TOTAL-EFFECT TEST IS SUPERFLUOUS FOR ESTABLISHING MEDIATION IN THE CLASSIC MEDIATION MODEL

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Supplementary Material

In this supplementary material, we present the details for constructing the transformed data matrix $\tilde{\mathcal{D}}$ and the detailed proof for Lemma 2.

S1 Details for Constructing the Transformed Data Matrix $\tilde{\mathcal{D}}$

In this section, we provide the details for the construction of the transformed data matrix $\tilde{\mathcal{D}}$ from the original data matrix $\mathcal{D}$. Actually, we only need to show how to construct a upper triangular matrix $\tilde{D}$. Re-scaling $\tilde{D}$ is trivial.

Given the original data matrix $\mathcal{D} = (\mathbf{1}, \mathbf{X}, \mathbf{M}, \mathbf{Y})$ in the classic mediation model, which is a column full rank matrix with $rank(\mathcal{D}) = 4$, we can always find an orthogonal matrix Q via the standard Gram-Schmidt process to transfer $\mathcal{D}$ to an upper triangular matrix $\tilde{\mathcal{D}} = Q^T\mathcal{D}$. Please see section 5.2.7 in Golub and Van Loan (1996).

Below, we demonstrate the Gram-Schmidt process by a concrete numerical example. Suppose

$$\mathcal{D} = (\mathbf{1}, \mathbf{X}, \mathbf{M}, \mathbf{Y}) = \begin{pmatrix} 1 & 1 & 2.1 & 3.1 \\ 1 & 2 & 2.9 & 5.2 \\ 1 & 3 & 4.2 & 6.9 \\ 1 & 4 & 4.9 & 8.9 \\ 1 & 5 & 5.9 & 10.9 \end{pmatrix}.$$

The Gram-Schmidt process contains the steps below. Firstly, calculate the first columns of $\tilde{\mathcal{D}}$ and Q.

$$\tilde{\mathcal{D}}(1,1) = ||\mathbf{1}||_2 = \sqrt{5},$$

$$Q(:,1) = \mathbf{1}/\tilde{\mathcal{D}}(1,1) = (1/\sqrt{5}, 1/\sqrt{5}, 1/\sqrt{5}, 1/\sqrt{5}, 1/\sqrt{5}).$$

Then, we can generate the k-th column of $\tilde{\mathcal{D}}$ and Q for $k = 2, 3, 4$ in turn according to the following algorithm:

$$\tilde{\mathcal{D}}(1:k-1,k) = Q(:,1:k-1)'\mathcal{D}(:,k),$$

$$z = \mathcal{D}(:,k) - Q(:,1:k-1)\tilde{\mathcal{D}}(1:k-1,k),$$

$$\tilde{\mathcal{D}}(k,k) = ||z||_2,$$

$$Q(:,k) = z/\tilde{\mathcal{D}}(k,k).$$

And the numerical results are:

$$\tilde{\mathcal{D}} = \begin{pmatrix} 2.236 & 6.708 & 8.944 & 15.652 \\ 0 & 3.162 & 3.035 & 6.103 \\ 0 & 0 & 0.253 & -0.150 \\ 0 & 0 & 0 & 0.092 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Q = \begin{pmatrix} 0.447 & -0.632 & 0.079 & -0.306 & 0.547 \\ 0.447 & -0.316 & -0.553 & 0.510 & -0.365 \\ 0.447 & 0.000 & 0.790 & 0.204 & -0.365 \\ 0.447 & 0.316 & -0.237 & -0.714 & -0.365 \\ 0.447 & 0.632 & -0.079 & 0.306 & 0.547 \end{pmatrix}.$$

It's easy to check that $Q^TQ = I_5$, i.e., Q is an orthogonal matrix, and $\tilde{\mathcal{D}}$ upper triangular matrix with positive diagonals.

S2 Detailed Proof for Lemma 2

For simplicity of notation, we use **1**, **X**, **M** and **Y** to represent the transformed data matrices.

Here, we calculate LSE estimates $\hat{a}$, $\hat{b}$, $\hat{d}$ and $\hat{c}$:

$$\begin{pmatrix} \hat{i}_M \\ \hat{a} \end{pmatrix} = \left(\begin{pmatrix} \mathbf{1}^T \\ \mathbf{X}^T \end{pmatrix} \begin{pmatrix} \mathbf{1}, \mathbf{X} \end{pmatrix} \right)^{-1} \begin{pmatrix} \mathbf{1}^T \\ \mathbf{X}^T \end{pmatrix} \mathbf{M}$$
$$= \frac{1}{x_2^2} \begin{pmatrix} x_1^2 + x_2^2 & -x_1 \\ -x_1 & 1 \end{pmatrix} \begin{pmatrix} m_1 \\ x_1 m_1 + x_2 m_2 \end{pmatrix} = \begin{pmatrix} * \\ m_2/x_2 \end{pmatrix},$$

$$\begin{pmatrix} \hat{i}_Y \\ \hat{d} \\ \hat{b} \end{pmatrix} = \left(\begin{pmatrix} \mathbf{1}^T \\ \mathbf{X}^T \\ \mathbf{M}^T \end{pmatrix} \begin{pmatrix} \mathbf{1}, \mathbf{X}, \mathbf{M} \end{pmatrix} \right)^{-1} \begin{pmatrix} \mathbf{1}^T \\ \mathbf{X}^T \\ \mathbf{M}^T \end{pmatrix} \mathbf{Y}$$
$$= \frac{1}{x_2^2 m_3^2} \begin{pmatrix} * & * & * \\ x_2 m_1 m_2 - x_1 m_2^2 - x_1 m_3^2 & m_2^2 + m_3^2 & -x_2 m_2 \\ x_1 x_2 m_2 - x_2^2 m_1 & -x_2 m_2 & x_2^2 \end{pmatrix} \begin{pmatrix} y_1 \\ x_1 y_1 + x_2 y_2 \\ m_1 y_1 + m_2 y_2 + m_3 y_3 \end{pmatrix}$$
$$= \begin{pmatrix} * \\ (m_3 y_2 - m_2 y_3)/x_2 m_3 \\ y_3/m_3 \end{pmatrix},$$

$$\begin{pmatrix} \hat{i}_Y^* \\ \\ \hat{c} \end{pmatrix} = \left(\begin{pmatrix} \mathbf{1}^T \\ \\ \mathbf{X}^T \end{pmatrix} \begin{pmatrix} \mathbf{1}, \mathbf{X} \end{pmatrix} \right)^{-1} \begin{pmatrix} \mathbf{1}^T \\ \\ \mathbf{X}^T \end{pmatrix} \mathbf{Y}$$
$$= \frac{1}{x_2^2} \begin{pmatrix} x_1^2 + x_2^2 & -x_1 \\ \\ -x_1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ \\ x_1 y_1 + x_2 y_2 \end{pmatrix} = \begin{pmatrix} * \\ \\ y_2/x_2 \end{pmatrix},$$

where the symbol $*$ stands for terms we are not interested in.

By projecting data matrix onto subspaces, we have

$$\mathbf{M_1} = (m_1, 0, \cdots, 0), \; \mathbf{M_{1,X}} = (m_1, m_2, 0, \cdots, 0),$$
$$\mathbf{Y_1} = (y_1, 0, \cdots, 0), \; \mathbf{Y_{1,X}} = (y_1, y_2, 0, \cdots, 0), \; \mathbf{Y_{1,M,X}} = (y_1, y_2, y_3, 0, \cdots, 0),$$
$$\mathbf{Y_{1,M}} = \left(y_1, \frac{m_2 y_2 + m_3 y_3}{m_2^2 + m_3^2} \cdot m_2, \frac{m_2 y_2 + m_3 y_3}{m_2^2 + m_3^2} \cdot m_3, 0, \cdots, 0 \right).$$

Let $r = |m_2|/m_3$, $p = |y_3|/y_4$, $q = |y_2|/y_4$, $r_{n,\alpha} = [\lambda_{1,n-2}(\alpha)/(n-2)]^{1/2}$ and $p_{n,\alpha} = [\lambda_{1,n-3}(\alpha)/(n-3)]^{1/2}$. The rejection regions for a, b, c and d are as follows.

$$\mathcal{R}_a(\alpha) = \left\{ \frac{||\mathbf{M_{1,X}} - \mathbf{M_1}||/(2-1)}{||\mathbf{M} - \mathbf{M_{1,X}}||/(n-2)} > \lambda_{1,n-2}(\alpha) \right\} = \left\{ \frac{m_2^2}{m_3^2} > \frac{\lambda_{1,n-2}(\alpha)}{n-2} \right\}$$
$$= \{ r > r_{n,\alpha} \},$$
$$\mathcal{R}_b(\alpha) = \left\{ \frac{||\mathbf{Y_{1,M,X}} - \mathbf{Y_{1,X}}||/(3-2)}{||\mathbf{Y} - \mathbf{Y_{1,M,X}}||/(n-3)} > \lambda_{1,n-3}(\alpha) \right\} = \left\{ \frac{y_3^2}{y_4^2} > \frac{\lambda_{1,n-3}(\alpha)}{n-3} \right\}$$
$$= \{ p > p_{n,\alpha} \},$$

$$\mathcal{R}_c(\alpha) = \left\{ \frac{||\mathbf{Y_{1,X}} - \mathbf{Y_1}||/(2-1)}{||\mathbf{Y} - \mathbf{Y_{1,X}}||/(n-2)} > \lambda_{1,n-2}(\alpha) \right\} = \left\{ \frac{y_2^2}{y_3^2 + y_4^2} > \frac{\lambda_{1,n-2}(\alpha)}{n-2} \right\}$$
$$= \left\{ \frac{q^2}{1+p^2} > r_{n,\alpha}^2 \right\} = \left\{ q > r_{n,\alpha}\sqrt{1+p^2} \right\},$$
$$\mathcal{R}_d(\alpha) = \left\{ \frac{||\mathbf{Y_{1,M,X}} - \mathbf{Y_{1,M}}||/(3-2)}{||\mathbf{Y} - \mathbf{Y_{1,M,X}}||/(n-3)} > \lambda_{1,n-3}(\alpha) \right\} = \left\{ \frac{(m_2 y_3 - m_3 y_2)^2}{(m_2^2 + m_3^2) y_4^2} > \frac{\lambda_{1,n-3}(\alpha)}{n-3} \right\}$$
$$= \begin{cases} \left\{ \frac{(q-rp)^2}{1+r^2} > p_{n,\alpha}^2 \right\} = \left\{ |q-rp| > p_{n,\alpha}\sqrt{r^2+1} \right\}, & \text{if } m_2 y_2 y_3 \geq 0; \\ \left\{ \frac{(q+rp)^2}{1+r^2} > p_{n,\alpha}^2 \right\} = \left\{ |q+rp| > p_{n,\alpha}\sqrt{r^2+1} \right\}, & \text{if } m_2 y_2 y_3 < 0. \end{cases}$$

Bibliography

Golub, G. and Van Loan, C. (1996). *Matrix Computations.* Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, Baltimore.