A THRESHOLDING-BASED PREWHITENED LONG-RUN VARIANCE ESTIMATOR AND ITS DEPENDENCE-ORACLE PROPERTY

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Supplementary Material

S1. Appendix A: Proofs of Lemmas 1–6

Proof. (Lemma 1) Since the second claim follows easily by applying Lemma 1 of Liu and Wu (2010), we shall here omit the details and only provide the proof for the first claim, namely $\varphi \in (-1, 1)$. For this, it suffices to prove that φ cannot takes values in $\{-1, 1\}$, as autocorrelations are always bounded between ± 1. However, if $\varphi = 1$, then due to the stationarity, one must have $X_i = X_{i-1} = \cdots = X_0$, violating the short-range dependence condition that $\Theta_{0,2} < \infty$. The case for $\varphi = -1$ can be similarly argued, and thus $\varphi \notin \{-1, 1\}$. □

Proof. (Lemma 2) Let $\tilde{U}_i = X_i - \tilde{\varphi} X_{i-1}$, $i = 2, \ldots, n$, and

$$\hat{\gamma}_{\tilde{U},k} = \frac{1}{n-1} \sum_{i=2}^{n-|k|} (\tilde{U}_i - \bar{\tilde{U}}_{n-1})(\tilde{U}_{i+|k|} - \bar{\tilde{U}}_{n-1}), \quad \bar{\tilde{U}}_{n-1} = \frac{1}{n-1} \sum_{i=2}^{n} \tilde{U}_i.$$

then $\tilde{V}_i = \tilde{U}_i - (1-\tilde{\varphi})\bar{X}_n$ and $\bar{\tilde{V}}_{n-1} = \bar{\tilde{U}}_{n-1} - (1-\tilde{\varphi})\bar{X}_n$. Note that sample autocovariances are shift-invariant, we have $\hat{\gamma}_{\tilde{V},k} = \hat{\gamma}_{\tilde{U},k}$, $|k| < n-1$, and thus it suffices to prove the same result for $(\tilde{U}_i)$. For this, let $\tilde{D}_i = (\tilde{U}_i - \bar{\tilde{U}}_{n-1}) - (U_i - \bar{U}_{n-1})$, $i = 2, \ldots, n$, be the sequence of centered differences, then by elementary calculation $\tilde{D}_i = -(\tilde{\varphi} - \varphi)(X_{i-1} - \bar{X}_{n-1})$ and

$$\hat{\gamma}_{\tilde{U},k} - \hat{\gamma}_{U,k} = \frac{1}{n-1} \sum_{i=2}^{n-|k|} \{\tilde{D}_i (U_{i+|k|} - \bar{U}_{n-1}) + \tilde{D}_{i+|k|}(U_i - \bar{U}_{n-1}) + \tilde{D}_i \tilde{D}_{i+|k|}\} := \mathrm{I}_k + \mathrm{II}_k + \mathrm{III}_k,$$

where

$$\begin{aligned}
\mathrm{I}_k &= -(\tilde{\varphi}-\varphi)\frac{1}{n-1}\sum_{i=2}^{n-|k|}(X_{i+|k|-1}-\bar{X}_{n-1})(U_i-\bar{U}_{n-1});\\
\mathrm{II}_k &= -(\tilde{\varphi}-\varphi)\frac{1}{n-1}\sum_{i=2}^{n-|k|}(X_{i-1}-\bar{X}_{n-1})(U_{i+|k|}-\bar{U}_{n-1});\\
\mathrm{III}_k &= (\tilde{\varphi}-\varphi)^2\frac{1}{n-1}\sum_{i=2}^{n-|k|}(X_{i-1}-\bar{X}_{n-1})(X_{i+|k|-1}-\bar{X}_{n-1}).
\end{aligned}$$

We shall here provide uniform bounds for I_k, II_k and III_k, $|k| < n-1$, for which we need the following preparation. Let $\mathcal{F}_{i,j} = (\epsilon_i, \ldots, \epsilon_j)$, $i \le j$, with the convention that $\mathcal{F}_{i,j} = \emptyset$ if $i > j$, and define

$$\vartheta_{k,l} = E(U_k \mid \mathcal{F}_{k-l,k}) - E(U_k \mid \mathcal{F}_{k-l+1,k}).$$

Then for any fixed $l \in \mathbb{Z}$, $\vartheta_{k,l}$, $k = 2, \ldots, n$, form a sequence of martingale differences, and

$$\begin{aligned}
\|\vartheta_{k,l}\| &= \|E(U_l \mid \mathcal{F}_{0,l}) - E(U_l \mid \mathcal{F}_{1,l})\|\\
&\le \|E\{G(\mathcal{F}_l) - G(\mathcal{F}_l^\star) \mid \mathcal{F}_{0,l}\}\| + |\varphi| \cdot \|E\{G(\mathcal{F}_{l-1}) - G(\mathcal{F}_{l-1}^\star) \mid \mathcal{F}_{0,l}\}\|\\
&\le \theta_{l,2} + |\varphi|\theta_{l-1,2}.
\end{aligned}$$

Note that $E(U_i) = (1-\varphi)\mu$, by Doob's inequality we obtain that

$$\begin{aligned}
\left\|\max_{2\le k\le n}\left|\sum_{i=2}^{k}\{U_i-(1-\varphi)\mu\}\right|\right\| &= \left\|\max_{2\le k\le n}\left|\sum_{i=2}^{k}\sum_{l=0}^{\infty}\vartheta_{i,l}\right|\right\|\\
&\le \sum_{l=0}^{\infty}\left\|\max_{2\le k\le n}\left|\sum_{i=2}^{k}\vartheta_{i,l}\right|\right\|\\
&\le 2\sum_{l=0}^{\infty}\left(\sum_{i=2}^{n}\|\vartheta_{i,l}\|^2\right)^{1/2} \le 2(n-1)^{1/2}(1+|\varphi|)\Theta_{0,2}.
\end{aligned}$$

As a result, we have

$$\begin{aligned}
&\left\|\max_{|k|<n-1}\left|\frac{1}{n-1}\sum_{i=2}^{n-|k|}(U_i-\bar{U}_{n-1})\right|\right\|\\
\le&\left\|\max_{|k|<n-1}\left|\frac{1}{n-1}\sum_{i=2}^{n-|k|}\{U_i-(1-\varphi\mu)\}\right|\right\| + \left\|\frac{1}{n-1}\sum_{i=2}^{n}\{U_i-(1-\varphi\mu)\}\right\|\\
\le&\frac{4(1+|\varphi|)\Theta_{0,2}}{(n-1)^{1/2}} = O(n^{-1/2}),
\end{aligned}$$

and thus

$$\max_{|k|<n-1}\left|\mathrm{I}_k + (\tilde{\varphi}-\varphi)\frac{1}{n-1}\sum_{i=2}^{n-|k|}(X_{i+|k|-1}-\mu)(U_i-\bar{U}_{n-1})\right| = O_p(n^{-3/2}).$$

By a similar argument, one can obtain that

$$E\left\{\max_{|k|<n-1}\left|\frac{1}{n-1}\sum_{i=2}^{n-|k|}(X_{i+|k|-1}-\mu)(U_i-\bar{U}_{n-1})-\Gamma_{n,k,1}\right|\right\} \le \frac{2\Theta_{0,2}}{(n-1)^{1/2}}\cdot\frac{2(1+|\varphi|)\Theta_{0,2}}{(n-1)^{1/2}},$$

and thus

$$\max_{|k|<n-1}|\mathrm{I}_k + (\tilde{\varphi}-\varphi)\Gamma_{n,k,1}| = O_p(n^{-3/2}).$$

Following a similar martingale decomposition argument for II_k and III_k, we have

$$\max_{|k|<n-1}|\mathrm{II}_k + (\tilde{\varphi}-\varphi)\Gamma_{n,k,2}| = O_p(n^{-3/2})$$

and

$$\max_{|k|<n-1}\left|\mathrm{III}_k - (\tilde{\varphi}-\varphi)^2\Gamma_{n,k,3}\right| = O_p(n^{-2}),$$

Lemma 2 follows. □

Proof. (Lemma 3) Let $H(\mathcal{F}_i) = G(\mathcal{F}_i) - \varphi G(\mathcal{F}_{i-1})$, then $U_i = H(\mathcal{F}_i)$ and its functional dependence measure satisfies

$$\theta_{U,k,q} = \|H(\mathcal{F}_k) - H(\mathcal{F}_k^{\star})\|_q \le \theta_{k,q} + |\varphi|\theta_{k-1,q}.$$

Since $\theta_{k,q} = O(k^{-\delta})$ for some $\delta > 3/2$ as assumed, we have $\theta_{U,k,q} = O(k^{-\delta})$ and

$$\Theta_{U,k,q} = \sum_{i=k}^{\infty}\theta_{U,i,q} = O(k^{1-\delta}), \quad \Psi_{U,k,q} = \left(\sum_{i=k}^{\infty}\theta_{U,i,q}^2\right)^{1/2} = O(k^{1/2-\delta}).$$

As a result,

$$\begin{aligned}\Delta_{U,k,q} &= \sum_{i=0}^{\infty}\min(\Psi_{U,k,q},\theta_{U,i,q})\\ &= O[k^{1/2-\delta}k^{1-1/(2\delta)} + k^{\{1-1/(2\delta)\}(1-\delta)}] = O[k^{\{1-1/(2\delta)\}(1-\delta)}].\end{aligned}$$

Since $\|U_0\|_4 \le (1+|\varphi|)\|X_0\|_4$, by Lemma 6 of Xiao and Wu (2012) we obtain that

$$\lim_{n\to\infty}\mathrm{pr}\left\{\max_{|k|<n-1}|\hat{\gamma}_{U,k} - E(\hat{\gamma}_{U,k})| \le c_q^{\star}\left(\frac{\log n}{n-1}\right)^{1/2}\right\} = 1. \quad \text{(S1.1)}$$

Without loss of generality, assume that $\mu = E(X_0) = 0$. Then

$$\hat{\gamma}_{U,k} = \frac{1}{n-1}\sum_{i=2}^{n-|k|}U_iU_{i+|k|} + \left(1-\frac{|k|}{n-1}\right)\bar{U}_{n-1}^2 - \frac{1}{n-1}\sum_{i=2}^{n-|k|}(U_i+U_{i+|k|})\bar{U}_{n-1}, \quad \text{(S1.2)}$$

and by Lemma 1 of Liu and Wu (2010), there exists a constant $c_0 < \infty$ such that

$$\max_{|k|<n-1} \left\| \hat{\gamma}_{U,k} - \frac{1}{n-1} \sum_{i=1}^{n-|k|} U_i U_{i+|k|} \right\|_1 \leq c_0 n^{-1}.$$

Therefore, we have $\max_{|k|<n-1} |E(\hat{\gamma}_{U,k}) - \{1-|k|/(n-1)\}\gamma_{U,k}| = O(n^{-1})$ and

$$\lim_{n\to\infty} \mathrm{pr}\left\{ \max_{|k|<n-1} |E(\hat{\gamma}_{U,k})| \cdot \left| \frac{1}{\hat{\gamma}_{U,0}} - \frac{1}{\gamma_{U,0}} \right| \leq c_q^\star \left(\frac{\log\log n}{n-1} \right)^{1/2} \right\} = 1. \tag{S1.3}$$

Note that $(\log\log n)^{1/2} = o\{(\log n)^{1/2}\}$ and $(\xi+1)/2 > 1$, by (S1.1) and (S1.3),

$$\lim_{n\to\infty} \mathrm{pr}\left\{ \max_{|k|<n-1} \left| \hat{\rho}_{U,k} - \left(1 - \frac{|k|}{n-1}\right) \rho_{U,k} \right| \leq \frac{c_q^\star(\xi+1)}{2\hat{\gamma}_{U,0}} \left(\frac{\log n}{n-1} \right)^{1/2} \right\} = 1.$$

Since $\gamma_{U,0} = (1+\varphi^2)\gamma_0 - 2\varphi\gamma_1$ and $\xi > (\xi+1)/2 > 1$, Lemma 3 follows by (S1.3). □

Proof. (Lemma 4) Let $\nu_n = c_q\{(\log n)/n\}^{1/2}$ and $\rho^\circ_{U,k,n} = \{1-|k|/(n-1)\}\rho_{U,k}$, $|k| < n-1$. Note that $\lambda_n - \nu_n(\psi-1)/2 = \nu_n(\psi+1)/2 > \nu_n$, by Lemma 3 we have

$$\lim_{n\to\infty} \mathrm{pr}\left\{ \max_{l_n<|k|<n-1} |\hat{\rho}_{U,k} - \rho^\circ_{U,k,n}| \leq (\psi+1)\nu_n/2 \right\} = 1,$$

and thus

$$\lim_{n\to\infty} \mathrm{pr}\left\{ \sum_{l_n<|k|<n-1} (\hat{\rho}_{U,k} - \rho^\circ_{U,k,n}) \mathbb{1}_{\{|\hat{\rho}_{U,k}| \geq \lambda_n,\ |\rho^\circ_{U,k,n}| \leq \nu_n(\psi-1)/2\}} = 0 \right\} = 1.$$

On the other hand, since $|\rho^\circ_{U,k,n}| \leq |\rho_{U,k}|$ for all $|k| < n-1$, we can obtain that

$$\begin{aligned} & \sum_{l_n<|k|<n-1} (\hat{\rho}_{U,k} - \rho^\circ_{U,k,n}) \mathbb{1}_{\{|\hat{\rho}_{U,k}| \geq \lambda_n,\ |\rho^\circ_{U,k,n}| > \nu_n(\psi-1)/2\}} \\ \leq & \max_{l_n<|k|<n-1} |\hat{\rho}_{U,k} - \rho^\circ_{U,k,n}| \sum_{l_n<|k|<n-1} \frac{2|\rho^\circ_{U,k,n}|}{\nu_n(\psi-1)} = O_p\left(\sum_{l_n<|k|<n-1} |\rho^\circ_{U,k,n}| \right). \end{aligned}$$

Therefore, by using the fact that

$$\left| \sum_{l_n<|k|<n-1} \rho^\circ_{U,k,n} \mathbb{1}_{\{|\hat{\rho}_{U,k}| \geq \lambda_n\}} \right| \leq \sum_{l_n<|k|<n-1} |\rho^\circ_{U,k,n}| = O_p\left(\sum_{|k|>l_n} |\rho_{U,k}| \right),$$

we have

$$\sum_{|k|<n-1} \hat{\rho}_{U,k} \mathbb{1}_{\{|\hat{\rho}_{U,k}| \geq \lambda_n\}} = \sum_{|k|\leq l_n} \hat{\rho}_{U,k} \mathbb{1}_{\{|\hat{\rho}_{U,k}| \geq \lambda_n\}} + O_p\left(\sum_{|k|>l_n} |\rho_{U,k}| \right). \tag{S1.4}$$

We shall now deal with the sum for $|k| \le l_n$. For this, by Lemma 3, we have

$$\lim_{n\to\infty} \mathrm{pr}\left(\sum_{|k|\le l_n} \hat\rho_{U,k}\mathbb{1}_{\{|\hat\rho_{U,k}|\ge\lambda_n\}} = \sum_{|k|\le l_n} \hat\rho_{U,k}\mathbb{1}_{\{|\hat\rho_{U,k}|\ge\lambda_n,\ \rho_{U,k}\neq 0\}}\right) = 1,$$

and

$$\lim_{n\to\infty} \mathrm{pr}\left(\sum_{|k|\le l_n} \hat\rho_{U,k}\mathbb{1}_{\{|\hat\rho_{U,k}|<\lambda_n,\ \rho_{U,k}\neq 0\}} = \sum_{|k|\le l_n} \hat\rho_{U,k}\mathbb{1}_{\left\{|\hat\rho_{U,k}|<\lambda_n,\ |\rho^\circ_{U,k,n}|<2\lambda_n,\ \rho_{U,k}\neq 0\right\}}\right) = 1.$$

Therefore, by using the fact that

$$\left|\sum_{|k|\le l_n} \hat\rho_{U,k}\mathbb{1}_{\left\{|\hat\rho_{U,k}|<\lambda_n,\ |\rho^\circ_{U,k,n}|<2\lambda_n,\ \rho_{U,k}\neq 0\right\}}\right| \le \lambda_n \sum_{|k|\le l_n} \mathbb{1}_{\left\{|\rho^\circ_{U,k,n}|<2\lambda_n,\ \rho_{U,k}\neq 0\right\}},$$

we have

$$\sum_{|k|\le l_n} \hat\rho_{U,k}\mathbb{1}_{\{|\hat\rho_{U,k}|\ge\lambda_n\}} = \sum_{|k|\le l_n} \hat\rho_{U,k}\mathbb{1}_{\{\rho_{U,k}\neq 0\}} + O_p\left(\lambda_n \sum_{|k|\le l_n} \mathbb{1}_{\left\{|\rho^\circ_{U,k,n}|<2\lambda_n,\ \rho_{U,k}\neq 0\right\}}\right).$$

Hence, in combination with (S1.4), we have

$$\begin{aligned}\sum_{|k|<n-1} \hat\rho_{U,k}\mathbb{1}_{\{|\hat\rho_{U,k}|\ge\lambda_n\}} &= \sum_{|k|\le l_n} \hat\rho_{U,k}\mathbb{1}_{\{\rho_{U,k}\neq 0\}} + O_p\Bigg(\sum_{|k|>l_n} |\rho_{U,k}| \\ &\qquad + \lambda_n \sum_{|k|\le l_n} \mathbb{1}_{\left\{|\rho^\circ_{U,k,n}|<2\lambda_n,\ \rho_{U,k}\neq 0\right\}}\Bigg), \qquad \text{(S1.5)}\end{aligned}$$

and (i) follows by the fact that $\sum_{|k|\le l_n} \mathbb{1}_{\left\{|\rho^\circ_{U,k,n}|<2\lambda_n,\ \rho_{U,k}\neq 0\right\}} \le 2l_n + 1$. We shall now prove (ii), for which we need the following preparation. Let

$$\mathcal{P}_j\cdot = E(\cdot \mid \mathcal{F}_j) - E(\cdot \mid \mathcal{F}_{j-1}), \quad j \in \mathbb{Z},$$

be the projection operator, and define $\zeta_{k,j} = \mathcal{P}_j U_k$. Then $\|\zeta_{k,j}\| \le \theta_{k-j,2} + |\varphi|\theta_{k-j-1,2}$, and $\zeta_{k,j}$ and $\zeta_{k,j'}$ are orthogonal in the sense that $E(\zeta_{k,j}\zeta_{k,j'}) = 0$ if $j \neq j'$. Therefore, we have

$$\begin{aligned}|\mathrm{cov}(U_i, U_{i+|k|})| &= \left|E\left(\sum_{j\in\mathbb{Z}} \zeta_{i,j} \sum_{j'\in\mathbb{Z}} \zeta_{i+|k|,j'}\right)\right| \\ &\le \sum_{j\in\mathbb{Z}} \|\zeta_{i,j}\| \cdot \|\zeta_{i+|k|,j}\| \\ &\le \sum_{j=1}^{\infty} (\theta_{j,2} + |\varphi|\theta_{j-1,2})(\theta_{j+|k|,2} + |\varphi|\theta_{j+|k|-1,2}),\end{aligned}$$

because $\theta_{s,2} = 0$ if $s < 0$. Hence, if the functional dependence measure have a sparse structure, namely there exists a positive integer $M < \infty$ such that $\theta_{s,2} = 0$ for all $|s| > M$, then by the above inequality $\mathrm{cov}(U_i, U_{i+|k|}) = 0$ if $|k| > M$, and thus

$$\lim_{n\to\infty} \mathrm{pr}\left(\sum_{|k|\le l_n} \mathbb{1}_{\{|\rho^{\circ}_{U,k,n}|<2\lambda_n,\ \rho_{U,k}\neq 0\}} = \sum_{|k|\le M} \mathbb{1}_{\{|\rho^{\circ}_{U,k,n}|<2\lambda_n,\ \rho_{U,k}\neq 0\}}\right) = 1. \tag{S1.6}$$

Note that for any fixed $M < \infty$, the minimum absolute value of nonzero autocorrelations with lag $|k| \le M$ satisfies

$$\varepsilon_M = \min_{|k|\le M} \{|\rho_{U,k}| :\ \rho_{U,k} \neq 0\} > 0,$$

and thus

$$\begin{aligned} \varepsilon^{\circ}_{M,n} &= \min_{|k|\le M} \{|\rho^{\circ}_{U,k,n}| :\ \rho_{U,k} \neq 0\} \\ &\ge \{1 - M/(n-1)\}\varepsilon_M > \varepsilon_M/2 \end{aligned}$$

for all large n. Since $\lambda_n \to 0$ as $n \to \infty$, we have $\varepsilon^{\circ}_{M,n} \ge 2\lambda_n$ for all large n, and thus

$$\lim_{n\to\infty} \mathrm{pr}\left(\sum_{|k|\le M} \mathbb{1}_{\{|\rho^{\circ}_{U,k,n}|<2\lambda_n,\ \rho_{U,k}\neq 0\}} = 0\right) = 1. \tag{S1.7}$$

Then (ii) follows by (S1.5), (S1.6) and (S1.7). □

Proof. (Lemma 5) Recall that

$$\Gamma_{n,k,3} = \frac{1}{n-1}\sum_{i=2}^{n-|k|} (X_{i-1} - \mu)(X_{i+|k|-1} - \mu) = \frac{1}{n-1}\sum_{i=1}^{(n-1)-|k|} (X_i - \mu)(X_{i+|k|} - \mu),$$

then by the proof of (S1.1), we have

$$\max_{|k|<n-1} |\Gamma_{n,k,3} - E(\Gamma_{n,k,3})| = O_p\{n^{-1/2}(\log n)^{1/2}\}. \tag{S1.8}$$

Similarly, we can obtain that

$$\max_{|k|<n-1} |(\Gamma_{n,k,1} + \Gamma_{n,k,2}) - E(\Gamma_{n,k,1} + \Gamma_{n,k,2})| = O_p\{n^{-1/2}(\log n)^{1/2}\},$$

and thus by Lemma 2,

$$\max_{|k|<n-1} |\hat{\gamma}_{\tilde{V},k} - \hat{\gamma}_{U,k}| = O_p(n^{-1/2}).$$

Recall the definition of ν_n and $\rho^{\circ}_{U,k,n}$ from the proof of Lemma 4, then by Lemma 3 and the assumption that $\gamma_0 > 0$, we have

$$\lim_{n\to\infty} \mathrm{pr}\left\{\max_{|k|<n-1} |\hat{\rho}_{\tilde{V},k} - \rho^{\circ}_{U,k,n}| \le (\psi + 1)\nu_n/2\right\} = 1. \tag{S1.9}$$

Hence, by the proof of Lemma 4, we can obtain that

$$\sum_{|k|<n-1}\hat{\rho}_{\tilde{V},k}\mathbb{1}_{\{|\hat{\rho}_{\tilde{V},k}|\geq\lambda_n\}} = \sum_{|k|\leq l_n}\hat{\rho}_{\tilde{V},k}\mathbb{1}_{\{\rho_{U,k}\neq 0\}} + O_p\left(\sum_{|k|>l_n}|\rho_{U,k}| + \lambda_n\sum_{|k|\leq l_n}\mathbb{1}_{\{|\rho^{\circ}_{U,k,n}|<2\lambda_n,\ \rho_{U,k}\neq 0\}}\right),$$

and thus

$$\sum_{|k|<n-1}\hat{\gamma}_{\tilde{V},k}\mathbb{1}_{\{|\hat{\rho}_{\tilde{V},k}|\geq\lambda_n\}} = \sum_{|k|\leq l_n}\hat{\gamma}_{\tilde{V},k}\mathbb{1}_{\{\gamma_{U,k}\neq 0\}} + O_p\left(\sum_{|k|>l_n}|\gamma_{U,k}| + \lambda_n\sum_{|k|\leq l_n}\mathbb{1}_{\{|\rho^{\circ}_{U,k,n}|<2\lambda_n,\ \rho_{U,k}\neq 0\}}\right).$$

Since $n^{1/2}\lambda_n \to \infty$ as $n \to \infty$, it suffices to prove that

$$\sum_{|k|\leq l_n}\hat{\gamma}_{\tilde{V},k}\mathbb{1}_{\{\gamma_{U,k}\neq 0\}} - \sum_{|k|\leq l_n}\hat{\gamma}_{U,k}\mathbb{1}_{\{\gamma_{U,k}\neq 0\}} = O_p(n^{-1/2} + l_n/n).$$

For this, by Lemma 2 and (S1.8), we have

$$\sum_{|k|\leq l_n}\hat{\gamma}_{\tilde{V},k}\mathbb{1}_{\{\gamma_{U,k}\neq 0\}} - \sum_{|k|\leq l_n}\hat{\gamma}_{U,k}\mathbb{1}_{\{\gamma_{U,k}\neq 0\}} = -(\tilde{\varphi}-\varphi)\sum_{|k|\leq l_n}(\Gamma_{n,k,1}+\Gamma_{n,k,2}) + O_p(l_n/n).$$

Note that

$$\begin{aligned}\sum_{|k|\leq l_n}\Gamma_{n,k,1} &= \frac{1}{n-1}\sum_{|k|\leq l_n}\sum_{i=2}^{n-|k|}(X_{i-1}-\mu)\{U_{i+|k|}-(1-\varphi)\mu\} \\ &= \frac{1}{n-1}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}(X_i-\mu)\{U_{j+1}-(1-\varphi)\mu\}\mathbb{1}_{\{|i-j|\leq l_n\}},\end{aligned}$$

then by the m-dependence approximation as in the proof of Lemma A.2 of Zhang and Wu (2012) we obtain that

$$\sum_{|k|\leq l_n}\{\Gamma_{n,k,1} - E(\Gamma_{n,k,1})\} = O_p\{(l_n/n)^{1/2}\}.$$

A similar argument can be made on the sum of $\Gamma_{n,k,2}$, and as a result,

$$(\tilde{\varphi}-\varphi)\sum_{|k|\leq l_n}(\Gamma_{n,k,1}+\Gamma_{n,k,2}) = O_p(n^{-1/2} + n^{-1}l_n^{1/2}) = O_p(n^{-1/2} + l_n/n),$$

Lemma 5 follows. □

Proof. (Lemma 6) Let $s(\infty) = \sum_{k=0}^{\infty} \mathbb{1}_{\{\theta_{k,2}\neq 0\}}$ be the number of nonzero functional dependence measures, then $s(\infty) = \infty$ and $s(\infty) < \infty$ correspond to cases (i) and (ii) respectively. If $\tilde{\varphi} \geq \tau_n$, then $\hat{V}_i = \tilde{V}_i$ and thus by Lemma 5,

$$\sum_{|k|<n-1} \hat{\gamma}_{\hat{V},k} \mathbb{1}_{\{|\hat{\rho}_{\hat{V},k}|\geq\lambda_n\}} = \sum_{|k|\leq l_n} \hat{\gamma}_{U,k} \mathbb{1}_{\{\gamma_{U,k}\neq 0\}} + O_p\left[n^{-1/2} + l_n/n + \sum_{|k|>l_n} |\gamma_{U,k}| + \lambda_n l_n \mathbb{1}_{\{s(\infty)=\infty\}}\right].$$

On the other hand, if $\tilde{\varphi} < \tau_n$, then $\hat{V}_i = X_i - \bar{X}_n = U_i - \bar{X}_n$. Since sample autocovariances are shift-invariant, we have by Lemma 4,

$$\sum_{|k|<n-1} \hat{\gamma}_{\hat{V},k} \mathbb{1}_{\{|\hat{\rho}_{\hat{V},k}|\geq\lambda_n\}} = \sum_{|k|\leq l_n} \hat{\gamma}_{U,k} \mathbb{1}_{\{\gamma_{U,k}\neq 0\}} + O_p\left[\sum_{|k|>l_n} |\gamma_{U,k}| + \lambda_n l_n \mathbb{1}_{\{s(\infty)=\infty\}}\right].$$

We shall here derive a stochastic error bound for the term $\sum_{|k|\leq l_n} \hat{\gamma}_{U,k} \mathbb{1}_{\{\gamma_{U,k}\neq 0\}}$. For this, without loss of generality, assume that the mean $\mu = E(X_0) = 0$. Then by (S1.2) and the proof of Lemma 5, we have

$$\begin{aligned}
\sum_{|k|\leq l_n} \hat{\gamma}_{U,k} \mathbb{1}_{\{\gamma_{U,k}\neq 0\}} &= \frac{1}{n-1} \sum_{|k|\leq l_n} \sum_{i=2}^{n-|k|} U_i U_{i+|k|} \mathbb{1}_{\{\gamma_{U,k}\neq 0\}} + O_p(l_n/n) \\
&= \frac{1}{n-1} \sum_{|k|\leq l_n} \sum_{i=2}^{n-|k|} \gamma_{U,k} \mathbb{1}_{\{\gamma_{U,k}\neq 0\}} + O_p\{(l_n/n)^{1/2} + l_n/n\} \\
&= \sum_{|k|\leq l_n} \left(1 - \frac{|k|}{n-1}\right) \gamma_{U,k} + O_p\{(l_n/n)^{1/2}\},
\end{aligned}$$

and (i) follows. On the other hand, if there exists an $M < \infty$ such that $\theta_{k,2} = 0$ for all $k > M$ as in case (ii), then by the proof of Lemma 4 we have

$$\lim_{n\to\infty} \mathrm{pr}\left(\sum_{|k|\leq l_n} \hat{\gamma}_{U,k} \mathbb{1}_{\{\gamma_{U,k}\neq 0\}} = \sum_{|k|\leq M} \hat{\gamma}_{U,k} \mathbb{1}_{\{\gamma_{U,k}\neq 0\}}\right) = 1.$$

Note that

$$\sum_{|k|\leq M} \hat{\gamma}_{U,k} \mathbb{1}_{\{\gamma_{U,k}\neq 0\}} = \sum_{|k|\leq M} \left(1 - \frac{|k|}{n-1}\right) \gamma_{U,k} + O_p\{n^{-1/2}\},$$

(ii) follows. □

S2. Appendix B: Additional Details on Simulation

In our Monte Carlo simulations, we consider the linear process

$$\text{Model I}: X_i = \sum_{k=1}^{\infty} a_k \epsilon_{i-k+1} = a_1\epsilon_i + a_2\epsilon_{i-1} + a_3\epsilon_{i-2} + \cdots;$$

and its nonlinear generalization

$$\text{Model II}: X_i = a_1\epsilon_i|\epsilon_i| + \sum_{k=2}^{\infty} a_k\epsilon_{i-k+1} = a_1\epsilon_i|\epsilon_i| + a_2\epsilon_{i-1} + a_3\epsilon_{i-2} + \cdots ,$$

whose long-run variances are given by

$$g_X = \left(\sum_{k=1}^{\infty} a_k\right)^2 \text{var}(\epsilon_0)$$

and

$$g_X = \left(\sum_{k=2}^{\infty} a_k\right)^2 \text{var}(\epsilon_0) + 2a_1\left(\sum_{k=2}^{\infty} a_k\right)\text{cov}(\epsilon_0, \epsilon_0|\epsilon_0|) + a_1^2\text{var}(\epsilon_0|\epsilon_0|)$$

for Models I and II respectively. When generating the above processes and computing their long-run variances, we use the approximation that $\sum_{k=2}^{\infty} a_k\epsilon_{i-k+1} \approx \sum_{k=2}^{n} a_k\epsilon_{i-k+1}$ and $\sum_{k=2}^{\infty} a_k \approx \sum_{k=2}^{n} a_k$. For the P01 and PP12H estimates, we use the trapezoidal lag-window, and the associated bandwidth is selected by the empirical rule described in Appendix A of Paparoditis and Politis (2012). Note that the PP12T and PP12H estimates require the selection of a threshold, and Paparoditis and Politis (2012) in their Section 3.2 suggested a choice of $2\psi\hat{\gamma}_{X,0}\{(\log_{10} n)/n\}^{1/2}$ where $\psi > 1$ corresponds to effective thresholding; see for example conditions in their Theorem 1. For the PP12T estimate, we follow the rule-of-thumb choice of Paparoditis and Politis (2012) and use $\psi = 1.5$. For the PP12H estimate, we use $\psi = 1$ due to its superior performance for sparse linear processes as observed by Paparoditis and Politis (2012).

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