

# SIMPLE OPTIMAL TESTS FOR CIRCULAR REFLECTIVE SYMMETRY ABOUT A SPECIFIED MEDIAN DIRECTION

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*Abstract:* In this paper we propose optimal tests for circular reflective symmetry about a fixed median direction. The distributions against which optimality is achieved are the  $k$ -sine-skewed distributions of Umbach and Jammalamadaka (2009). We first show that sequences of  $k$ -sine-skewed models are locally and asymptotically normal in the vicinity of reflective symmetry. Following the Le Cam methodology, we construct optimal (in the maximin sense) parametric tests for reflective symmetry, which we render semi-parametric by a studentization argument. These asymptotically distribution-free tests happen to be uniformly optimal (under any reference density) and are moreover of a simple and intuitive form. They furthermore exhibit nice small sample properties, as we show through a Monte Carlo simulation study. Our new tests also allow us to re-visit the famous red wood ants data set of Jander (1957). The choice of  $k$ -sine-skewed alternatives, which are the circular analogues of the Azzalini-type linear skew-symmetric distributions, permits us a Fisher singularity analysis *à la* Hallin and Ley (2012) with the result that only the prominent sine-skewed von Mises distribution suffers from these inferential drawbacks. We conclude the paper by discussing the unspecified location case.

*Key words and phrases:* Circular statistics, Fisher information singularity, skewed distributions, tests for symmetry.

## 1. Introduction

Symmetry is a fundamental and ubiquitous structural assumption in statistics, underpinning most classical inferential methods, be it for univariate data on the real line or for circular data. Its acceptance generally simplifies the statistician's task, both in the elaboration of new theoretical tools and in the analysis of a given set of observations. For instance, the classical models for circular data, such as the von Mises, cardioid, wrapped normal, or wrapped Cauchy distributions (see Mardia and Jupp (2000, Sec. 3.5)) are all symmetric about their unique mode. This form of symmetry on the circle is called *reflective symmetry*. However, quoting Mardia (1972, p.10), "symmetrical distributions on the circle are comparatively rare", and recent years have seen an increasing interest in

non-symmetric models (see, e.g., Umbach and Jammalamadaka (2009), Kato and Jones (2010), Abe and Pewsey (2011) or Jones and Pewsey (2012)). It is then all the more important to be able to test whether the hypothesis of symmetry holds or not to know whether the classical or the modern models should be used. Since circular distributions are encountered in several domains of scientific investigation, with particular emphasis on the analysis of phases of periodic phenomena (physics, biology, etc.) and on directions (animal movements as a response to some stimulus, pigeon homing, earth sciences, etc.), practical examples needing tests for circular symmetry are common. In this paper, we are interested in those settings where the experimental setup suggests a specific direction about which to test symmetry, e.g. in animal orientation problems.

While testing for symmetry about a fixed center (the median) is a classical issue on the real line and has generated an important number of publications, the situation is quite different in the circular case. Indeed, the null hypothesis of circular reflective symmetry is relatively unexplored in the literature. There exist essentially three proposals for such tests (not to be confused with the tests for  $\ell$ -fold symmetry on the circle, see Jupp and Spurr (1983) or Mardia and Jupp (2000, p.146)):

- Schach (1969) constructs locally optimal linear rank tests against rotation alternatives, the circular analogue of a linear shift alternative. His construction comprises the circular sign and Wilcoxon tests.
- Universally consistent tests from the linear setting have been adapted to the circular case (such as the runs tests, see Pewsey (2004)).
- A “true” test for circular symmetry has been studied in Pewsey (2004) by having recourse to the second sine moment about the fixed median direction, a classical measure of circular skewness, see Batschelet (1965).

The scarcity of existing tests for circular symmetry might at first sight seem puzzling as one may be tempted to say that all tests for (linear) univariate symmetry should be adaptable to the circular setup (such as done for runs tests), replacing the real line by  $[-\pi, \pi)$ . However, this translation from one setup to the other is not straightforward, due to several facts including that the points at  $\pi$  and  $-\pi$  coincide as periodicity is an essential feature of circular distributions. Moreover, when observations are distributed on a large arc of the circle, it is likely that adapted tests suffer from a loss of power (Pewsey (2004)). It also seems unlikely that optimal tests on the real line would retain their optimality features on the circle, nothing *a priori* ensures that they behave well against the complicated wrapped versions of the univariate skew distributions they were designed for. Thus, except against rotation alternatives, there exist no optimal tests for reflective symmetry.

Our aim in the present paper is to fill this gap by proposing tests for circular reflective symmetry about a fixed center that behave very well against certain (general) skew alternatives. More specifically, we build locally and asymptotically optimal (in the maximin sense) tests for symmetry against *k-sine-skewed alternatives* (Umbach and Jammalamadaka (2009), Abe and Pewsey (2011)), a broad class of recently proposed skew circular distributions that has received an increasing interest over the past few years (see Section 2 for a description). These skew distributions are obtained by perturbation of a base symmetric distribution via a factor involving sines and a parameter to regulate skewness. The motivations for this choice are mainly twofold: they are circular analogues of the skew-symmetric distributions on the real line (see Azzalini and Capitanio (2003) or Wang, Boyer, and Genton (2004)) inspired by the skew-normal distribution proposed in the seminal paper Azzalini (1985), and the resulting test statistics are based on the (trigonometric) sine moments. We thus provide these classical measures of circular skewness as well as the test of Pewsey (2004) with so far not known optimality properties. Our findings also enable us to discuss Fisher singularity issues exactly as in the linear case.

The backbone of our approach is the Le Cam methodology which, although of linear nature, lends itself well for a transcription to circular settings (and even, with much more complications, to data living on unit hyperspheres in higher dimensions, see Ley et al. (2013)) In a first stage, we obtain optimal parametric tests, and then, by means of studentization arguments, we turn them into semi-parametric ones, valid under the entire null hypothesis of symmetry and *uniformly* optimal under *any* given symmetric base distribution. We hence derive, as Schach (1969), a family of fully efficient semi-parametric tests which, in our case, are always optimal. For a given density, our tests behave asymptotically like the likelihood ratio tests, but they improve on the latter by their simplicity and the fact that, thanks to the Le Cam approach, one can derive explicit power expressions against sequences of contiguous skew alternatives.

The paper is organized as follows. In Section 2, we first describe the family of *k-sine-skewed* distributions, then establish their *ULAN* property in the vicinity of symmetry, the crucial step in the Le Cam approach, and discuss some aspects of this property. In Section 3, we construct our optimal tests for reflective symmetry about a known center and investigate their asymptotic properties. The finite-sample performances of our tests for reflective symmetry are evaluated and compared to existing tests in a large Monte Carlo simulation study, see Section 4. Application to the famous red wood ants data set of Jander (1957) is reported on in Section 5. The Fisher information singularity issue is tackled in Section 6. Section 7 concludes the paper with final comments and an outlook on the case

where the median direction is not specified, as considered e.g., in Pewsey (2002). An Appendix collects the proofs.

## 2. $k$ -sine-skewed Distributions and the ULAN Property

In this section, we first describe in detail the class of  $k$ -sine-skewed circular distributions and then establish their ULAN property. Throughout, all angles are in radians and we choose without loss of generality the zero direction as initial direction and anti-clock-wise orientation of the unit circle. To stress the difference between symmetry and asymmetry, we consider the interval  $[-\pi, \pi)$  instead of  $[0, 2\pi)$ , and define quantities such as the cumulative distribution function (cdf) accordingly. This can lead to some differences with the commonly adopted notation of, e.g., Mardia and Jupp (2000, Chap. 3), but they do not affect the mathematical outcomes.

### 2.1. $k$ -sine-skewed densities

The  $k$ -sine-skewed distributions are obtained by perturbation of a base symmetric density. Define the collection

$$\mathcal{F} := \left\{ f_0 : f_0(\theta) > 0 \text{ a.e.}, f_0(\theta + 2\pi k) = f_0(\theta) \forall k \in \mathbb{Z}, f_0(-\theta) = f_0(\theta), \right. \\ \left. f_0 \text{ unimodal at } 0, \int_{-\pi}^{\pi} f_0(\theta) d\theta = 1 \right\}$$

of unimodal reflectively symmetric (about the zero direction) circular densities. The periodicity requirement is both classical and essential when dealing with circular distributions. The best-known representatives of the collection  $\mathcal{F}$  are the von Mises, cardioid, and wrapped Cauchy distributions, with respective densities  $f_{\text{VM}_\kappa}(\theta) := (1/2\pi I_0(\kappa)) \exp(\kappa \cos(\theta))$  for  $\kappa > 0$  ( $I_0$  stands for the modified Bessel function of the first kind and order zero),  $f_{\text{CA}_\ell}(\theta) := (1/2\pi)(1 + \ell \cos(\theta))$  for  $\ell \in (0, 1)$ , and  $f_{\text{WC}_\rho}(\theta) := [(1 - \rho^2)/2\pi][1/(1 + \rho^2 - 2\rho \cos(\theta))]$  for  $\rho \in (0, 1)$ . A location parameter  $\mu \in [-\pi, \pi)$  is readily introduced as center of symmetry, leading to densities  $f(\theta - \mu), \theta \in [-\pi, \pi)$ , with mode  $\mu$ . Inspired by the classical one-dimensional skewing method of Azzalini and Capitanio (2003), Umbach and Jammalamadaka (2009) have skewed such symmetric densities  $f_0$  by transforming them into

$$2f_0(\theta - \mu)G(\omega(\theta - \mu)), \quad \theta \in [-\pi, \pi),$$

where  $G(\theta) = \int_{-\pi}^{\theta} g(y)dy$  is the cdf of some circular symmetric density  $g$  and  $\omega$  is a weighting function satisfying, for all  $\theta \in [-\pi, \pi)$ , the three conditions  $\omega(-\theta) = -\omega(\theta)$ ,  $\omega(\theta + 2\pi k) = \omega(\theta) \forall k \in \mathbb{Z}$ , and  $|\omega(\theta)| \leq \pi$ . This construction being too general, and for the sake of mathematical tractability, Umbach and

Jammalamadaka particularized their choice to  $G(\theta) = (\pi + \theta)/(2\pi)$ , the cdf of the uniform circular distribution, and  $\omega(\theta) = \lambda\pi \sin(k\theta)$ ,  $k \in \mathbb{N}_0$ , with  $\lambda \in (-1, 1)$  playing the role of a skewness parameter. This yields what we call the *k-sine-skewed densities*

$$f_{\mu,\lambda}^k(\theta) := f_0(\theta - \mu)(1 + \lambda \sin(k(\theta - \mu))), \quad \theta \in [-\pi, \pi), \quad (2.1)$$

with location parameter  $\mu \in [-\pi, \pi)$  and skewness parameter  $\lambda \in (-1, 1)$ . When  $\lambda = 0$ , no perturbation occurs and we retrieve the base symmetric density, otherwise (2.1) is skewed to the left ( $\lambda > 0$ ) or to the right ( $\lambda < 0$ ). Further properties of *k-sine-skewed* distributions are that  $f_{\mu,\lambda}^k(\mu - \theta) = f_{\mu,-\lambda}^k(\mu + \theta)$ ,  $f_{\mu,\lambda}^k(\mu) = f_0(0)$  whatever the value of  $\lambda$ , and the two endpoints,  $f_{\mu,\lambda}^k(\mu - \pi)$  and  $f_{\mu,\lambda}^k(\mu + \pi)$ , coincide. For  $k \geq 2$ ,  $f_{\mu,\lambda}^k$  is multimodal whereas, for  $k = 1$ , multimodality only rarely occurs. This explains why Abe and Pewsey (2011) restricted their attention to the study of the densities

$$f_{\mu,\lambda}(\theta) := f_0(\theta - \mu)(1 + \lambda \sin(\theta - \mu)), \quad \theta \in [-\pi, \pi), \quad (2.2)$$

which they called sine-skewed circular densities (hence our terminology *k-sine-skewed densities* for general  $k$ ). Abe and Pewsey have given the conditions under which the densities (2.2) are multimodal. In the present paper, we establish all our theoretical results and propose tests for general *k-sine-skewed* distributions. Note that, when  $f_0$  is the circular uniform density, then (2.2) is the cardioid density  $f_{\text{CA}\lambda}$  with mode at  $\mu + \pi/2 \pmod{2\pi}$ , hence, in passing, we will as well consider an optimal test for uniformity against the cardioid distribution.

Sine-skewed (and *k-sine-skewed*) distributions lend themselves well to modeling real data phenomena. Aside from Abe and Pewsey (2011), these skew-circular distributions have been used, *inter alia*, in the analysis of the CO<sub>2</sub> daily cycle in the low atmosphere at a rural site (Pérez et al. (2012)) and of forest disturbance regimes (Abe et al. (2012)). This, combined with the motivations stated in the Introduction, makes *k-sine-skewed* distributions an appealing choice as asymmetric alternatives in the construction of tests for circular reflective symmetry.

## 2.2. The ULAN property for *k-sine-skewed* densities

Let  $\theta_1, \dots, \theta_n$  be i.i.d. circular observations with common density (2.1). For any symmetric base density  $f_0 \in \mathcal{F}$  and any  $k \in \mathbb{N}_0$ , denote by  $P_{\boldsymbol{\vartheta};f_0,k}^{(n)}$ , where  $\boldsymbol{\vartheta} := (\mu, \lambda)' \in [-\pi, \pi) \times (-1, 1)$ , the joint distribution of the  $n$ -tuple  $\theta_1, \dots, \theta_n$ . Since, for  $\lambda = 0$ , the density  $f_{\mu,\lambda}^k$  reduces to  $f_0$  and hence does not depend on  $k$ , we drop the index  $k$  and simply write  $P_{\boldsymbol{\vartheta};f_0}^{(n)}$  at  $\boldsymbol{\vartheta} = \boldsymbol{\vartheta}_0 := (\mu, 0)'$ . Any pair  $(f_0, k)$  induces the parametric location-skewness model

$$\mathcal{P}_{f_0,k}^{(n)} := \left\{ P_{\boldsymbol{\vartheta};f_0,k}^{(n)} : \boldsymbol{\vartheta} \in [-\pi, \pi) \times (-1, 1) \right\},$$

whereas any  $k \in \mathbb{N}_0$  induces the semi-parametric location-skewness model  $\mathcal{P}_k^{(n)} := \cup_{f_0 \in \mathcal{F}} \mathcal{P}_{f_0, k}^{(n)}$ .

The first step in our construction of tests for symmetry about a fixed center consists in establishing the *Uniform Local Asymptotic Normality (ULAN)* property, in the vicinity of symmetry (i.e., at  $\lambda = 0$ ), of the parametric model  $\mathcal{P}_{f_0, k}^{(n)}$ . This paves the way to numerous other applications of the Le Cam theory such as the construction of tests for symmetry about an unspecified center or of the one-step optimal estimators, see e.g., van der Vaart (2002). ULAN requires the following mild regularity condition on the base densities  $f_0$ .

**Assumption (A).** The function  $f_0(\theta)$  is a.e.- $\mathcal{C}^1$  over  $[-\pi, \pi)$  (or equivalently over  $\mathbb{R}$  by periodicity) with a.e.-derivative  $\dot{f}_0$ .

Most classical reflectively symmetric densities satisfy this requirement. Note that the  $\mathcal{C}^1$  condition over a bounded set combined with the fact that  $f_0 > 0$  and the periodicity condition entails that, letting  $\varphi_{f_0} = -\dot{f}_0/f_0$ , the Fisher information quantity for location  $I_{f_0} := \int_{-\pi}^{\pi} \varphi_{f_0}^2(\theta) f_0(\theta) d\theta$  is finite. ULAN of the parametric model  $\mathcal{P}_{f_0, k}^{(n)}$  with respect to  $\boldsymbol{\vartheta} = (\mu, \lambda)'$ , in the vicinity of symmetry, then takes the following form.

**Theorem 1.** *Let  $f_0 \in \mathcal{F}$  and  $k \in \mathbb{N}_0$ , and assume that Assumption (A) holds. Then, for any  $\mu \in [-\pi, \pi)$ , the parametric family of densities  $\mathcal{P}_{f_0, k}^{(n)}$  is ULAN at  $\boldsymbol{\vartheta}_0 = (\mu, 0)'$  with central sequence*

$$\begin{aligned} \Delta_{f_0, k}^{(n)}(\mu) &:= \begin{pmatrix} \Delta_{f_0, k; 1}^{(n)}(\mu) \\ \Delta_{k; 2}^{(n)}(\mu) \end{pmatrix} \\ &:= \frac{1}{\sqrt{n}} \sum_{i=1}^n \begin{pmatrix} \varphi_{f_0}(\theta_i - \mu) \\ \sin(k(\theta_i - \mu)) \end{pmatrix}, \end{aligned}$$

and corresponding Fisher information matrix

$$\mathbf{\Gamma}_{f_0, k} := \begin{pmatrix} \Gamma_{f_0, k; 11} & \Gamma_{f_0, k; 12} \\ \Gamma_{f_0, k; 12} & \Gamma_{f_0, k; 22} \end{pmatrix},$$

where  $\Gamma_{f_0, k; 11} := I_{f_0}$ ,  $\Gamma_{f_0, k; 12} := -\int_{-\pi}^{\pi} \sin(k\theta) \dot{f}_0(\theta) d\theta$ , and  $\Gamma_{f_0, k; 22} := \int_{-\pi}^{\pi} \sin^2(k\theta) f_0(\theta) d\theta$ . More precisely, for any  $\mu^{(n)} = \mu + O(n^{-1/2})$  and for any bounded sequence  $\boldsymbol{\tau}^{(n)} = (\tau_1^{(n)}, \tau_2^{(n)})' \in \mathbb{R}^2$  such that  $n^{-1/2} \tau_2^{(n)}$  belongs to  $(-1, 1)$ , we have, letting  $\Lambda^{(n)} := \log(d\mathbb{P}_{(\mu^{(n)} + n^{-1/2} \tau_1^{(n)}, n^{-1/2} \tau_2^{(n)})'; f_0, k}^{(n)} / d\mathbb{P}_{(\mu^{(n)}, 0)'; f_0, k}^{(n)})$ ,

$$\Lambda^{(n)} = \boldsymbol{\tau}^{(n)'} \Delta_{f_0, k}^{(n)}(\mu^{(n)}) - (1/2) \boldsymbol{\tau}^{(n)'} \mathbf{\Gamma}_{f_0, k} \boldsymbol{\tau}^{(n)} + o_P(1) \tag{2.3}$$

and  $\Delta_{f_0, k}^{(n)}(\mu^{(n)}) \xrightarrow{\mathcal{L}} \mathcal{N}_2(\mathbf{0}, \mathbf{\Gamma}_{f_0, k})$ , both under  $\mathbb{P}_{(\mu^{(n)}, 0)'; f_0, k}^{(n)}$  as  $n \rightarrow \infty$ .

The proof is given in the Appendix. One easily sees that the Fisher information for skewness  $\Gamma_{f_0,k;22}$ , and hence the cross-information quantity  $\Gamma_{f_0,k;12}$ , is finite by bounding  $\sin^2$  by 1 under the integral sign. Note that the constant  $k$  has no effect on the validity of Theorem 1 and that  $\Delta_{k;2}^{(n)}(\mu)$  does not depend on  $f_0$ , a fact that will become of great interest in the sequel. With this ULAN property in hand, we are ready to derive our optimal tests for reflective symmetry about a fixed center  $\theta$ , as explained below in Section 2.3. Moreover, since we do not fix  $\mu$  in Theorem 1, our result also paves the way for deriving optimal tests for symmetry about an unknown center; see Section 7.

We conclude the section by noting, in view of the proof of Lemma A.1 in the Appendix, that Assumption (A) can be weakened to

Assumption (A<sup>min</sup>). The mapping  $\theta \mapsto f_0^{1/2}(\theta)$  is differentiable in quadratic mean over  $[-\pi, \pi)$  (or equivalently over  $\mathbb{R}$  by periodicity) with quadratic mean or weak derivative  $(f_0^{1/2})'(\theta)$  and, letting  $\psi_{f_0}(\theta) = -2(f_0^{1/2})'(\theta)/f_0^{1/2}(\theta)$ , the Fisher information quantity for location  $J_{f_0} := \int_{-\pi}^{\pi} \psi_{f_0}^2(\theta) f_0(\theta) d\theta$  is finite.

Quadratic mean differentiability of  $f_0^{1/2}$ , a classical requirement in the Le Cam framework, means that  $\int_{-\pi}^{\pi} (f_0^{1/2}(\theta + h) - f_0^{1/2}(\theta) - h\psi_{f_0}(\theta))^2 d\theta = o(h^2)$  as  $h \rightarrow 0$ , which corresponds exactly to the integral (A.1) of the proof of Theorem 1 with  $h = -t$  and hence is the minimal condition in order to have the ULAN property of the parametric model  $\mathcal{P}_{f_0,k}^{(n)}$ . Under Assumption (A), these two derivatives coincide a.e., as well as  $\psi_{f_0}$  and  $\varphi_{f_0}$ , and  $J_{f_0} = I_{f_0}$  (the  $\mathcal{C}^1$  condition ensures finiteness of  $I_{f_0}$ , while in (A<sup>min</sup>) one requires that  $J_{f_0} < \infty$ ).

### 2.3. Constructing Le Cam optimal tests from the ULAN property

The ULAN property allows us to deduce that (see Le Cam (1986) for details) our parametric location-skewness model  $\mathcal{P}_{f_0,k}^{(n)}$  is locally (around  $(\mu, 0)'$ ) and asymptotically (for large sample sizes) equivalent to a simple Gaussian shift model. Intuitively, this follows from the fact that the likelihood ratio expansion (2.3), up to the remainder terms, strongly resembles the likelihood ratio of a Gaussian shift model  $\mathcal{N}_2(\mathbf{\Gamma}_{f_0,k}\boldsymbol{\tau}^{(n)}, \mathbf{\Gamma}_{f_0,k})$  with a single observation denoted by  $\boldsymbol{\Delta}_{f_0,k}^{(n)}$ . Since the optimal procedures for Gaussian shift experiments are well-known, we can translate them into our circular location-skewness model and hence obtain inferential procedures that are asymptotically optimal in the maximin sense. Recall that a test  $\phi^*$  is called maximin in the class  $\mathcal{C}_\alpha$  of level- $\alpha$  tests for the null  $\mathcal{H}_0$  against the alternative  $\mathcal{H}_1$  if (i)  $\phi^*$  has level  $\alpha$  and (ii) the power of  $\phi^*$  is such that

$$\inf_{P \in \mathcal{H}_1} \mathbb{E}_P[\phi^*] \geq \sup_{\phi \in \mathcal{C}_\alpha} \inf_{P \in \mathcal{H}_1} \mathbb{E}_P[\phi].$$

We employ this scheme for testing the null hypothesis  $\mathcal{H}_0^\mu$  of symmetry about a known central direction  $\mu \in [-\pi, \pi)$ . Our procedures are (asymptotically) optimal against a fixed  $k$ -sine-skewed alternative (2.1). We first construct  $f_0$ -parametric tests for  $\mathcal{H}_{0;f_0}^\mu = \mathbb{P}_{(\mu,0)';f_0}^{(n)}$ : the optimality of these tests under the base density  $f_0$  is thwarted by the fact that they meet the asymptotic level- $\alpha$  constraint only under  $f_0$ . To avoid this non-validity beyond  $f_0$ , we make use of a classical *studentization argument* allowing us to turn our parametric tests into tests for the semi-parametric null hypothesis  $\mathcal{H}_0^\mu = \cup_{f_0 \in \mathcal{F}} \mathbb{P}_{(\mu,0)';f_0}^{(n)}$ .

### 3. The Test Statistic and Its Asymptotic Properties

Fix  $\mu \in [-\pi, \pi)$ . The  $f_0$ -parametric test  $\phi_{f_0;k}^{(n);\mu}$  for circular reflective symmetry about a known central direction  $\mu$  we propose rejects  $\mathcal{H}_{0;f_0}^\mu$  at asymptotic level  $\alpha$  whenever the statistic

$$Q_{f_0;k}^{(n);\mu} := \frac{|\Delta_{k;2}^{(n)}(\mu)|}{\Gamma_{f_0,k;22}^{1/2}} = \frac{|n^{-1/2} \sum_{i=1}^n \sin(k(\theta_i - \mu))|}{\Gamma_{f_0,k;22}^{1/2}} \tag{3.1}$$

exceeds  $z_{\alpha/2}$ , the  $\alpha/2$  upper quantile of the standard normal distribution. Tests for reflective symmetry against one-sided alternatives of the form  $\lambda > 0$  or  $\lambda < 0$  are built similarly. It follows from the Le Cam theory that this test is locally and asymptotically maximin for testing the null  $\mathcal{H}_{0;f_0}^\mu$  against  $\mathcal{H}_{1;f_0,k}^\mu := \cup_{\lambda \neq 0 \in (-1,1)} \mathbb{P}_{(\mu,\lambda)';f_0,k}^{(n)}$ . This optimality does not hold against  $k'$ -sine-skewed laws with  $k' \neq k$ , each value of  $k$  leads to a distinct optimal test.

Consider  $g_0 \in \mathcal{F}$ . Under  $\mathbb{P}_{(\mu,0)';g_0}^{(n)}$ ,  $\Delta_{k;2}^{(n)}(\mu)$  is asymptotically normal with mean 0 and variance  $\Gamma_{g_0,k;22} \neq \Gamma_{f_0,k;22}$ . It is therefore natural to consider the *studentized* test  $\phi_k^{*(n);\mu}$  that rejects (at asymptotic level  $\alpha$ ) the null of circular reflective symmetry  $\mathcal{H}_0^\mu$  when

$$Q_k^{*(n);\mu} := \frac{|\sum_{i=1}^n \sin(k(\theta_i - \mu))|}{(\sum_{i=1}^n \sin^2(k(\theta_i - \mu)))^{1/2}} \tag{3.2}$$

exceeds  $z_{\alpha/2}$ . This very simple test statistic does not depend on  $f_0$  (hence the omission of the index  $f_0$  in  $\phi_k^{*(n);\mu}$ ) since the central sequence for skewness,  $\Delta_{k;2}^{(n)}(\mu)$ , does *not* depend on  $f_0$ . This remarkable fact implies that all parametric tests  $\phi_{f_0;k}^{(n);\mu}$ ,  $k \in \mathbb{N}$ , lead to the same studentized test statistic  $\phi_k^{*(n);\mu}$ , which therefore inherits optimality from its parametric antecedents under *any* base symmetric distribution! This is all summarized in the following result (see the Appendix for a proof).

**Theorem 2.** *Let  $k \in \mathbb{N}_0$ . Then,*



- (i) under  $\cup_{f_0 \in \mathcal{F}} P_{(\mu,0)';f_0}^{(n)}$ ,  $Q_k^{*(n);\mu} \xrightarrow{D} \mathcal{N}(0,1)$  as  $n \rightarrow \infty$ , so that the test  $\phi_k^{*(n);\mu}$  has asymptotic level  $\alpha$  under the same hypothesis;
- (ii) under  $P_{(\mu, n^{-1/2}\tau_2^{(n)})';f_0, k'}^{(n)}$  with  $f_0 \in \mathcal{F}$  and  $k' \in \mathbb{N}_0$ ,  $Q_k^{*(n);\mu}$  is asymptotically normal with mean  $\Gamma_{f_0, k; 22}^{-1/2} C_{f_0}(k, k') \tau_2$  and variance 1, where  $\tau_2 = \lim_{n \rightarrow \infty} \tau_2^{(n)}$  and  $C_{f_0}(k, k') := \int_{-\pi}^{\pi} \sin(k\theta) \sin(k'\theta) f_0(\theta) d\theta$  (which is finite);
- (iii) for all  $f_0 \in \mathcal{F}$ ,  $Q_k^{*(n);\mu} = Q_{f_0, k}^{(n);\mu} + o_P(1)$  as  $n \rightarrow \infty$  under  $P_{(\mu,0)';f_0}^{(n)}$ , so that the studentized test  $\phi_k^{*(n);\mu}$  is locally and asymptotically maximin, at asymptotic level  $\alpha$ , when testing  $\mathcal{H}_0^\mu$  against alternatives of the form  $\cup_{\lambda \neq 0 \in (-1,1)} \cup_{f_0 \in \mathcal{F}} P_{(\mu, \lambda)';f_0, k}^{(n)}$ .

Theorem 2(i) shows that the studentized test  $\phi_k^{*(n);\mu}$  is indeed valid under the entire null hypothesis  $\mathcal{H}_0^\mu$ , hence is asymptotically distribution-free. Note the uniform (in  $f_0$ , not in  $k$ ) optimality of our studentized test.

We have also considered above alternatives where  $k \in \mathbb{N}$  is replaced by some  $k' \in \mathbb{N}$  possibly different from the  $k$  used in the construction of the tests. Point (ii) of Theorem 2 allows us to give the explicit asymptotic power of  $\phi_k^{*(n);\mu}$  against the local alternatives  $P_{(\mu, n^{-1/2}\tau_2^{(n)})';f_0, k'}^{(n)}$ :

$$1 - \Phi\left(z_{\alpha/2} - (\Gamma_{f_0, k; 22})^{-1/2} C_{f_0}(k, k') \tau_2\right) + \Phi\left(-z_{\alpha/2} - (\Gamma_{f_0, k; 22})^{-1/2} C_{f_0}(k, k') \tau_2\right),$$

where  $\Phi$  is the standard Gaussian cdf. In Figure 1, we have plotted this power as a function of  $\tau_2$  for  $f_0$  the von Mises density with concentration parameter 1 and for  $k = 2$  and  $k' = 1, 2, 3$ . The plot shows that the power of the test is lower if  $k'$  is not correctly chosen.

For  $f_0(\theta) = 1/2\pi$ , the uniform density, (2.2) with  $k = 1$  corresponds to the cardioid density with mode at  $\mu + \pi/2 \in [-\pi, \pi)$ . Hence, for fixed  $\mu$ , an optimal test for testing the null hypothesis of uniformity against cardioid alternatives shall be based on (3.1) with  $k = 1$  and  $\Gamma_{f_0, k; 22} = \frac{1}{2}$ . It can be easily shown that the test statistic coincides with the Rayleigh (1919) test statistic, known to be optimal for the null hypothesis of uniformity against cardioid alternatives (see Jammalamadaka and Sengupta (2001, p.133)). When  $k = 2$ ,  $\phi_2^{*(n);\mu}$  coincides with the so-called ‘‘b2-star’’ test proposed in Pewsey (2004). We have thus shown that that test enjoys maximin optimality features against 2-sine-skewed alternatives, and provided its asymptotic powers against contiguous alternatives. This not only complements, but also gives further insight into the b2-star test. Finally, our tests are easy to interpret, being based on sine moments, the classical measures of skewness for circular data (see, e.g., Batschelet (1965)).

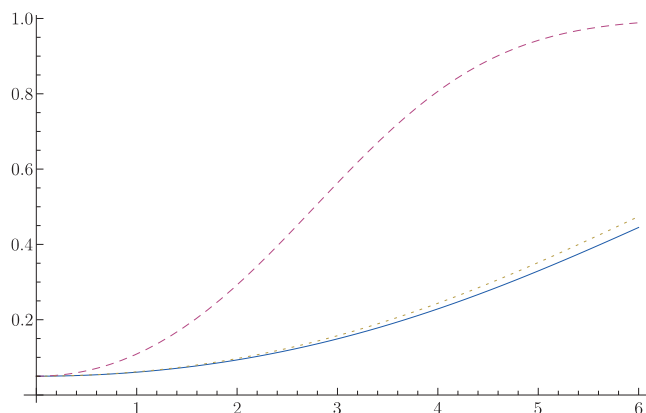


Figure 1. Power curves, as a function of  $\tau_2$ , of the studentized test  $\phi_k^{*(n);\mu}$  for  $k = 2$  against local alternatives  $P_{(\mu, n^{-1/2}\tau_2^{(n)})'; f_0, k'}^{(n)}$  for  $f_0$  the von Mises density with concentration parameter 1 and for  $k'$  equal to 1 (solid line), 2 (dashed line) and 3 (dotted line).

#### 4. Monte Carlo Simulation Study

In this section we report on the finite-sample properties of the proposed testing procedures for reflective symmetry. We check the nominal level constraint under distinct forms of reflective symmetry and determine the power properties under various forms of asymmetry. For this, we generated  $N = 10,000$  independent samples of sizes  $n = 30$  and  $n = 100$  from reflectively symmetric and increasingly skewed ( $\lambda > 0$ ) circular distributions, and ran our tests (which contain Pewsey's b2star test), as well as the modified runs test of Pewsey (2004), under two-sided form at the asymptotic level  $\alpha = 5\%$ .

Without loss of generality, we fix the center of symmetry  $\mu$  to 0. We ran our tests  $\phi_1^{*(n);0}$ ,  $\phi_2^{*(n);0}$  and  $\phi_3^{*(n);0}$  as well as the modified runs test  $\phi_{\text{modrun}}^{(n)}$  (with  $p = 0.6$ , see Pewsey (2004)). We consider  $k = 1, 2, 3$  for our tests because these values are able to capture both skew unimodality ( $k = 1$ ) and multimodality, but do not lead to too many oscillations of the sines within  $[-\pi, \pi)$  which can lead to numerical difficulties.

As reflectively symmetric distributions representing the null hypothesis, we considered the von Mises laws  $f_{VM_1}$  and  $f_{VM_{10}}$ , the cardioid  $f_{CA_{0.5}}$ , the wrapped Cauchy  $f_{WC_{0.5}}$  as well as a mixture of two  $f_{VM_1}$  and two  $f_{VM_{10}}$  von Mises laws with, in each case, respective centers at  $-\pi/4$  and  $\pi/4$  and mixing probability 0.5. The latter mixture is meant to assess the performances of our tests under bimodality. The densities  $f_{VM_1}$  and  $f_{VM_{10}}$  were then turned into their 1-, 2- and 3-sine-skewed versions, whereas  $f_{CA_{0.5}}$  and  $f_{WC_{0.5}}$  were 1- and 2-sine-skewed. The skewness parameter  $\lambda$  was increased from zero to successively positive values.

Table 1. Empirical rejection probabilities, out of  $N = 10,000$  replications and for the sample sizes  $n = 30$  and  $n = 100$ , under various reflectively symmetric and 1-sine-skewed distributions, of the optimal tests  $\phi_1^{*(n);0}$ ,  $\phi_2^{*(n);0}$  and  $\phi_3^{*(n);0}$  as well as of the modified runs test  $\phi_{\text{modrun}}^{(n)}$  with  $p = 0.6$ . Tests were performed at level  $\alpha = 5\%$ .

Test	$n = 30/n = 100$	$n = 30/n = 100$	$n = 30/n = 100$	$n = 30/n = 100$
1-sine-skewed $f_{VM_1}$				
	$\lambda = 0$	$\lambda = 0.2$	$\lambda = 0.4$	$\lambda = 0.6$
$\phi_1^{*(n);0}$	0.047/0.048	0.110/0.266	0.311/0.779	0.608/0.988
$\phi_2^{*(n);0}$	0.051/0.053	0.063/0.090	0.101/0.228	0.156/0.449
$\phi_3^{*(n);0}$	0.046/0.054	0.047/0.053	0.051/0.062	0.056/0.074
$\phi_{\text{modrun}}^{(n)}$	0.054/0.054	0.064/0.072	0.112/0.156	0.204/0.375
1-sine-skewed $f_{VM_{10}}$				
	$\lambda = 0$	$\lambda = 0.2$	$\lambda = 0.4$	$\lambda = 0.6$
$\phi_1^{*(n);0}$	0.046/0.048	0.058/0.095	0.091/0.235	0.164/0.458
$\phi_2^{*(n);0}$	0.047/0.048	0.059/0.093	0.090/0.234	0.164/0.451
$\phi_3^{*(n);0}$	0.048/0.047	0.058/0.087	0.086/0.212	0.151/0.408
$\phi_{\text{modrun}}^{(n)}$	0.053/0.051	0.054/0.060	0.070/0.089	0.096/0.150
1-sine-skewed $f_{CA_{0.5}}$				
	$\lambda = 0$	$\lambda = 0.2$	$\lambda = 0.4$	$\lambda = 0.6$
$\phi_1^{*(n);0}$	0.051/0.051	0.111/0.292	0.341/0.824	0.666/0.994
$\phi_2^{*(n);0}$	0.049/0.046	0.054/0.065	0.070/0.107	0.089/0.185
$\phi_3^{*(n);0}$	0.052/0.048	0.046/0.051	0.052/0.050	0.049/0.048
$\phi_{\text{modrun}}^{(n)}$	0.043/0.051	0.061/0.064	0.094/0.128	0.163/0.262
1-sine-skewed $f_{WC_{0.5}}$				
	$\lambda = 0$	$\lambda = 0.2$	$\lambda = 0.4$	$\lambda = 0.6$
$\phi_1^{*(n);0}$	0.048/0.049	0.100/0.227	0.270/0.693	0.539/0.972
$\phi_2^{*(n);0}$	0.050/0.052	0.060/0.083	0.091/0.197	0.143/0.382
$\phi_3^{*(n);0}$	0.050/0.049	0.048/0.056	0.060/0.085	0.067/0.125
$\phi_{\text{modrun}}^{(n)}$	0.051/0.052	0.061/0.074	0.106/0.153	0.200/0.345

The bimodal mixture of von Mises laws was skewed by shifting the center  $\pi/4$  to  $\pi/4 + \lambda$ . To investigate other forms of perturbation of symmetry, we applied the Moebius transform of Kato and Jones (2010) to  $f_{VM_1}$  and  $f_{VM_{10}}$  with  $r = 0.5$ : this transforms each  $\theta_i$ ,  $i = 1, \dots, n$ , into  $\lambda + 2 \arctan(\omega_r \tan((\theta_i - \lambda)/2))$  with  $\omega_r = (1 - r)/(1 + r)$ . The empirical rejection probabilities are reported in Table 1 for 1-sine-skewed alternatives, Table 2 for 2-sine-skewed alternatives, and Table 3 for Moebius, von Mises mixtures and 3-sine-skewed alternatives.

Whatever sample sizes considered, all four tests met the 5% nominal level constraint under each reflectively symmetric density considered, even under bimodality, and appeared unbiased. Under  $k$ -sine-skewed alternatives, the optimality features of our tests  $\phi_k^{*(n);0}$  were confirmed, whereas under certain  $k'$ -sine-skewed densities the test  $\phi_k^{*(n);0}$  for  $k \neq k'$  exhibited low powers (especially when combining the indices 1 and 3). When the observations were highly concentrated

Table 2. Empirical rejection probabilities, out of  $N = 10,000$  replications and for the sample sizes  $n = 30$  and  $n = 100$ , under various reflectively symmetric and 2-sine-skewed distributions, of the optimal tests  $\phi_1^{*(n);0}$ ,  $\phi_2^{*(n);0}$  and  $\phi_3^{*(n);0}$  as well as of the modified runs test  $\phi_{\text{modrun}}^{(n)}$  with  $p = 0.6$ . Tests were performed at level  $\alpha = 5\%$ .

Test	$n = 30/n = 100$	$n = 30/n = 100$	$n = 30/n = 100$	$n = 30/n = 100$
2-sine-skewed $f_{VM_1}$				
	$\lambda = 0$	$\lambda = 0.2$	$\lambda = 0.4$	$\lambda = 0.6$
$\phi_1^{*(n);0}$	0.048/0.048	0.064/0.103	0.107/0.254	0.186/0.491
$\phi_2^{*(n);0}$	0.049/0.049	0.115/0.295	0.347/0.825	0.669/0.994
$\phi_3^{*(n);0}$	0.047/0.047	0.060/0.097	0.102/0.242	0.173/0.482
$\phi_{\text{modrun}}^{(n)}$	0.048/0.048	0.056/0.067	0.081/0.122	0.131/0.244
2-sine-skewed $f_{VM_{10}}$				
Test	$\lambda = 0$	$\lambda = 0.2$	$\lambda = 0.4$	$\lambda = 0.6$
$\phi_1^{*(n);0}$	0.049/0.049	0.084/0.175	0.201/0.572	0.414/0.901
$\phi_2^{*(n);0}$	0.050/0.050	0.086/0.179	0.205/0.577	0.419/0.907
$\phi_3^{*(n);0}$	0.050/0.049	0.083/0.173	0.201/0.553	0.405/0.890
$\phi_{\text{modrun}}^{(n)}$	0.049/0.054	0.065/0.068	0.117/0.176	0.211/0.405
2-sine-skewed $f_{CA_{0.5}}$				
Test	$\lambda = 0$	$\lambda = 0.2$	$\lambda = 0.4$	$\lambda = 0.6$
$\phi_1^{*(n);0}$	0.049/0.048	0.051/0.060	0.067/0.106	0.086/0.186
$\phi_2^{*(n);0}$	0.048/0.049	0.121/0.292	0.336/0.820	0.670/0.993
$\phi_3^{*(n);0}$	0.047/0.051	0.055/0.065	0.066/0.113	0.089/0.187
$\phi_{\text{modrun}}^{(n)}$	0.046/0.050	0.059/0.071	0.097/0.137	0.171/0.289
2-sine-skewed $f_{WC_{0.5}}$				
Test	$\lambda = 0$	$\lambda = 0.2$	$\lambda = 0.4$	$\lambda = 0.6$
$\phi_1^{*(n);0}$	0.051/0.051	0.066/0.096	0.105/0.223	0.165/0.456
$\phi_2^{*(n);0}$	0.053/0.055	0.113/0.277	0.315/0.788	0.642/0.992
$\phi_3^{*(n);0}$	0.046/0.048	0.066/0.094	0.109/0.264	0.192/0.524
$\phi_{\text{modrun}}^{(n)}$	0.051/0.047	0.056/0.067	0.082/0.123	0.139/0.262

( $f_{VM_{10}}$  case), the differences in performance between the three tests vanished. All our tests were powerful under the Moebius transformed skew densities, and even under skewed von Mises mixture distributions with high concentration parameter  $\kappa$ . This suggests that the proposed tests also perform well under other skew laws. Our three tests generally outperformed the modified runs test.

## 5. A Real Data Application

We applied our optimal tests for reflective symmetry to a well-known data set from an animal orientation experiment. This data set stems from an experiment with 730 red wood ants (*Formica rufa* L.) described in Jander (1957). Each ant was individually placed in the center of an arena with a black target positioned at an angle of  $180^\circ$  from the zero direction, and the initial direction in which each ant moved upon release was recorded to the nearest  $10^\circ$ . Thus it is clear that

Table 3. Empirical rejection probabilities, out of  $N = 10,000$  replications and for the sample sizes  $n = 30$  and  $n = 100$ , under various reflectively symmetric and various skewed distributions, of the optimal tests  $\phi_1^{*(n);0}$ ,  $\phi_2^{*(n);0}$  and  $\phi_3^{*(n);0}$  as well as of the modified runs test  $\phi_{\text{modrun}}^{(n)}$  with  $p = 0.6$ . Tests were performed at level  $\alpha = 5\%$ .

Test	$n = 30/n = 100$	$n = 30/n = 100$	$n = 30/n = 100$	$n = 30/n = 100$
Moebius transformed $f_{VM_1}$				
	$\lambda = 0$	$\lambda = 0.2/3$	$\lambda = 0.4/3$	$\lambda = 0.2$
$\phi_1^{*(n);0}$	0.051/0.051	0.074/0.127	0.142/0.351	0.239/0.639
$\phi_2^{*(n);0}$	0.050/0.049	0.082/0.153	0.169/0.453	0.304/0.776
$\phi_3^{*(n);0}$	0.047/0.048	0.082/0.154	0.163/0.460	0.302/0.771
$\phi_{\text{modrun}}^{(n)}$	0.049/0.053	0.057/0.059	0.074/0.074	0.086/0.116
Moebius transformed $f_{VM_{10}}$				
Test	$\lambda = 0$	$\lambda = 0.02$	$\lambda = 0.04$	$\lambda = 0.06$
$\phi_1^{*(n);0}$	0.046/0.046	0.092/0.215	0.244/0.641	0.464/0.937
$\phi_2^{*(n);0}$	0.047/0.048	0.092/0.215	0.245/0.644	0.466/0.938
$\phi_3^{*(n);0}$	0.047/0.048	0.093/0.215	0.247/0.646	0.469/0.940
$\phi_{\text{modrun}}^{(n)}$	0.050/0.052	0.069/0.081	0.133/0.204	0.237/0.457
Skewed $f_{VM_1}$ mixtures				
Test	$\lambda = 0$	$\lambda = 0.4$	$\lambda = 0.8$	$\lambda = 1.2$
$\phi_1^{*(n);0}$	0.048/0.053	0.067/0.102	0.072/0.149	0.066/0.095
$\phi_2^{*(n);0}$	0.048/0.049	0.052/0.054	0.072/0.123	0.107/0.256
$\phi_3^{*(n);0}$	0.050/0.048	0.050/0.051	0.048/0.052	0.047/0.053
$\phi_{\text{modrun}}^{(n)}$	0.049/0.049	0.053/0.057	0.073/0.078	0.066/0.086
Skewed $f_{VM_{10}}$ mixtures				
Test	$\lambda = 0$	$\lambda = 0.2$	$\lambda = 0.4$	$\lambda = 0.6$
$\phi_1^{*(n);0}$	0.049/0.050	0.066/0.121	0.103/0.253	0.130/0.341
$\phi_2^{*(n);0}$	0.053/0.051	0.055/0.065	0.138/0.361	0.502/0.957
$\phi_3^{*(n);0}$	0.050/0.050	0.255/0.676	0.835/0.999	0.990/1.00
$\phi_{\text{modrun}}^{(n)}$	0.051/0.053	0.132/0.238	0.407/0.802	0.734/0.995
3-sine-skewed $f_{VM_1}$				
Test	$\lambda = 0$	$\lambda = 0.2$	$\lambda = 0.4$	$\lambda = 0.6$
$\phi_1^{*(n);0}$	0.046/0.052	0.050/0.052	0.057/0.063	0.058/0.080
$\phi_2^{*(n);0}$	0.051/0.050	0.060/0.098	0.104/0.240	0.176/0.482
$\phi_3^{*(n);0}$	0.047/0.048	0.117/0.290	0.340/0.828	0.676/0.995
$\phi_{\text{modrun}}^{(n)}$	0.051/0.046	0.059/0.065	0.083/0.132	0.128/0.280
3-sine-skewed $f_{VM_{10}}$				
Test	$\lambda = 0$	$\lambda = 0.2$	$\lambda = 0.4$	$\lambda = 0.6$
$\phi_1^{*(n);0}$	0.049/0.046	0.099/0.230	0.266/0.689	0.548/0.970
$\phi_2^{*(n);0}$	0.050/0.047	0.103/0.245	0.280/0.722	0.574/0.980
$\phi_3^{*(n);0}$	0.049/0.049	0.105/0.255	0.287/0.743	0.593/0.984
$\phi_{\text{modrun}}^{(n)}$	0.052/0.050	0.067/0.079	0.132/0.201	0.264/0.511

the experimental design suggests the black target as natural median direction, a fact that is clearly corroborated by the graphical representation of the data in Figure 2. The question of interest is whether the directions chosen by the ants

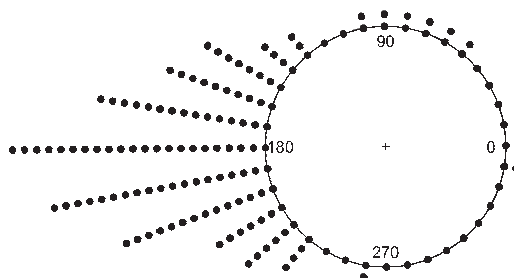


Figure 2. Raw circular plot of the Jander (1957) data set recorded during an orientation experiment with 730 red wood ants. Each dot represents the direction chosen by five ants.

are symmetrically distributed around the median direction representing the black target, allowing us to know *inter alia* whether the classical symmetric or the more recent skew distributions better model this data set. Papers that investigated this problem include Pewsey (2004), Umbach and Jammalamadaka (2009) and Abe and Pewsey (2011).

This data set is a good candidate for testing circular symmetry about a known median direction. The data plot in Figure 2 indicates that the underlying density might be multimodal rather than unimodal, indicating that the tests  $\phi_2^{*(n);0}$  and  $\phi_3^{*(n);0}$  might be more powerful in the present situation than  $\phi_1^{*(n);0}$  (see Abe and Pewsey (2011) for a discussion on conditions under which 1-sine-skewed distributions are unimodal or multimodal). Indeed,  $\phi_1^{*(n);0}$  yields a p-value of 0.778, while  $\phi_2^{*(n);0}$  and  $\phi_3^{*(n);0}$ , respectively, give p-values 0.011 and 0.013. The latter two p-values provide evidence that the data are in fact not symmetrically distributed around the median direction of  $180^\circ$ . Pewsey (2004) obtained the same conclusion with his b2star test. Abe and Pewsey (2011) find that neither the symmetric nor the 1-sine-skewed distributions provide an adequate fit to these data. Their findings are not a surprise: according to  $\phi_1^{*(n);0}$ , 1-sine-skewed densities show no improvement over symmetric ones, and our other tests reject the hypothesis of reflective symmetry at just above the 1% level. As a conclusion, the ant data look to be better fitted by 2- or 3-sine-skewed distributions.

## 6. Singularity of the Location-Skewness Fisher Information Matrix

The skew-normal distribution of Azzalini (1985) is also famous for having a singular Fisher information matrix in the vicinity of symmetry, due to the collinearity of the scores for location and skewness in its initial parameterization. A large literature has been devoted to the analysis of the reasons for this singularity, to its negative impact on inferential procedures, to possible cures

(reparameterizations), and to the study of which other skew-symmetric distributions suffer from the same drawback. For a recent overview see Hallin and Ley (2012), where the class of skew-symmetric distributions suffering from the Fisher singularity is determined.

The present section can be inscribed into this stream of literature. We solve the same problem for  $k$ -sine-skewed circular distributions. Moreover, our results are very important when one considers the construction of optimal tests about an unknown center  $\mu$ , as will be briefly discussed in the final section. Recall that the information matrix in the vicinity of symmetry is given by

$$\mathbf{\Gamma}_{f_0,k} = \begin{pmatrix} \int_{-\pi}^{\pi} \varphi_{f_0}^2(\theta) f_0(\theta) d\theta & \int_{-\pi}^{\pi} \sin(k\theta) \varphi_{f_0}(\theta) f_0(\theta) d\theta \\ \int_{-\pi}^{\pi} \sin(k\theta) \varphi_{f_0}(\theta) f_0(\theta) d\theta & \int_{-\pi}^{\pi} \sin^2(k\theta) f_0(\theta) d\theta \end{pmatrix}.$$

This matrix is singular if and only if

$$\left( \int_{-\pi}^{\pi} \varphi_{f_0}^2(\theta) f_0(\theta) d\theta \right) \left( \int_{-\pi}^{\pi} \sin^2(k\theta) f_0(\theta) d\theta \right) = \left( \int_{-\pi}^{\pi} \sin(k\theta) \varphi_{f_0}(\theta) f_0(\theta) d\theta \right)^2. \quad (6.1)$$

The Cauchy-Schwarz inequality readily yields that the equality sign “=” in (6.1) can be replaced by “ $\geq$ ” with equality holding if and only if  $\varphi_{f_0}(\theta) = a \sin(k\theta)$  for some real constant  $a$ . The latter easy-to-solve first-order differential equation then shows that an information singularity can only occur for base symmetric densities  $f_0$  of the form  $c \exp((a/k) \cos(k\theta))$  for  $a \in \mathbb{R}$  and  $c > 0$  a normalizing constant. The class of base densities  $\mathcal{F}$  we consider contains the condition of unimodality on  $f_0$ , which rules out  $k \geq 2$  and forces  $a$  to be positive. Hence, the only base symmetric density for which the Fisher information matrix  $\mathbf{\Gamma}_{f_0,k}$  is singular is  $f_0(\theta) = c \exp(\kappa \cos(\theta))$  with  $\kappa = a/k > 0$  a concentration parameter, the von Mises circular density. We formalize this result in the following proposition.

**Proposition 1.** *Let  $f_0$  be a symmetric base density belonging to  $\mathcal{F}$  and satisfying Assumption (A), and consider  $k$ -sine-skewed densities of the form  $f_0(\theta - \mu)(1 + \lambda \sin(k(\theta - \mu)))$ . Then the Fisher information matrix associated with the parameters  $\mu \in [-\pi, \pi)$  and  $\lambda \in (-1, 1)$  is singular in the vicinity of symmetry (that is, at  $\lambda = 0$ ) if and only if one is considering sine-skewed von Mises densities.*

A referee raised the question of the existence of a parameterization that avoids this singularity, as is the case for skew-normal distributions with the Centered Parameterization (Azzalini (1985)) or the parameterization proposed in Hallin and Ley (2014). Mimicking these constructions, one obtains such a

singularity-free parameterization, but this is beyond the scope of the present paper.

## 7. Final Comments

The tests we propose are uniformly (over the null hypothesis) locally and asymptotically maximin against  $k$ -sine-skewed alternatives, asymptotically distribution-free and moreover of a very simple form. They furthermore exhibit nice finite sample behaviors. Now, as already mentioned before, It would be of interest to adapt our procedures to the case of an unspecified center, and our general ULAN property provides the required theoretical background for constructing such tests. The crucial difference here lies in the fact that we need to replace the unknown location  $\mu$  with an estimator  $\hat{\mu}$ . If the information matrix  $\Gamma_{f_0,k}$  were diagonal, then the substitution of  $\hat{\mu}$  for  $\mu$  would have no influence, asymptotically, on the behavior of the central sequence for skewness  $\Delta_{k;2}^{(n)}(\mu)$ . As this is rarely the case, a local perturbation of  $\mu$  has the same asymptotic impact on  $\Delta_{k;2}^{(n)}(\mu)$  as a local perturbation of  $\lambda = 0$ . Thus the cost of not knowing  $\mu$  is strictly positive when performing inference on  $\lambda$ ; the stronger the correlation between  $\mu$  and  $\lambda$ , the larger the cost. The worst case occurs when the information matrix is singular (see Section 6), which leads to asymptotic local powers equal to the nominal level  $\alpha$ ; more precisely, this situation entails that the best possible test is the trivial test, that is, the test discarding the observations and rejecting the null of reflective symmetry at level  $\alpha$  whenever an auxiliary Bernoulli variable with parameter  $\alpha$  takes value one.

To take into account the cost of not knowing  $\mu$ , one can replace the central sequence  $\Delta_{k;2}^{(n)}(\mu)$  with the, in Le Cam terminology (see, e.g., Le Cam (1986)), *efficient central sequence*

$$\begin{aligned} \Delta_{f_0,k;2}^{(n)eff}(\mu) &:= \Delta_{k;2}^{(n)}(\mu) - \frac{\Gamma_{f_0,k;12}}{\Gamma_{f_0,k;11}} \Delta_{f_0,k;1}^{(n)}(\mu) \\ &= n^{-1/2} \sum_{i=1}^n \left( \sin(k(\theta_i - \mu)) - \frac{\Gamma_{f_0,k;12}}{\Gamma_{f_0,k;11}} \varphi_{f_0}(\theta_i - \mu) \right). \end{aligned}$$

This is the orthogonal projection of  $\Delta_{k;2}^{(n)}(\mu)$  onto the subspace orthogonal to  $\Delta_{f_0,k;1}^{(n)}(\mu)$ , which ensures that  $\Delta_{f_0,k;2}^{(n)eff}(\mu)$  and  $\Delta_{f_0,k;1}^{(n)}(\mu)$  are asymptotically uncorrelated. An asymptotic test can then be obtained by considering a studentized version of  $\Delta_{f_0,k;2}^{(n)eff}(\hat{\mu})$ . Unfortunately, by doing so, it can be shown that, only under  $f_0$ , there is no asymptotic effect when  $\mu$  is replaced with  $\hat{\mu}$  (this fails to hold for  $g_0 \neq f_0$ ). Therefore, rather than having as in the present paper a test that is valid under any density  $f_0 \in \mathcal{F}$  with a fixed location  $\mu$ , we would obtain a test



which is valid for any value of  $\mu$  but only under a single  $f_0$  (complete parametric test). Constructing tests that are completely distribution-free (with respect to both the underlying base density and the location parameter) is an ongoing research project.

As to the choice of  $k$  in the statistic (3.2), we showed that, for a fixed  $k$ , the test based on  $Q_k^{*(n);\mu}$  is asymptotically optimal against  $k$ -sine-skewed alternatives with the same  $k$ . Without a particular value of  $k$  in mind, we suggest two possibilities. First, one can consider several tests  $\phi_k^{*(n);\mu}$  and compare their outcomes. Second, a test could be performed using the asymptotic joint distribution of  $\mathbf{Q}^{*(n);\mu} := (Q_1^{*(n);\mu}, Q_2^{*(n);\mu}, \dots, Q_q^{*(n);\mu})'$  for a certain  $q \in \mathbb{N}_0$ . The asymptotic distribution of  $\mathbf{Q}^{*(n);\mu}$  under the null can be derived using Theorem 2. However, an asymptotic test based on  $\mathbf{Q}^{*(n);\mu}$  clearly loses the optimality property against all the alternatives considered here. One can also raise the question of choosing  $q \in \mathbb{N}_0$  in  $\mathbf{Q}^{*(n);\mu}$ . This issue is therefore beyond the scope of the present paper.

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### Appendix A

Our proof relies on Lemma 1 of Swensen (1985) and more precisely on its extension in Garel and Hallin (1995). The sufficient conditions for ULAN in those results follow from standard arguments once it is shown that  $(\mu, \lambda)' \mapsto (f_{\mu, \lambda}^k)^{1/2}(\theta)$  (see (2.1)) is quadratic mean differentiable at any  $(\mu, 0)'$ , which we establish in the following lemma.

**Lemma A.1.** Let  $f_0 \in \mathcal{F}$  and  $k \in \mathbb{N}_0$ , and assume that Assumption (A) holds. Define

$$D_\theta(f_{\mu, 0}^k)^{1/2}(\theta) := -\frac{1}{2} \frac{\dot{f}_0(\theta - \mu)}{f_0^{1/2}(\theta - \mu)}$$

and

$$D_\lambda(f_{\mu, \lambda}^k)^{1/2}(\theta)|_{\lambda=0} := \frac{1}{2} f_0^{1/2}(\theta - \mu) \sin(k(\theta - \mu)).$$

Then, for any  $\mu \in [-\pi, \pi)$ , we have that, as  $(t, \ell) \rightarrow (0, 0)$ ,

$$(i) \int_{-\pi}^{\pi} \left( (f_{\mu+t, 0}^k)^{1/2}(\theta) - (f_{\mu, 0}^k)^{1/2}(\theta) - t D_\mu(f_{\mu, 0}^k)^{1/2}(\theta) \right)^2 d\theta = o(t^2),$$

- (ii)  $\int_{-\pi}^{\pi} \left( (f_{\mu+t,\ell}^k)^{1/2}(\theta) - (f_{\mu+t,0}^k)^{1/2}(\theta) - \ell D_{\lambda}(f_{\mu+t,\lambda}^k)^{1/2}(\theta)|_{\lambda=0} \right)^2 d\theta = o(\ell^2),$
- (iii)  $\int_{-\pi}^{\pi} \left( D_{\lambda}(f_{\mu+t,\lambda}^k)^{1/2}(\theta)|_{\lambda=0} - D_{\lambda}(f_{\mu,\lambda}^k)^{1/2}(\theta)|_{\lambda=0} \right)^2 d\theta = o(1),$
- (iv)  $\int_{-\pi}^{\pi} \left\{ (f_{\mu+t,\ell}^k)^{1/2}(\theta) - (f_{\mu,0}^k)^{1/2}(\theta) - \binom{t}{\ell}' \begin{pmatrix} D_{\mu}(f_{\mu,0}^k)^{1/2}(\theta) \\ D_{\lambda}(f_{\mu,\lambda}^k)^{1/2}(\theta)|_{\lambda=0} \end{pmatrix} \right\}^2 d\theta = o(\|(t, \ell)'\|^2).$

**Proof of Lemma A.1.** (i) By definition of  $f_{\mu,0}^k$  we can rewrite the left-hand side of (i) as

$$\int_{-\pi}^{\pi} \left( f_0^{1/2}(\theta - \mu - t) - f_0^{1/2}(\theta - \mu) + \frac{1}{2}t \frac{\dot{f}_0(\theta - \mu)}{f_0^{1/2}(\theta - \mu)} \right)^2 d\theta. \tag{A.1}$$

The a.e.-differentiability of  $f_0$  (Assumption (A)) combined with the Mean Value Theorem turns (A.1) into

$$\begin{aligned} & \int_{-\pi}^{\pi} \left( \frac{1}{2}t \frac{\dot{f}_0(\theta - \mu^*)}{f_0^{1/2}(\theta - \mu^*)} - \frac{1}{2}t \frac{\dot{f}_0(\theta - \mu)}{f_0^{1/2}(\theta - \mu)} \right)^2 d\theta \\ &= \frac{1}{4}t^2 \int_{-\pi}^{\pi} \left( \frac{\dot{f}_0(\theta - \mu^*)}{f_0^{1/2}(\theta - \mu^*)} - \frac{\dot{f}_0(\theta - \mu)}{f_0^{1/2}(\theta - \mu)} \right)^2 d\theta \end{aligned} \tag{A.2}$$

with  $\mu^* \in (\mu, \mu + t)$ . Assumption (A) and the periodicity requirement ensure that  $\dot{f}_0(\theta)/f_0^{1/2}(\theta)$  is continuous over  $[-\pi, \pi]$ , hence its square can be bounded by a sufficiently large constant; consequently, the Lebesgue Dominated Convergence Theorem implies that (A.2) is  $o(t^2)$ .

(ii) Similarly, the left-hand side integral in (ii) can be re-expressed as

$$\int_{-\pi}^{\pi} f_0(\theta - \mu - t) \left( (1 + \ell \sin(k(\theta - \mu - t)))^{1/2} - 1 - \ell \frac{1}{2} \sin(k(\theta - \mu - t)) \right)^2 d\theta.$$

Exactly as for (i), the differentiability of  $\sin(k\theta)$  allows us to re-write this integral under the form

$$\frac{1}{4}\ell^2 \int_{-\pi}^{\pi} f_0(\theta - \mu - t) \sin^2(k(\theta - \mu - t)) \left( \frac{1}{(1 + \ell^* \sin(k(\theta - \mu - t)))^{1/2}} - 1 \right)^2 d\theta$$

with  $\ell^* \in (0, \ell)$ . Since  $\sin^2(k\theta)f_0(\theta)$  is integrable and  $(1 + \ell^* \sin(k(\theta - \mu - t)))^{-1}$  is bounded by a constant not depending on  $\ell$  (indeed, we can take  $\ell^* \leq \ell < 1/2$  as  $\ell \rightarrow 0$ , hence  $1 + \ell^* \sin(k(\theta - \mu - t)) \geq 1/2$  over  $[-\pi, \pi)$  which does not depend on

$\ell$ ), Lebesgue's Dominated Convergence Theorem applies and yields the desired  $o(\ell^2)$  quantity.

(iii) The left-hand side in (iii) is

$$\frac{1}{4} \int_{-\pi}^{\pi} \left( f_0^{1/2}(\theta - (\mu + t)) \sin(k(\theta - (\mu + t))) - f_0^{1/2}(\theta - \mu) \sin(k(\theta - \mu)) \right)^2 d\theta. \quad (\text{A.3})$$

Since  $f_0^{1/2}(\theta) \sin(k\theta)$  is square-integrable, the quadratic mean continuity entails that (A.3) tends to zero as  $t \rightarrow 0$ , hence is an  $o(1)$  quantity.

(iv) The left-hand side in (iv) is bounded by  $C(S_1 + S_2 + \ell^2 S_3)$ , where

$$S_1 = \int_{-\pi}^{\pi} \left( (f_{\mu+t,0}^k)^{1/2}(\theta) - (f_{\mu,0}^k)^{1/2}(\theta) - t D_{\mu}(f_{\mu,0}^k)^{1/2}(\theta) \right)^2 d\theta,$$

$$S_2 = \int_{-\pi}^{\pi} \left( (f_{\mu+t,\ell}^k)^{1/2}(\theta) - (f_{\mu+t,0}^k)^{1/2}(\theta) - \ell D_{\lambda}(f_{\mu+t,\lambda}^k)^{1/2}(\theta)|_{\lambda=0} \right)^2 d\theta$$

and

$$S_3 = \int_{-\pi}^{\pi} \left( D_{\lambda}(f_{\mu+t,\lambda}^k)^{1/2}(\theta)|_{\lambda=0} - D_{\lambda}(f_{\mu,\lambda}^k)^{1/2}(\theta)|_{\lambda=0} \right)^2 d\theta.$$

The result then follows from (i), (ii) and (iii).

## Appendix B

**Proofs of Theorem 2.** Fix  $f_0 \in \mathcal{F}$ . Part (i) of the theorem trivially follows from the Central Limit Theorem combined with the fact that

$$Q_k^{*(n);\mu} = \frac{|n^{-1/2} \sum_{i=1}^n \sin(k(\theta_i - \mu))|}{(n^{-1} \sum_{i=1}^n \sin^2(k(\theta_i - \mu)))^{1/2}} = \frac{|n^{-1/2} \sum_{i=1}^n \sin(k(\theta_i - \mu))|}{(\Gamma_{f_0,k;22})^{1/2}} + o_{\mathbb{P}}(1) \quad (\text{A.4})$$

as  $n \rightarrow \infty$  under  $\mathbb{P}_{(\mu,0)';f_0}^{(n)}$ . Part (ii) can be readily handled by using the "Third Lemma of Le Cam" (see Le Cam (1986)). Under  $\mathbb{P}_{(\mu,n^{-1/2}\tau_2^{(n)})';f_0,k'}^{(n)}$ , the asymptotic normality of  $\Delta_{k;2}^{(n)}(\mu)$  with mean  $C_{f_0}(k, k')\tau_2$  and variance  $\Gamma_{f_0,k;22}$  is obtained by establishing the joint normality of  $\Delta_{k;2}^{(n)}(\mu)$  and  $\log(d\mathbb{P}_{(\mu,n^{-1/2}\tau_2^{(n)})';f_0,k'}^{(n)}/d\mathbb{P}_{(\mu,0)';f_0}^{(n)})$  under  $\mathbb{P}_{(\mu,0)';f_0}^{(n)}$  and then applying Le Cam's third Lemma (which holds thanks to the ULAN property). Part (ii) follows immediately since (A.4) also holds under  $\mathbb{P}_{(\mu,n^{-1/2}\tau_2^{(n)})';f_0,k'}^{(n)}$  by contiguity. Finally, Part (iii) trivially follows from (A.4) and the optimality features of the parametric test  $\phi_{f_0;k}^{(n);\mu}$  for all  $f_0 \in \mathcal{F}$ .

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