying Lemma S2 and Lemma 6.1 in Liu (2013), it is easy to obtain that,

$$\max_{(i,j),(i',j')\in\mathcal{H}_{03}} \mathsf{P}\left(|T_{i,j}|\ge h, |T_{i',j'}|\ge h\right) = [1 + O\{(\log q)^{-1-\gamma}\}]G^2(h),$$

with $\gamma = \min\{\xi, 1/2\}$. Thus, we have that

$$\begin{aligned}&\frac{\sum_{(i,j),(i',j')\in\mathcal{H}_{02}}\{\mathsf{P}(|T_{i,j}|\ge h, |T_{i',j'}|\ge h) - \mathsf{P}(|T_{i,j}|\ge h)\mathsf{P}(|T_{i',j'}|\ge h)\}}{q^2G^2(h)}\\ &= O\{(\log q)^{-1-\gamma}\}. \qquad \text{(S12)}\end{aligned}$$

Combining (S10), (S11) and (S12), we have

$$\int_0^{h_q}\left[\frac{C}{qG(h)} + \frac{C}{q^{1-\nu}\{G(h)\}^{2r'/(1+r')}} + C(\log q)^{-1-\gamma}\right]dh = o(v_q).$$

This proves (S8). Along with (S9), we prove Theorem 3. □

A5 Proof of Proposition 1

It sufficies to show that

$$\mathsf{P}_{H_{0,i,j}}\left(|T_{i,j}|\ge h, |A_{i,j}|\ge\lambda\right) = \{1+o(1)\}G(h)\mathsf{P}\left(|N(0,1) + a_{i,j}|\ge\lambda\right) + O(q^{-M}),$$

uniformly for $0 \le h \le C\sqrt{\log q}$, $0 \le \lambda \le C\sqrt{\log q}$, and $1 \le i < j \le p$. By the fact that N is fixed, the second part then directly follows.

Note that $G[h + o\{(\log q)^{-1/2}\}]/G(h) = 1 + o(1)$ uniformly in $0 \le h \le c(\log q)^{1/2}$ for any constant c. By the proof of Theorem 1, it suffices to show that,

$$\mathsf{P}\left(|U_{i,j}| \ge t, |Q_{i,j}| \ge \lambda\right) = \{1 + o(1)\}G(h)\mathsf{P}\left(|N(0,1)| \ge \lambda\right) + O(q^{-M}),$$

where

$$U_{i,j} = \frac{\bar{S}_{1,i,j} - \bar{S}_{2,i,j}}{(\sigma^2_{w,i,1} + \sigma^2_{w,i,2})^{1/2}}, \quad \text{and} \quad Q_{i,j} = \frac{\bar{S}_{1,i,j} - s_{1,i,j} + (\sigma^2_{w,i,1}/\sigma^2_{w,i,2})(\bar{S}_{2,i,j} - s_{2,i,j})}{\sqrt{\sigma^2_{w,i,1}(1 + \sigma^2_{w,i,1}/\sigma^2_{w,i,2})}},$$

with $\sigma^2_{w,i,d} = \mathsf{Var}(S_{d,l,i,j})/n_d$, $d = 1, 2$. Note that $U_{i,j}$ and $Q_{i,j}$ are uncorrelated with each other.

We next truncate $V_{i,j}$ and $Q_{i,j}$, respectively, by τ_n as defined in Theorem 1 with rate $\{\log(q + n)\}^{1+\epsilon}$ for a sufficiently small $\epsilon > 0$. Then we have the truncated $\hat{V}_{i,j}$ and $\hat{Q}_{i,j}$ satisfy that,

$$\mathsf{P}\left\{\max_{1 \le i \le m} |V_i - \hat{V}_i| \ge (\log q)^{-1}\right\} \le \mathsf{P}\left(\max_{1 \le i \le q} \max_{1 \le l \le n_1 + n_2} |U_{l,i}| \ge \tau_n\right) = O(q^{-M}),$$

and

$$\mathsf{P}\left\{\max_{1 \le i \le m} |Q_i - \hat{Q}_i| \ge (\log q)^{-1}\right\} = O(q^{-M}).$$

Thus, it suffices to show that

$$\mathsf{P}\left(|\hat{U}_{i,j}| \ge h, |\hat{Q}_{i,j}| \ge \lambda\right) = \{1 + o(1)\}G(h)G(\lambda) + O(q^{-M}), \tag{S13}$$

uniformly for $0 \le h \le C\sqrt{\log q}$ and $0 \le \lambda \le C\sqrt{\log q}$. It follows from Lemma S2 that

$$\mathsf{P}\left(|\hat{U}_{i,j}| \ge h, |\hat{Q}_{i,j}| \ge \lambda\right)$$

$$\leq \mathsf{P}\left\{|N_1| \geq h - \epsilon_n(\log q)^{-1/2}, |N_2| \geq \lambda - \epsilon_n(\log q)^{-1/2}\right\} + c_1 \exp\left\{-\frac{n^{1/2}\epsilon_n}{c_2\tau_n(\log q)^{1/2}}\right\},$$

where $c_1 > 0$ and $c_2 > 0$ are constants, $\epsilon_n \to 0$ is to be specified later, and $\boldsymbol{N} = (N_1, N_2)$ is a normal random vector with $\mathsf{E}(\boldsymbol{N}) = 0$ and $\mathsf{Cov}(N_1, N_2) = 0$. Since $\log q = o(n^{1/c})$ for some $c > 5$, we let $\epsilon_n \to 0$ sufficiently slowly, so that for any large $M > 0$,

$$c_1 \exp\left\{-\frac{n^{1/2}\epsilon_n}{c_2\tau_n(\log q)^{1/2}}\right\} = O(q^{-M}).$$

Thus, we have

$$\mathsf{P}\left(|\hat{U}_{i,j}| \geq h, |\hat{Q}_{i,j}| \geq \lambda\right) \leq \mathsf{P}\left\{|N_1| \geq h - \epsilon_n(\log q)^{-1/2}, |N_2| \geq \lambda - \epsilon_n(\log q)^{-1/2}\right\} + O(q^{-M}).$$

Similarly, using Lemma S2 again, we have

$$\mathsf{P}\left(|\hat{U}_{i,j}| \geq h, |\hat{Q}_{i,j}| \geq \lambda\right) \geq \mathsf{P}\left\{|N_1| \geq h + \epsilon_n(\log q)^{-1/2}, |N_2| \geq \lambda + \epsilon_n(\log q)^{-1/2}\right\} - O(q^{-M}).$$

This proves (S13), then also Proposition 1. □

A6 Proof of Theorems 4 and 5

By the proofs of Theorems 1 and 2 in Xia et al. (2020), it suffices to check the asymptotic normality, the weak dependency, and the asymptotic independence assumptions. By the construction of $T_{i,j}$ and the proof of Proposition 1, we have that $T_{i,j}$ is asymptotic normal. In addition, the assumptions of Theorem 3 ensures the weak dependency. Finally, Proposition 1 proves that the asymptotic independence holds. Thus Theorems 4 and 5 follow. □

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