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THE MEAN AND STANDARD DEVIATION OF THE RUN LENGTH DISTRIBUTION OF $\bar{X}$ CHARTS WHEN CONTROL LIMITS ARE ESTIMATED

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Abstract: Control charts with estimated control limits are widely used in practice. Common practice in control chart theory is to estimate the control limits using data from the process and once the process is determined to be in control to treat the resulting control limits as though fixed. Little is known about the run length distributions of these charts when the fact that control limits are estimated is taken into account. For example, no calculation has ever been done to find the mean and standard deviation of the run length distribution of $\bar{X}$ charts when process mean μ and standard deviation σ are estimated. In this paper, we derive and evaluate the mean and standard deviation of the $\bar{X}$ charts when control limits are estimated in three different ways. The results are then used to discuss the inadequacy of the widely followed empirical rules for choosing the number of samples m and the sample size n.

Key words and phrases: Average run length, control chart, standard deviation.

1. Introduction

For a process variable with a $N(\mu, \sigma^2)$ distribution, let Y_{ij}, $i = 1, 2, 3, \ldots$ and $j = 1, \ldots, n$ denote independent random samples of size n taken in sequence. When the mean μ and the standard deviation σ are known, the process can be monitored by plotting the sample means

$$\bar{Y}_i = \frac{1}{n}\sum_{j=1}^{n} Y_{ij}, \quad i = 1, 2, 3, \ldots,$$

on a Shewhart chart with 3σ control limits

$$UCL = \mu + 3\frac{\sigma}{\sqrt{n}}, \quad CL = \mu, \quad LCL = \mu - 3\frac{\sigma}{\sqrt{n}}. \tag{1.1}$$

Define E_i to be the event that the ith sample mean $\bar{Y}_i$ is either above UCL or below LCL. Then the events $\{E_i\}$ are independent and for all $i \geq 1$,

$$P(E_i) = P(\bar{Y}_i < LCL \text{ or } \bar{Y}_i > UCL) = 1 - \Phi(3) + \Phi(-3) = 0.0027,$$

where Φ is the distribution function of a $N(0,1)$ random variable. If we define U to be the number of samples until the first E_i occurs, then U is known as the run length of the chart and has a geometric distribution with parameter $p = P(E_i) = 0.0027$. It follows that the average run length (ARL) or the mean and the standard deviation (SD) of U are given by

$$E(U) = \frac{1}{p} = 370.4 \qquad \text{and} \qquad SD(U) = \frac{\sqrt{1-p}}{p} = 369.9. \tag{1.2}$$

When the mean μ and the standard deviation σ are unknown, the control limits in (1.1) need to be estimated. Suppose that X_{ij}, $i = 1, \ldots, m$ and $j = 1, \ldots, n$, are m independent samples of size n taken when the process is believed to be in control; we usually estimate UCL, CL and LCL by

$$\widehat{UCL} = \bar{\bar{X}} + 3\frac{\hat{\sigma}}{\sqrt{n}}, \quad \widehat{CL} = \bar{\bar{X}}, \quad \widehat{LCL} = \bar{\bar{X}} - 3\frac{\hat{\sigma}}{\sqrt{n}}, \tag{1.3}$$

where μ is estimated by the grand sample mean

$$\bar{\bar{X}} = \frac{1}{m}\sum_{i=1}^{m}\left(\frac{1}{n}\sum_{j=1}^{n} X_{ij}\right) = \frac{1}{m}\sum_{i=1}^{m}\bar{X}_i,$$

and σ can be estimated in at least three different ways. One way is to base the estimation on the average range

$$\bar{R} = \frac{1}{m}(R_1 + R_2 + \cdots + R_m), \tag{1.4}$$

where R_i is the range of the ith sample; and estimate σ by $\hat{\sigma} = \bar{R}/d_2$, where d_2 is a function of the sample size n defined by

$$d_2 = d_2(n) = E(Z_{(n)} - Z_{(1)}),$$

with $Z_{(n)}$ and $Z_{(1)}$ the largest term and the smallest term, respectively, in a random $N(0,1)$ sample $Z_1, \ldots, Z_n$. Another way is to base the estimation on the average standard deviation

$$\bar{S} = \frac{1}{m}(S_1 + S_2 + \cdots + S_m), \tag{1.5}$$

where S_i is the ith sample standard deviation

$$S_i = \left\{\frac{1}{n-1}\sum_{j=1}^{n}(X_{ij} - \bar{X}_i)^2\right\}^{1/2},$$

and estimate σ by $\hat{\sigma} = \bar{S}/c_4$, where c_4 is a function of the sample size n defined by

$$c_4 = c_4(n) = \left(\frac{2}{n-1}\right)^{1/2} \frac{\Gamma(n/2)}{\Gamma[(n-1)/2]}.$$

The third way is to base the estimation on the pooled sample standard deviation

$$S_p = \left\{\frac{1}{m}\sum_{i=1}^{m} S_i^2\right\}^{1/2} = \left\{\frac{1}{m(n-1)}\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij} - \bar{X}_i)^2\right\}^{1/2}, \qquad (1.6)$$

and estimate σ by $\hat{\sigma} = S_p/c_4[m(n-1)+1]$. Note that our estimator $\bar{\bar{X}}$ of μ is independent of our estimator $\hat{\sigma}$ of σ in each case.

Similar to the situation where the control limits are known, we define F_i to be the event that the ith sample mean $\bar{Y}_i$ is either above $\widehat{UCL}$ or below $\widehat{LCL}$, and define V to be the number of samples until the first F_i occurs. We find, however, that the behavior of $\{F_i\}$ and V is quite different from the behavior of $\{E_i\}$ and U: $\{E_i\}$ is an independent sequence, but $\{F_i\}$ is not; the distribution of U is well known as the geometric distribution, but the distribution of V is unknown. In fact, no calculation, to my knowledge, has ever been done to find the ARL and SD of V.

In practice, the well known rules for choosing m and n in order to estimate μ and σ "adequately" are to choose m between 20 and 30 with n equal to 4 or 5 (Montgomery (1991), Quesenberry (1993)). However, these rules are based primarily on empirical evidence. On the other hand, studies in control chart theory usually ignore the effect of estimating control limits, except in Hillier (1969) where the effect of estimation is considered explicitly in setting up $\bar{X}$ and R charts, and in Proschan and Savage (1960) where the maximum value of m is tabulated for a given n. But, as with many other studies, the above two studies are primarily concerned with controlling the Type I Error at desired levels. If control limits do not need to be estimated, knowing Type I Errors allows one to find the ARL and SD of the run length distribution of the control charts quite easily through equation (1.2). What is not fully realized is that when control limits must be estimated, knowing Type I Errors does not permit one to find ARL and SD using equation (1.2), even with the approach of Hillier (1969), or Proschan and Savage (1960). This point is first explored in Quesenberry (1993), where the dependence among the F_i is well documented, and through a simulation study, the inadequacy of the above mentioned empirical rules is addressed. Our work is directly motivated by the work of Quesenberry (1993).

In this paper, the ARL and SD of the run length distribution of the $\bar{X}$ charts are found when control limits are estimated in three different ways. Expressions for the ARL and SD are derived in Section 2. Numerical evaluation of these expressions are carried out in Section 3. Some discussions and conclusions regarding the choice of m and n are given in Section 4.

2. Derivation

Let X_{ij}, $i = 1, \ldots, m$ and $j = 1, \ldots, n$ denote historical data and let Y_{ij}, $i = 1, 2, 3, \ldots,$ and $j = 1, \ldots, n$ denote current or future data. In order to handle both the in-control and out-of-control cases, let $X_{ij} \sim N(\mu, \sigma^2)$ and let $Y_{ij} \sim N(\mu + a\sigma, b^2\sigma^2)$, where a and b are constants. When $a = 0$ and $b = 1$, the process is in control, otherwise the process is shifted and/or changed. Because $\bar{\bar{X}} \sim N(\mu, \sigma^2/(mn))$ and $\bar{Y}_i \sim (\mu + a\sigma, b^2\sigma^2/n)$, for any given $\bar{\bar{X}} = \bar{\bar{x}}$ and any given $\hat{\sigma}$, we have

$$
\begin{aligned}
&P(F_i \mid \bar{\bar{x}},\ \hat{\sigma}) \\
&= P\left(\bar{Y}_i < \widehat{LCL} \text{ or } \bar{Y}_i > \widehat{UCL} \mid \bar{\bar{x}},\ \hat{\sigma}\right) \\
&= 1 - \Phi\left(\frac{\bar{\bar{x}} + 3\hat{\sigma}/\sqrt{n} - \mu - a\sigma}{b\sigma/\sqrt{n}}\right) + \Phi\left(\frac{\bar{\bar{x}} - 3\hat{\sigma}/\sqrt{n} - \mu - a\sigma}{b\sigma/\sqrt{n}}\right) \\
&= 1 - \Phi\left(\frac{1}{b\sqrt{m}}\left(\frac{\bar{\bar{x}} - \mu}{\sigma/\sqrt{mn}}\right) + \frac{3}{b}\frac{\hat{\sigma}}{\sigma} - \frac{a}{b}\sqrt{n}\right) \\
&\quad + \Phi\left(\frac{1}{b\sqrt{m}}\left(\frac{\bar{\bar{x}} - \mu}{\sigma/\sqrt{mn}}\right) - \frac{3}{b}\frac{\hat{\sigma}}{\sigma} - \frac{a}{b}\sqrt{n}\right) \\
&= 1 - \Phi\left(\frac{z}{b\sqrt{m}} + \frac{3}{b}w - \frac{a}{b}\sqrt{n}\right) + \Phi\left(\frac{z}{b\sqrt{m}} - \frac{3}{b}w - \frac{a}{b}\sqrt{n}\right) \\
&= h(z, w; a, b), \text{ say,}
\end{aligned}
\tag{2.1}
$$

where $z = (\bar{\bar{x}} - \mu)/(\sigma/\sqrt{mn})$ and $w = \hat{\sigma}/\sigma$. Because $\{F_i\}$, given $\bar{\bar{x}}$ and $\hat{\sigma}$, are independent, from the equations (2.1) and (1.2) and the property of conditional expectation, we find the first two moments of V as

$$
\begin{aligned}
E(V) &= E\left\{E(V \mid \bar{\bar{X}},\ \hat{\sigma})\right\} \\
&= \int_{-\infty}^{+\infty}\int_{0}^{+\infty} \frac{1}{h(z, w; a, b)} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right) f(w)\,dz\,dw,
\end{aligned}
\tag{2.2}
$$

$$
\begin{aligned}
E(V^2) &= E\left\{E(V^2 \mid \bar{\bar{X}},\ \hat{\sigma})\right\} \\
&= \int_{-\infty}^{+\infty}\int_{0}^{+\infty} \left(\frac{2 - h(z, w; a, b)}{h^2(z, w; a, b)}\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right) f(w)\,dz\,dw,
\end{aligned}
\tag{2.3}
$$

where $f(w)$ is the probability density of $W = \hat{\sigma}/\sigma$ whose form depends on the normality assumption and m and n only. The standard deviation of V is now found according to

$$
SD(V) = \left\{E(V^2) - E^2(V)\right\}^{1/2}. \tag{2.4}
$$

3. Evaluation

In order to evaluate $E(V)$ and $E(V^2)$, we need $f(w)$ when σ is estimated by $\bar{R}/d_2(n)$, $\bar{S}/c_4(n)$ and $S_p/c_4[m(n-1)+1]$, respectively. For the first two cases, $f(w)$ is the result of an m-fold convolution of a known probability density, but unfortunately, its form is too complicated for numerical computation. For the third case, $f(w)$ is the density of a scaled χ distribution with $m(n-1)$ degrees of freedom. Our approach is therefore to approximate $f(w)$ in the first two cases and to use the exact $f(w)$ in the third case.

Because all three estimators of σ give close estimates and in the third case a scaled χ density is the exact density, we decide to use scaled χ densities to approximate $f(w)$ in the first two cases. Patnaik (1950) gives steps to follow if one wants to approximate the distribution of the average range by that of a $c\chi_\nu/\sqrt{\nu}$ random variable, where c and ν are constants to be determined, and χ_ν is a χ random variable with ν degrees of freedom. The approach of Patnaik (1950) can be applied to approximate the distribution of the average standard deviation; we make a small modification to give a unified presentation below.

Let $\mathrm{Var}\,(Z_{(n)}-Z_{(1)}) = v_2(n)$, $\mathrm{Var}\,(\bar{R}/[d_2(n)\sigma]) = M_1$ and $\mathrm{Var}\,(\bar{S}/[c_4(n)\sigma]) = M_2$. Then $M_1 = v_2(n)/[md_2^2(n)]$ and $M_2 = [1-c_4^2(n)]/[mc_4^2(n)]$ are known constants. For any positive constant M, let

$$r = \left\{-2+2\sqrt{1+2M}\right\}^{-1}, \tag{3.1}$$

$$t = M + \frac{1}{16r^3}, \tag{3.2}$$

$$\nu = \left\{-2+2\sqrt{1+2t}\right\}^{-1}, \tag{3.3}$$

$$c = 1 + \frac{1}{4\nu} + \frac{1}{32\nu^2} - \frac{5}{128\nu^3}. \tag{3.4}$$

Then the probability density of $c\chi_\nu/\sqrt{\nu}$ is

$$f(w;\nu,c) = \left(\frac{2}{c}\right)\frac{(\nu/2)^{\nu/2}}{\Gamma(\nu/2)}\left(\frac{w}{c}\right)^{\nu-1}\exp\left[-\frac{\nu}{2}\left(\frac{w}{c}\right)^2\right], \quad 0<w<\infty. \tag{3.5}$$

To approximate the density of $W = \hat{\sigma}/\sigma = \bar{R}/(d_2(n)\sigma)$ with (3.5), replace M in (3.1) to (3.4) by M_1 to obtain ν and c; to approximate the density of $W = \hat{\sigma}/\sigma = \bar{S}/(c_4(n)\sigma)$ with (3.5), replace M in (3.1) to (3.4) by M_2 to obtain ν and c. The exact density of $W = \hat{\sigma}/\sigma = S_p/\{c_4[m(n-1)+1]\sigma\}$ is given by (3.5) with $\nu = m(n-1)$ and $c = \{c_4[m(n-1)]+1\}^{-1}$.

The accuracy of the above approximations to the densities of $\bar{R}/(d_2(n)\sigma)$ and $\bar{S}/(c_4(n)\sigma)$ is studied through simulation. Without loss of generality, we take $\sigma = 1$ and simulate 10,000 combinations with $m = 5$ and $n = 4$ to generate 10,000 estimates of σ based on (1.4) and (1.5), respectively. The histograms of these estimates together with their corresponding approximate densities are plotted in Figure 1. It can be seen from Figure 1 that approximation (3.5) is good. For larger m and n (plots are not shown here), this approximation is even better.

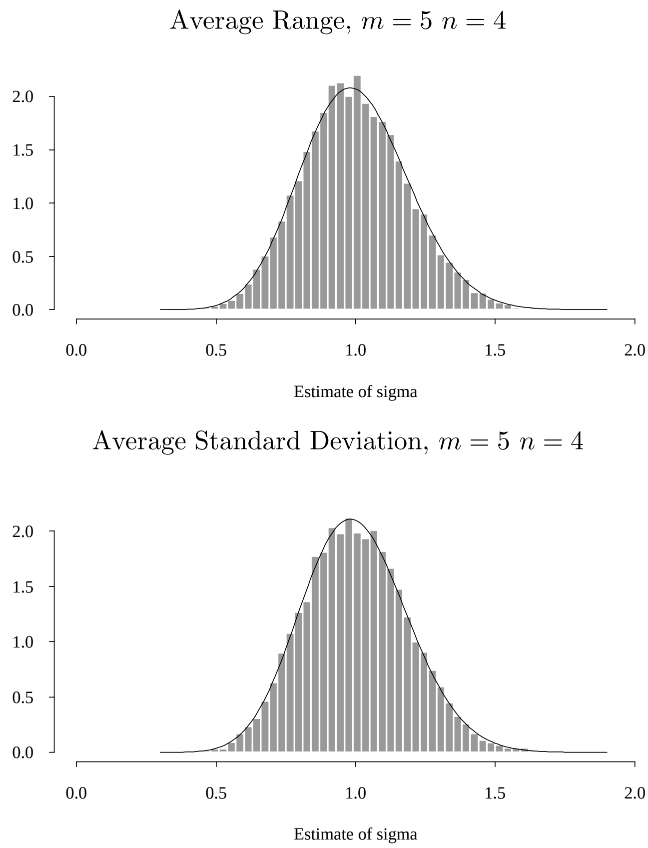

Figure 1. Approximation (3.5) to the densities of $\bar{R}/[d_2(4)\sigma]$ and $\bar{S}/[c_4(4)\sigma]$. Each histogram is based on 10,000 simulated point estimates of $\sigma = 1$. The solid line is the density (3.5) with $\nu = 13.9259$ and $c = 1.0181$ for the top plot and with $\nu = 14.2745$ and $c = 1.0177$ for the bottom plot.

With (3.5) ready for use, we evaluate (2.2) to (2.4) for various m, n, a and b using NAG routine D01DAF. Table 1 contains the results when the process is in control ($a = 0$ and $b = 1$), and Table 2 contains the results when there is a shift in the mean ($a \neq 0$, $b = 1$), or a shift in the variance ($a = 0$, $b \neq 1$), or a shift in both the mean and the variance ($a \neq 0$, $b \neq 1$).

Table 1. Average run length and standard deviation of the $\bar{X}$ charts with different estimators of σ, and when the process is in control. In each combination of m and n, the first value is when $\bar{R}/d_2(n)$ is used, the second value is when $\frac{\bar{S}}{c_4(n)}$ is used, and the third value is when $S_p/c_4[m(n-1)+1]$ is used.

	ARL							SD						
	n							n						
m	4	5	6	7	8	9	10	4	5	6	7	8	9	10
5	3071	1581	1024	775	643	564	512	45928	23152	12429	7225	4587	3179	2381
	2890	1426	904	679	562	493	448	43493	20332	9937	5284	3153	2125	1585
	2223	1191	806	630	534	475	435	33683	15740	7840	4319	2691	1887	1451
10	879	627	524	470	437	414	397	7277	2945	1728	1257	1021	881	790
	848	600	499	446	413	391	375	6684	2583	1497	1085	879	756	676
	754	564	481	435	406	386	371	4947	2146	1344	1012	836	727	657
20	520	454	420	400	387	378	371	1303	893	728	639	584	547	520
	513	445	411	391	377	368	361	1252	850	687	600	546	510	483
	492	435	405	387	375	366	359	1115	798	660	584	535	502	478
30	455	418	398	385	377	371	366	820	651	572	526	496	475	459
	451	413	392	379	371	365	360	800	631	551	504	474	453	437
	440	407	389	377	369	363	359	749	608	538	496	468	449	434
50	415	395	384	377	372	368	366	588	515	477	454	438	427	418
	413	392	380	373	368	365	362	580	505	466	442	426	415	406
	407	389	379	372	367	364	361	559	495	460	438	423	412	404
75	398	385	378	373	370	368	366	502	460	437	422	413	406	400
	397	383	376	371	368	365	364	497	454	430	416	405	398	392
	393	381	375	370	367	365	363	486	448	427	413	404	396	391
100	390	381	376	372	370	368	367	465	435	419	408	401	396	392
	389	380	374	371	368	366	365	461	431	414	403	396	390	386
	387	378	373	370	368	366	365	453	427	411	401	394	389	385
200	380	375	373	371	370	369	368	414	400	393	388	385	382	380
	379	375	372	370	369	368	367	412	399	391	386	382	379	377
	378	374	371	370	369	368	367	409	397	389	385	381	379	377
300	376	373	372	371	370	369	369	399	390	385	382	380	378	377
	376	373	371	370	369	369	368	398	389	384	380	378	376	375
	375	373	371	370	369	369	368	396	387	383	380	377	376	374
500	374	372	371	370	370	370	369	387	382	379	377	376	375	374
	374	372	371	370	370	369	369	386	381	378	376	375	374	373
	373	372	371	370	370	369	369	385	380	377	376	374	373	373
1000	372	371	371	370	370	370	370	378	376	374	373	373	372	372
	372	371	371	370	370	370	370	378	375	374	373	372	372	371
	372	371	370	370	370	370	370	377	375	374	373	372	372	371
∞	370							370						

Table 2. Average run length and standard deviation of the $\bar{X}$ charts with different estimators of σ when $n = 5$, and when the process is out of control. In each combination of m, a and b, the first value is when $\bar{R}/d_2(5)$ is used, the second value is when $\bar{S}/c_4(5)$ is used, and the third value is when $S_p/c_4[m(5-1)+1]$ is used.

	ARL							SD						
	a/b							a/b						
m	.3/1	.6/1	.9/1	0/1.2	0/1.4	.5/1.5	1/2	.3/1	.6/1	.9/1	0/1.2	0/1.4	.5/1.5	1/2
5	789.4	133.7	19.2	148.3	42.5	13.5	3.1	14322.8	3482.2	335.4	709.3	106.6	27.5	3.2
	716.4	123.8	18.4	141.4	41.4	13.3	3.1	12584.5	3065.1	297.5	643.7	100.0	26.3	3.2
	604.8	108.7	17.0	130.3	39.7	13.0	3.1	9753.1	2385.1	235.7	536.1	89.0	24.3	3.1
10	254.7	42.6	9.5	101.6	35.3	11.1	3.0	1354.0	181.1	21.9	216.5	54.0	14.9	2.7
	245.3	41.6	9.4	99.6	34.9	11.0	3.0	1197.6	164.9	20.8	204.0	52.2	14.7	2.6
	232.9	40.3	9.2	96.8	34.4	10.9	2.9	1008.6	145.0	19.3	187.6	49.9	14.2	2.6
20	156.2	28.4	7.5	88.6	32.9	10.1	2.9	334.0	49.1	9.9	125.8	39.9	11.3	2.4
	154.0	28.1	7.4	87.8	32.7	10.1	2.9	320.5	47.8	9.7	122.7	39.4	11.2	2.4
	151.2	27.8	7.4	86.1	32.5	10.1	2.9	304.5	46.2	9.6	118.8	38.6	11.1	2.4
30	133.9	25.3	7.0	85.4	32.2	9.8	2.9	221.7	35.5	8.1	107.4	36.4	10.4	2.4
	132.7	25.1	7.0	84.9	32.1	9.8	2.9	216.3	34.9	8.0	105.6	36.1	10.3	2.4
	131.3	25.0	6.9	84.2	32.0	9.8	2.9	210.1	34.3	7.9	103.5	35.7	10.3	2.4
50	118.6	23.2	6.6	83.2	31.7	9.6	2.8	160.6	28.0	7.0	95.1	33.9	9.7	2.3
	118.0	23.1	6.6	82.9	31.7	9.6	2.8	158.5	27.7	6.9	94.2	33.7	9.7	2.3
	117.4	23.0	6.6	82.5	31.6	9.6	2.8	156.0	27.4	6.9	93.1	33.5	9.6	2.3
75	111.8	22.3	6.5	82.2	31.5	9.5	2.8	136.8	25.0	6.5	89.7	32.8	9.4	2.3
	111.4	22.2	6.5	82.0	31.5	9.5	2.8	135.6	24.8	6.5	89.1	32.6	9.4	2.3
	111.0	22.2	6.5	81.8	31.4	9.5	2.8	134.2	24.7	6.4	88.4	32.5	9.3	2.3
100	108.6	21.8	6.4	81.7	31.4	9.4	2.8	126.2	23.6	6.3	87.1	32.2	9.2	2.3
	108.3	21.8	6.4	81.6	31.4	9.4	2.8	125.4	23.5	6.3	86.7	32.1	9.2	2.3
	108.0	21.7	6.4	81.4	31.4	9.4	2.8	124.5	23.4	6.2	86.2	32.0	9.2	2.3
200	103.9	21.2	6.3	81.1	31.3	9.4	2.8	111.8	21.7	6.0	83.5	31.4	9.0	2.3
	103.8	21.2	6.3	81.0	31.3	9.4	2.8	111.5	21.7	5.9	83.3	31.4	9.0	2.3
	103.7	21.1	6.3	81.0	31.2	9.3	2.8	111.1	21.7	5.9	83.0	31.3	9.0	2.3
300	102.4	21.0	6.3	80.9	31.2	9.3	2.8	107.4	21.2	5.9	82.3	31.1	8.9	2.3
	102.4	21.0	6.3	80.9	31.2	9.3	2.8	107.2	21.1	5.8	82.2	31.1	8.9	2.3
	102.3	20.9	6.3	80.8	31.2	9.3	2.8	106.9	21.1	5.8	82.0	31.1	8.9	2.3
500	101.2	20.8	6.2	80.7	31.2	9.3	2.8	103.8	20.7	5.8	81.3	30.9	8.8	2.3
	101.2	20.8	6.2	80.7	31.2	9.3	2.8	103.8	20.7	5.8	81.3	30.9	8.8	2.3
	101.2	20.8	6.2	80.7	31.2	9.3	2.8	103.7	20.7	5.8	81.2	30.9	8.8	2.3
1000	100.4	20.7	6.2	80.6	31.1	9.3	2.8	101.4	20.4	5.7	80.6	30.8	8.8	2.3
	100.4	20.7	6.2	80.6	31.1	9.3	2.8	101.4	20.4	5.7	80.6	30.8	8.8	2.3
	100.3	20.7	6.2	80.6	31.1	9.3	2.8	101.4	20.4	5.7	80.6	30.8	8.8	2.3
∞	99.5	20.6	6.2	80.5	31.1	9.3	2.8	99.0	20.1	5.7	80.0	30.6	8.8	2.3

4. Discussions and Conclusions

From Table 1 and Table 2, we observe that for the m and n combinations considered and in an average sense,

- Estimating σ to set up any one of the three $\bar{X}$ charts has noticeable effect on the ARL and SD of the chart. This effect is large when $m < 20$; is still fairly large when $20 \leq m < 50$; becomes small when $50 \leq m < 500$; and becomes very small when $m \geq 500$.
- As m increases, ARL approaches its limiting value faster than SD does.
- It is possible for ARL to be slightly smaller than its limiting value, but the same thing does not happen to SD.
- Among the three estimators of σ, $S_p/c_4[m(n-1)+1]$ performs uniformly better than $\bar{S}/c_4(n)$, and $\bar{S}/c_4(n)$ performs uniformly better than $\bar{R}/d_2(n)$, in terms of producing an SD that is close to its limiting value. The same statement is, however, not true for ARL when n is large.
- For fixed m, both ARL and SD are decreasing functions of n.

For the out-of-control case, we have presented the results for $n = 5$ only. The results for other values of n are qualitatively the same. Based on the above observations, we conclude that

1. If μ and σ need to be estimated and an $\bar{X}$ chart that performs as if μ and σ were known is desired, it is necessary to take m to be at least 100 when $n = 5$, and $m \geq 50$ when $n = 10$. This is not always possible in practice, but it is a result that should be better known.
2. The pooled estimator $S_p/c_4[m(n-1)+1]$ gives better control performance and should be preferred, unless the need for simplicity suggests the use of the average range $\bar{R}/d_2(n)$.
3. When the total number of measurements that can be taken to estimate control limits is fixed (i.e. m times n is a constant), increasing n and decreasing m has the desirable effect of getting run length distributions with smaller SD. For example, when $mn = 100$ and S_p is used, SD is 808 for $m = 20$ and $n = 5$, while SD is only 663 for $m = 10$ and $n = 10$, and both cases have acceptable ARL. When setting up $\bar{X}$ charts, this possibility should be explored together with other concerns, such as monitoring short-term versus long-term process variation.
4. The findings of this paper are meaningful in an average sense. They do not apply to specific individual $\bar{X}$ charts.

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