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OPTIMAL DESIGNS FOR FIRST-ORDER TRIGONOMETRIC REGRESSION ON A PARTIAL CYCLE

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Abstract: Trigonometric regression is commonly used to describe cyclic phenomena that occur in the engineering, biological, and medical sciences. Optimal designs for this model on a *complete* cycle have been studied extensively in the literature. However, much less attention has been paid to the design problem with a *partial* cycle. This paper solves this problem for the first-order trigonometric regression. Explicit D-, A-, and E-optimal designs are analytically derived. These designs are used to evaluate the D-, A-, and E-efficiencies of the equidistant sampling method commonly used in practice. Efficient and practical designs are then suggested. Some optimal exact designs and optimal designs for all nontrivial subsets of the coefficients are also obtained. A discussion is made on the ϕ_p-optimal designs ($p \in [-\infty, 1]$) for the general trigonometric regression on a partial cycle.

Key words and phrases: A-optimality, D-optimality, efficiency, E-optimality, equidistant sampling, exact design, ϕ_p-criterion.

1. Introduction

Consider the first-order trigonometric regression

$$y(x) = \beta_0 + \beta_1 \cos(x) + \beta_2 \sin(x) + \epsilon(x), \tag{1}$$

where $x \in \mathcal{X} = [\alpha_1, \alpha_2]$, with $-\pi \leq \alpha_1 < \alpha_2 \leq \pi$, and β_0, β_1, and β_2 are unknown parameters of interest. The error $\epsilon(x)$ is an unobservable random variable independent of x, with mean zero and unknown variance σ^2. If n uncorrelated observations $y(x)$ are to be taken, n_i at each $x_i \in \mathcal{X}$, where $i = 1, \ldots, l$ ($l \geq 1$), the corresponding experimental design can be viewed as a probability measure ξ on $\mathcal{X}$, with $\xi(x_i) = n_i/n$, for $i = 1, \ldots, l$. Let $\widehat{\beta}$ be the least squares estimator of $\beta = (\beta_0, \beta_1, \beta_2)'$. Then the covariance matrix of $\widehat{\beta}$ is $\mathrm{Cov}(\widehat{\beta}) = (\sigma^2/n)M^{-1}(\xi)$, where $M(\xi) = \int_{\mathcal{X}} f(x)f(x)' d\xi$ is the information matrix per observation of the design ξ, with $f(x) = (1, \cos x, \sin x)'$. In general, an (approximate) experimental design is a probability measure ξ on $\mathcal{X}$ that assigns all its mass to a finite number of points (Pukelsheim (1993, p.26)). To implement such a design in practice, certain approximation is often needed.

Some popular criteria for choosing a design ξ are the D-,A-, and E-optimality criteria, which belong to a class of ϕ_p-criteria (Pukelsheim (1993, Chap.9)). When $p=-\infty, -1$, and 0, we obtain the E-, A-, and D-criteria, respectively. Let λ_1, λ_2, and λ_3 be the eigenvalues of $M(\xi)$. Then the ϕ_p-criteria ($p \in [-\infty, 1]$) maximize the following ϕ_p functions:

$$\phi_p(M(\xi)) = \begin{cases} \min_{j\le 3}\lambda_j & \text{for } p=-\infty, \\ \left(\prod_{j\le 3}\lambda_j\right)^{1/3} & \text{for } p=0, \\ \left((1/3)\sum_{j\le 3}\lambda_j^p\right)^{1/p} & \text{for } p\in(-\infty,1] \text{ and } p\neq 0. \end{cases} \tag{2}$$

When observations can be taken from a complete cycle, that is, when $\mathcal{X} = [-\pi, \pi]$, the optimal design problem for the trigonometric regression of order m ($m \geq 1$),

$$y(x) = \beta_0 + \sum_{j=1}^{m} \beta_{2j-1}\cos(jx) + \sum_{j=1}^{m} \beta_{2j}\sin(jx) + \epsilon(x), \tag{3}$$

has been well discussed in the literature. See, for example, Hoel (1965), Karlin and Studden (1966, p.347), Fedorov (1972, p.94), Lau and Studden (1985), Riccomagno, Schwabe and Wynn (1997), and Dette and Haller (1998). It is well known (Pukelsheim (1993 p.241)) that, for any $n \geq 2m+1$, a design that assigns equal weight $1/n$ to each of n equispaced support points on $[-\pi, \pi]$ is ϕ_p-optimal, for all $p \in [-\infty, 1]$. However, much less attention has been paid to the design problem for Model (1) or (3) on a partial cycle $\mathcal{X} = [\alpha_1, \alpha_2]$, where $-\pi \leq \alpha_1 < \alpha_2 \leq \pi$: Hill (1978) obtained the D-optimal design for $(\beta_0, \beta_1, \beta_2)'$ for Model (1) with $\mathcal{X} = [-\pi/2, \pi/2]$; Karlin and Studden (1966, p.343) obtained the D-optimal design for $(\beta_0, \beta_1, \beta_3, \ldots, \beta_{2m-1})'$ for Model (3) with $\mathcal{X} = [0, \pi]$ and when all the sine terms are absent.

This paper will focus on Model (1), a widely used trigonometric regression for modeling cyclic phenomena in the engineering, physical, biological, and medical sciences. See, for example, Graybill (1976, pp.311 and 314), McCool (1979), and Kitsos, Titterington and Torsney (1988). In some applications, it is impossible or difficult to take observations on a complete cycle $[-\pi, \pi]$. For example, in mechanical and precision engineering (McCool (1979)), $\alpha_2 - \alpha_1$ is the included angle of a mechanical circular part, and the angle may range from $\pi/30$ (e.g., in the ball bearing industry) to 2π (e.g., a circular hole). Kitsos, Titterington and Torsney (1988) studied a design problem in rhythmometry involving circadian rhythm exhibited by peak expiratory flow, for which a complete cycle is a 24-hour period. However, in such circumstances the design region $[\alpha_1, \alpha_2]$ has to be restricted to "normal waking hours"—a partial cycle, especially if the people involved are ill or children. Thus, it is of practical importance to investigate optimal designs for model (1) on a partial cycle. They are useful as benchmarks in evaluating the performance of other designs, and also provide a means of identifying efficient and practical designs.

In this paper, the problem of constructing ϕ_p-optimal designs for Model (1) on a partial cycle is considered. It is first shown that the problem reduces to that of finding optimal designs for Model (1) with $\mathcal{X} = [-\alpha/2, \alpha/2]$, and that the requisite designs are symmetric. Then explicit D-, A-, and E-optimal designs are analytically derived. These designs are used to evaluate the efficiencies of the commonly used equidistant sampling method. Efficient and practical designs are then suggested. Some optimal exact designs and optimal designs for all nontrivial subsets of the coefficients are also obtained. A discussion is given on ϕ_p-optimal designs ($p \in [-\infty, 1]$) for the general trigonometric regression model (3) with $\mathcal{X} = [\alpha_1, \alpha_2]$, where $-\pi \le \alpha_1 < \alpha_2 \le \pi$. Technical proofs are given in the Appendix.

2. Symmetry of the Problem

For a $3\times t$ matrix A, denote its column space by $\mathcal{C}(A) = \{Ax : x \in R^t\}$, and a generalized inverse of A by A^- (satisfying $AA^-A = A$). Let K be a $3 \times s$ matrix of full column rank s ($s \le 3$), with $\mathcal{C}(K) \subset \mathcal{C}(M(\xi))$. Then, the information matrix for $K'\beta$ is given by $C_K(M(\xi)) = (K'M^-(\xi)K)^{-1}$.

For $\mathcal{X} = [-\alpha/2, \alpha/2]$, where $\alpha \in (0, 2\pi]$, the reflected design of ξ is $\xi^R(x) = \xi(-x)$ for $x \in \mathcal{X}$, and the symmetrized design of ξ is $\overline{\xi} = (\xi + \xi^R)/2$. We have $M(\xi^R) = QM(\xi)Q$, where $Q = \mathrm{diag}\,(1, 1, -1)$. If $K'K = I_s$, where I_s is the $s \times s$ identity matrix then, from Pukelsheim (1993, p.338), there exists an orthogonal matrix H_K such that $C_K(M(\xi^R)) = C_K(QM(\xi)Q) = H_K C_K(M(\xi))H_K'$. Thus $C_K(M(\xi^R))$ and $C_K(M(\xi))$ have the same eigenvalues. This implies that $\phi_p\left(C_K(M(\xi^R))\right) = \phi_p\left(C_K(M(\xi))\right)$, for $p \in [-\infty, 1]$. Then, by concavity of C_K and of ϕ_p (Pukelsheim (1993, pp.77 and 151)), we have

$$\begin{aligned}
\phi_p\left(C_K(M(\overline{\xi}))\right) &= \phi_p\left(C_K\left(M(\xi)/2 + M(\xi^R)/2\right)\right) \\
&\ge \phi_p\left(C_K\left(M(\xi)\right)/2 + C_K\left(M(\xi^R)\right)/2\right) \\
&\ge \phi_p\left(C_K(M(\xi))\right)/2 + \phi_p\left(C_K(M(\xi^R))\right)/2 = \phi_p\left(C_K(\xi)\right).
\end{aligned}$$

In particular, let $K'\beta$ designate a subset of the coefficients β_0, β_1, and β_2. Then $K'K = I_s$. The above derivation shows that it suffices to consider symmetric designs in order to obtain ϕ_p-optimal designs for any (nonempty) subset of the coefficients.

For $\mathcal{X} = [\alpha_1, \alpha_2]$, where $-\pi \le \alpha_1 < \alpha_2 \le \pi$, let $\alpha = \alpha_2 - \alpha_1$, $\gamma = (\alpha_1 + \alpha_2)/2$, $\widetilde{x} = x - \gamma$, $\widetilde{y}(\widetilde{x}) = y(x)$, and $\widetilde{\epsilon}(\widetilde{x}) = \epsilon(x)$. Then Model (1) becomes

$\widetilde{y}(\widetilde{x}) = \widetilde{\beta}_0 + \widetilde{\beta}_1 \cos\widetilde{x} + \widetilde{\beta}_2 \sin\widetilde{x} + \widetilde{\epsilon}(\widetilde{x})$, where $\widetilde{\beta} = (\widetilde{\beta}_0, \widetilde{\beta}_1, \widetilde{\beta}_2)' = (\beta_0,\ \beta_1\cos\gamma + \beta_2\sin\gamma,\ -\beta_1\sin\gamma + \beta_2\cos\gamma)' \equiv K_1\beta$, and $\widetilde{x} \in \widetilde{\mathcal{X}} = [-\alpha/2, \alpha/2]$. Note that $\beta_0 = \widetilde{\beta}_0$, $(\beta_1, \beta_2)' = K_2'\widetilde{\beta}$ with $K_2'K_2 = I_2$, and $\beta = K_1'\widetilde{\beta}$ with $K_1'K_1 = I_3$. By the argument in the above paragraph, it suffices to consider symmetric designs $\widetilde{\xi}$ on $\widetilde{\mathcal{X}}$ in order to obtain ϕ_p-optimal designs for β_0, $(\beta_1, \beta_2)'$, and β.

The focus of this paper is on Model (1), with $\mathcal{X} = [-\alpha/2, \alpha/2]$. Thus, it suffices to consider symmetric designs ξ, which give

$$M(\xi) = \begin{pmatrix} 1 & \mu & 0 \\ \mu & \nu & 0 \\ 0 & 0 & 1-\nu \end{pmatrix}, \quad \text{where} \quad \mu = \int_{\mathcal{X}} \cos(x)d\,\xi, \quad \nu = \int_{\mathcal{X}} \cos^2(x)d\,\xi. \tag{4}$$

The eigenvalues of $M(\xi)$ are $\lambda_1 = (1+\nu)/2 + \{(1-\nu)^2/4 + \mu^2\}^{1/2}$, $\lambda_2 = (1+\nu)/2 - \{(1-\nu)^2/4 + \mu^2\}^{1/2}$, and $\lambda_3 = 1-\nu$. Since $\phi_1(M(\xi)) \equiv 2/3$, the ϕ_1-criterion (the T-criterion) is useless.

3. D-, A-, and E-Optimal Designs

Throughout this paper, let $c = \cos(\alpha/2)$, and $\nu_m(u) = (1+c)u - c$, for $u \in [-1, 1]$. Let $\xi_u(x_1, \ldots, x_k)$ denote the design that assigns uniform weights $1/k$ to $x_1, \ldots, x_k \in \mathcal{X}$. For $w \in [0, 1]$, $x \in [0, \alpha/2]$, let $\xi(w, x)$ denote the symmetric design that assigns weights $w/2, 1-w$, and $w/2$ to $-x$, 0, and x, respectively. The following lemma shows that $\xi_m(w) \equiv \xi(w, \alpha/2)$ attains the largest possible value of ν among all symmetric designs with the same μ. The proof of the lemma is given in the Appendix.

Lemma 1. (a) *The design $\xi_m(w)$ gives $\mu = 1-(1-c)w$, and $\nu = 1-(1-c^2)w = \nu_m(\mu)$.*

(b) *Any symmetric design ξ has $\nu \in [\mu^2, \nu_m(\mu)]$.*

(c) *If a symmetric design ξ gives $\nu = \nu_m(\mu)$, the design is $\xi_m((1-\mu)/(1-c))$.*

(d) *For any given $\mu \in [c, 1]$ and $\nu \in [\mu^2, \nu_m(\mu)]$, there exist $w \in [0, 1]$ and $x \in [0, \alpha/2]$ such that, for $t = \cos(x)$,*

$$M(\xi(w,x)) = \begin{pmatrix} 1 & 1-w+wt & 0 \\ 1-w+wt & 1-w+wt^2 & 0 \\ 0 & 0 & w-wt^2 \end{pmatrix} = \begin{pmatrix} 1 & \mu & 0 \\ \mu & \nu & 0 \\ 0 & 0 & 1-\nu \end{pmatrix}. \tag{5}$$

Note that, for $\alpha \in [4\pi/3, 2\pi]$, any design ξ with $M(\xi)$ given by (4) and $(\mu, \nu) = (0, 1/2)$ (e.g., $\xi_u(-2\pi/3, 0, 2\pi/3)$) is ϕ_p-optimal, for all $p \in [-\infty, 1]$. This follows directly from Pukelsheim (1993, Sec. 9.16), because such a design is ϕ_p-optimal, for all $p \in [-\infty, 1]$, when $\mathcal{X} = [-\pi, \pi]$. Main results of this paper are the D-, A-, and E-optimal designs given in the following theorem, the proof of which is given in the Appendix.

Theorem 1. (a) *For* $\alpha \in (0, 4\pi/3)$, *the design* $\xi_m(2/3)$ *is* D*-optimal.*
(b) *For* $\alpha \in (0, 4\pi/3)$, *the design* $\xi_m(w_a)$ *is* A*-optimal, where*

$$w_a = (3+c)^{1/2}\{(3+c)^{1/2} + (1+c+c^2+c^3)^{1/2}\}^{-1}. \tag{6}$$

(c) *For* $\alpha \in (0, 4\pi/3)$, *the design* $\xi_m(w_e)$ *is* E*-optimal, where*

$$w_e = \begin{cases} (3+c)(5+2c+c^2)^{-1} & \text{if } \alpha \in (0, \alpha_*], \\ (1+3c)(1+3c-2c^2-2c^3)^{-1} & \text{if } \alpha \in (\alpha_*, 4\pi/3). \end{cases} \tag{7}$$

Here, $\alpha_* = 2\arccos(\sqrt{17}/2 - 5/2) \approx 1.2889\pi$.

Remark 1. For $\mathcal{X} = [\alpha_1, \alpha_2]$, where $-\pi \le \alpha_1 < \alpha_2 \le \pi$, let $\xi_m(w)$ denote the design that assigns weights $w/2, 1-w$, and $w/2$ to α_1, α_0, and α_2, respectively, where $\alpha_0 = (\alpha_1 + \alpha_2)/2$. Then, by the discussion in the third paragraph of Section 2, Theorem 1 still holds, with $\alpha = \alpha_2 - \alpha_1 \in (0, 4\pi/3)$. For $\alpha \in [4\pi/3, 2\pi]$, any design ξ on $\mathcal{X} = [\alpha_1, \alpha_2]$ with $M(\xi)$ given by (4) and $(\mu, \nu) = (0, 1/2)$ (e.g., $\xi_u(-2\pi/3 + \alpha_0, \alpha_0, 2\pi/3 + \alpha_0)$) is ϕ_p-optimal, for all $p \in [-\infty, 1]$. Again, this follows directly from Pukelsheim (1993, Sec. 9.16), because such a design is ϕ_p-optimal, for all $p \in [-\infty, 1]$, when $\mathcal{X} = [-\pi, \pi]$.

Figure 1 gives the values of $w_e/2$, $w_a/2$, and $1/3$, the probabilities assigned to each of the endpoints $\pm\alpha/2$, for the E-, A-, and D-optimal designs, respectively, for $\alpha \in (0, 4\pi/3)$.

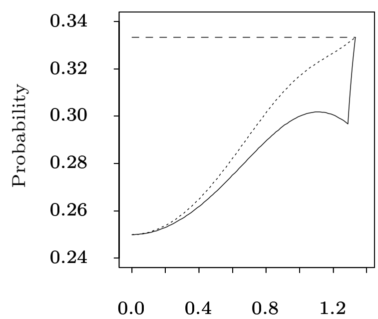

Figure 1. The probabilities assigned to each of the endpoints $\pm\alpha/2$, for the E (solid line), A (dotted line), and D (dashed line) optimal designs, respectively, for $\alpha \in (0, 4\pi/3)$.

For Model (3) with $m \ge 2$ and $\mathcal{X} = [\alpha_1, \alpha_2]$, where $-\pi \le \alpha_1 < \alpha_2 \le \pi$, it follows from the same argument as in Section 2 that, to obtain ϕ_p-optimal designs ($p \in [-\infty, 1)$), it suffices to consider $\mathcal{X} = [-\alpha/2, \alpha/2]$ $(0 < \alpha \le \pi)$ and symmetric designs on $\mathcal{X}$. (Note that the ϕ_1-optimality is useless since $\phi_1(M(\xi)) \equiv (m+1)/(2m+1)$ for any design ξ on $\mathcal{X}$.) For $\alpha \ge 4m\pi/(2m+1)$, it follows from Pukelsheim (1993, Sec. 9.16) that the equidistant sampling that assigns equal weights $1/(2m+1)$ to support points $x_j = 2\pi(-m+j-1)/(2m+1)$ $(j = 1, \ldots, 2m+1)$ is ϕ_p-optimal, for all $p \in [-\infty, 1)$. For $m \geq 2$ and $0 < \alpha < 4m\pi/(2m+1)$, by the results of Theorem 1, we conjecture that ϕ_p-optimal designs can be obtained from symmetric designs on $\mathcal{X}$ with $2m+1$ distinct support points, including a midpoint 0 and an endpoint pair $\pm\alpha/2$. In particular, D-optimal designs with $2m+1$ distinct support points must have equal weights $1/(2m+1)$ (Pukelsheim (1993, p.201)).

For $m = 2$ and D-optimality, the above conjecture leads to consideration of the design that assigns equal weights $1/5$ to support points 0, $\pm\theta$, and $\pm\alpha/2$, where $0 < \theta < \alpha/2 < 4\pi/5$. Let $x = \cos(\theta)$ and denote this design by ξ_x. Using the Maple programming language (Maple is a registered trademark of Waterloo Maple Inc.), we obtain that $\phi_0(M(\xi_x)) = (256/3125)(1-c)^3(1+c)(1-x)^3(1+x)(x-c)^4$, where $M(\xi_x) = \int_{\mathcal{X}} f(t)f(t)' d\xi_x(t)$, with $f(t) = (1, \cos(t), \sin(t), \cos(2t), \sin(2t))'$. Since $c < x < 1$, solving $(\partial/\partial x)\phi_0(M(\xi_x)) = 0$ leads to $4x^2 + (1-2c)x - (2+c) = 0$, which gives $x = x_0 = (2c-1)/8 + (1/8)(4c^2 + 12c + 33)^{1/2}$. By the Kiefer-Wolfowitz theorem (Pukelsheim (1993, p.212)), to show that ξ_{x_0} is D-optimal, it suffices to verify that $f(t)'\{M(\xi_{x_0})\}^{-1}f(t) \leq 5$ for $-\alpha/2 \leq t \leq \alpha/2$, with equality holding at every support point of ξ_{x_0}. We numerically and graphically verified this fact and thus informally showed that ξ_{x_0} is D-optimal. In general, for $m \geq 2$ and $0 < \alpha < 4m\pi/(2m+1)$, a ϕ_p-optimal design can be obtained using numerical algorithms based on the general equivalence theorem; see, for example, Atkinson and Donev (1992, pp.96 and 101).

4. Efficiency Comparisons

In this section, we study the efficiency of the equidistant sampling commonly used in practice (McCool (1979)), and then suggest designs that are both efficient and practically appealing. Let $\xi_{n,\alpha}$ denote the equidistant sampling with sample size n, and $\xi_{p,\alpha}$ the ϕ_p-optimal design for $p \in [-\infty, 1]$, on $\mathcal{X} = [-\alpha/2, \alpha/2]$. Let $v_p(\alpha) = \phi_p(M(\xi_{p,\alpha}))$. The ϕ_p-*efficiency* of $\xi_{n,\alpha}$ is defined by $\phi_p\text{-eff}(n,\alpha) = \phi_p(M(\xi_{n,\alpha}))/v_p(\alpha)$. When $p = -\infty, -1$, and 0, we obtain the E-, A-, and D-efficiency, respectively. If $\alpha \in [4\pi/3, 2\pi]$, then $M(\xi_{p,\alpha})$ is given by (4), with $(\mu, \nu) = (0, 1/2)$, and thus $v_{-\infty}(\alpha) = 1/2$, $v_{-1}(\alpha) = 3/5$, and $v_0(\alpha) = 4^{-1/3}$. For $\alpha \in (0, 4\pi/3)$, by Theorem 1, we have

$$
\begin{aligned}
v_{-\infty}(\alpha) &= (1+\nu_e)/2 - \{(1-\nu_e)^2/4 + \mu_e^2\}^{1/2},\\
v_{-1}(\alpha) &= 3(1-c)^2(1+c)w_a(1-w_a)\{3+c-(1-c)(2+2c+c^2)w_a\}^{-1},\\
v_0(\alpha) &= (4^{1/3}/3)(1-c)(1+c)^{1/3},
\end{aligned}
$$

where w_a is given by (6), w_e by (7), $\mu_e = 1-(1-c)w_e$, $\nu_e = (1+c)\mu_e - c$, and $c = \cos(\alpha/2)$.

The n support points of $\xi_{n,\alpha}$ are $x_j = -\alpha/2+(j-1)\alpha/(n-1)$, for $j = 1,\ldots,n$. If $\alpha = 2\pi(n-1)/n$, then $\xi_{n,\alpha}$ is an equidistant sampling on a complete cycle, because the angles between any neighbor pairs $(x_1, x_2),\ldots,(x_{n-1}, x_n),(x_n, x_1)$ all equal $2\pi/n$. For $\alpha \in [2\pi(n-1)/n, 2\pi]$, we can replace $\xi_{n,\alpha}$ by $\xi_{n,2\pi(n-1)/n}$, which is ϕ_p-optimal for $p \in [-\infty, 1]$ (by the discussion above Theorem 1), and thus has ϕ_p-efficiency 1. For $\alpha \in (0, 2\pi(n-1)/n)$,

$$\begin{aligned}
\phi_{-\infty}(M(\xi_{n,\alpha})) &= \min\left(1-\nu_{n,\alpha}, (1+\nu_{n,\alpha})/2 - \{(1-\nu_{n,\alpha})^2/4 + \mu_{n,\alpha}^2\}^{1/2}\right),\\
\phi_{-1}(M(\xi_{n,\alpha})) &= 3(1-\nu_{n,\alpha})(\nu_{n,\alpha}-\mu_{n,\alpha}^2)(1+\nu_{n,\alpha}-\mu_{n,\alpha}^2-\nu_{n,\alpha}^2)^{-1},\\
\phi_0(M(\xi_{n,\alpha})) &= (\nu_{n,\alpha}-\mu_{n,\alpha}^2)^{1/3}(1-\nu_{n,\alpha})^{1/3},
\end{aligned}$$

where $\mu_{n,\alpha} = (1/n)\Sigma_{j\le n}\cos(x_j)$, and $\nu_{n,\alpha} = (1/n)\Sigma_{j\le n}\cos^2(x_j)$. Note that $\lim_{n\to\infty}\mu_{n,\alpha} = (2/\alpha)\sin(\alpha/2)$ and $\lim_{n\to\infty}\nu_{n,\alpha} = 1/2+\sin\alpha/(2\alpha)$.

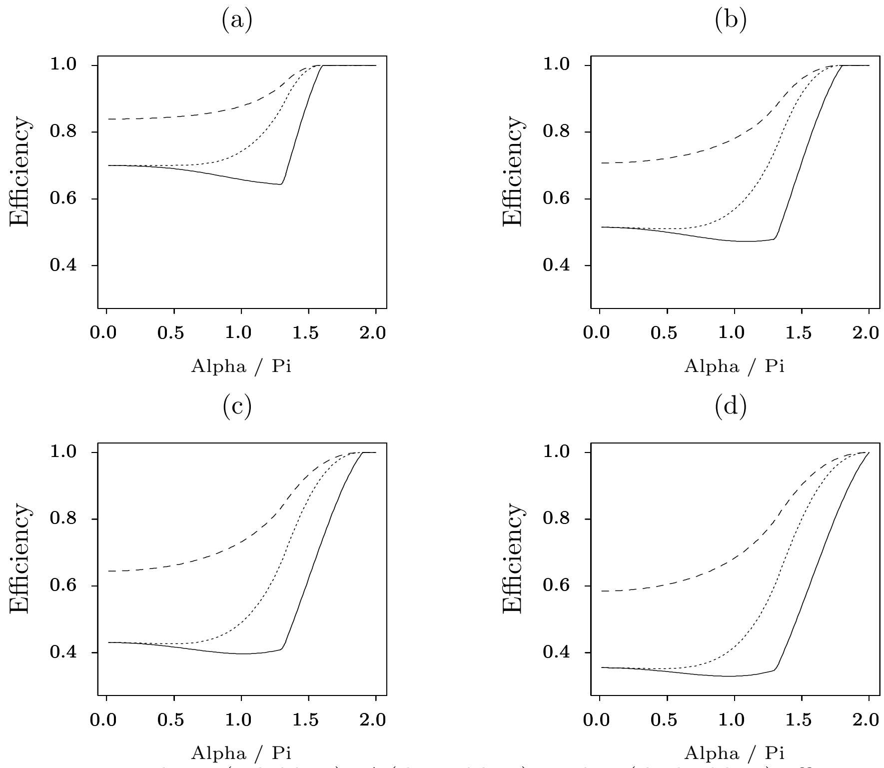

Figure 2. The E (solid line), A (dotted line), and D (dashed line) efficiencies of the equidistant sampling on $[-\alpha/2, \alpha/2]$ for (a) $n = 5$, (b) $n = 10$, (c) $n = 20$, and (d) $n \to \infty$.

Figure 2 gives the E-, A-, and D-efficiencies of the equidistant sampling for $\alpha \in (0, 2\pi]$ and $n = 5, 10, 20$, and $n \to \infty$. For $n \ge 20$, most of the E- and A-efficiencies are less than 0.6, and most of the D-efficiencies are less than 0.8. Thus, there is much room for efficiency improvement. For example, we can take $n/6$ measurements at the midpoint 0 and each of the endpoints $\pm\alpha/2$, and take the other half of the measurements using the equidistant sampling. This new method would increase all the low efficiency values (less than 0.65) by 20% to 80%, and would still provide sufficient information for checking model inadequacy.

The ϕ_p-*absolute efficiency* of a design ξ is defined by $\phi_p\text{-abseff}(\xi) = \phi_p(M(\xi))/m_p$, where $m_p = v_p(2\pi)$. When $p = -\infty, -1$, and 0, we have the E-, A-, and D-absolute efficiency, respectively. Note that, $m_{-\infty} = 1/2$, $m_{-1} = 3/5$, and $m_0 = 4^{-1/3}$. Figure 3 gives the E-, A-, and D-absolute efficiencies for the E-, A-, and D-optimal designs, respectively, and for the equidistant sampling (as $n \to \infty$), for $\alpha \in (0, 2\pi]$.

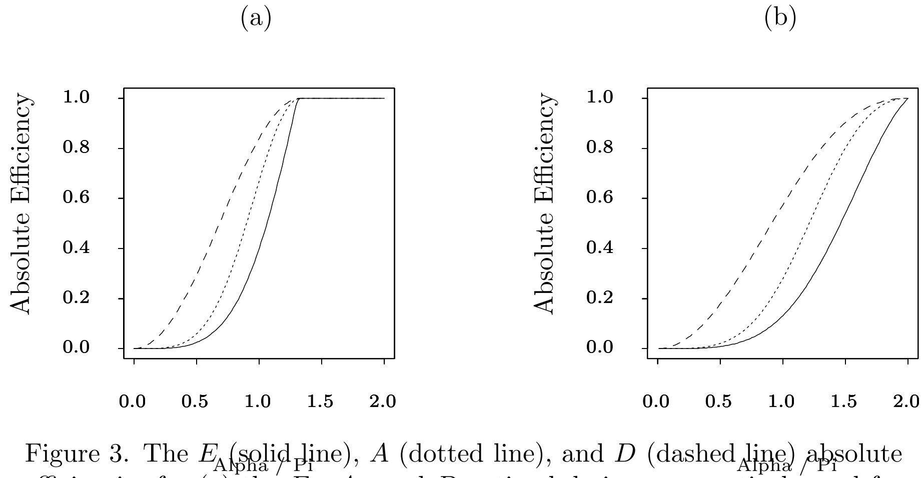

Figure 3. The E (solid line), A (dotted line), and D (dashed line) absolute efficiencies for (a) the E-, A-, and D-optimal designs, respectively, and for (b) the equidistant sampling (as $n \to \infty$).

5. Optimal Exact Designs

Consider Model (1) with $\mathcal{X} = [\alpha_1, \alpha_2]$, where $-\pi \le \alpha_1 < \alpha_2 \le \pi$. A design ξ on $\mathcal{X}$ is called an *exact* design of size n if it consists of n (not necessarily distinct) sampling points on $\mathcal{X}$ with equal weights $1/n$. An exact design ξ^* of size n is said to be ϕ_p-optimal ($p \in [-\infty, 1]$) if $\phi_p(M(\xi^*))$ attains the maximum value of $\phi_p(M(\xi))$ among all exact designs ξ of size n. To obtain ϕ_p-optimal exact designs for Model (1), by the same argument as in Section 2, it suffices to consider $\mathcal{X} = [-\alpha/2, \alpha/2]$, where $0 < \alpha \le 2\pi$.

It follows from Pukelsheim (1993, Sec. 9.16) that an exact design ξ of size n with $M(\xi)$ given by (4) and $(\mu, \nu) = (0, 1/2)$ is ϕ_p-optimal, for all $p \in [-\infty, 1]$; such designs will be called *orthogonal*. For $\alpha \ge 2\pi(n-1)/n$, let ξ_n denote an equidistant sampling design of size n on the complete cycle, with all of its support points on $\mathcal{X}$; then ξ_n is orthogonal. For $n = 3k$ ($k \ge 1$) and $\alpha \ge 4\pi/3$, a design consisting of k orthogonal designs ξ_3 is orthogonal and is denoted by ξ^*_{3k}. Similarly, for $\alpha \geq 3\pi/2$ (or $\alpha \geq 8\pi/5$), a design consisting of ξ^*_{3k} and ξ_4 (or ξ_5) is an orthogonal design of size $3k+4$ (or $3k+5$).

By Theorem 1, orthogonal designs of size $n = 3k$ exist if and only if $\alpha \geq 4\pi/3$. For $n = 3k+1$, consider the symmetric design consisting of $\pm x, \pm y$, $k-1$ pairs of $\pm z$, and $k-1$ midpoints 0, where $0 \leq x \leq y \leq z \leq \alpha/2$. Let $r = \cos(x), s = \cos(y)$, and $t = \cos(z)$ and denote this design by $\xi^*_{3k+1}(r,s,t)$. Note that $\xi^*_{3k+1}(r,s,t)$ is orthogonal if and only if $\mu = \{(k-1)(1+2t)+2r+2s\}/(3k+1) = 0$ and $\nu = \{(k-1)(1+2t^2)+2r^2+2s^2\}/(3k+1) = 1/2$. That is, r and s are the two roots of

$$g(v) = 8v^2 + 4(k-1)(1+2t)v + 4k(k-1)t^2 + 4(k-1)^2 t + k^2 - 3k - 2.$$

Since $1 \geq r \geq s \geq t \geq c$ and $r+s = (1-k)(1+2t)/2$, this is equivalent to the conditions that $c \leq (1-k)(1+2t)/4 \leq 1$, $g(c) \geq 0$, $g((1-k)(1+2t)/4) \leq 0$, and $g(1) \geq 0$. After some simplification, these conditions lead to (a) in Theorem 2 below. For $n = 3k+2$, consider $\xi^*_{3k+2}(r,s,t)$, the design consisting of $\xi^*_{3k+1}(r,s,t)$ and a midpoint 0. Using the same method as that for $n = 3k+1$, we obtain (b) of Theorem 2.

Theorem 2. (a) *For $n = 3k+1$ and $\alpha \geq \alpha^*_{3k+1} \equiv 2\arccos(t_{3k+1})$, where $t_{3k+1} = -1/2 - \{2k + (6k^2+2k)^{1/2}\}^{-1}$, there exists an orthogonal design $\xi^*_{3k+1}(r,s,t)$ with r and s given by*

$$r, s = (1-k)(1+2t)/4 \pm (1/4)\{4(1-k^2)t^2 - 4(1-k)^2 t - (k^2-4k-5)\}^{1/2},$$

where $t \leq -\sqrt{2}/2$ for $k = 1$ and $-1/2 - 2\{(k-1)+(3k^2-2k-1)^{1/2}\}^{-1} \leq t \leq t_{3k+1}$ for $k \geq 2$.
(b) *For $n = 3k+2$ and $\alpha \geq \alpha^*_{3k+2} \equiv 2\arccos(t_{3k+2})$, where $t_{3k+2} = -1/2 - \{k + (3k^2+2k)^{1/2}\}^{-1}$, there exists an orthogonal design $\xi^*_{3k+2}(r,s,t)$ with r and s given by*

$$r, s = (1-k)t/2 - k/4 \pm (1/4)\{4(1-k^2)t^2 - 4k(k-1)t - (k^2-2k-4)\}^{1/2},$$

where $-1/2 - (5/2)\{(k-1) + (6k^2-2k-4)^{1/2}\}^{-1} \leq t \leq t_{3k+2}$.

Table 1. Some Selected Values of α^*_{3k+1} and α^*_{3k+2}

k	1	2	3	4	5	10	20	50	∞
α^*_{3k+1}	1.5π	1.416π	1.388π	1.375π	1.366π	1.350π	1.342π	1.337π	$4\pi/3$
α^*_{3k+2}	1.6π	1.465π	1.421π	1.399π	1.386π	1.360π	1.347π	1.339π	$4\pi/3$

Table 1 gives some selected values of α^*_{3k+1} and α^*_{3k+2}. Note that the above orthogonal designs have a similar structure to those in Wu (1997). Motivated by

Wu's results, we conjecture that there exists no orthogonal design of size $3k+1$ (or $3k+2$) for $\alpha < \alpha^*_{3k+1}$ (or $\alpha < \alpha^*_{3k+2}$), and that the D-optimal designs in this case can be obtained among the designs $\xi_{3k+1}(r,c,c)$ (or $\xi_{3k+2}(r,c,c)$). The optimal value of r can be obtained by numerically maximizing $\phi_0(M(\xi_{3k+1}(r,c,c))$ (or $\phi_0(M(\xi_{3k+2}(r,c,c)))$ for $r \in [c,1]$. In particular, using the Maple programming language, we find that the optimal value of r is 1 for $n = 3k+1$ and $c \leq (1-k)/(2k)$, that the value is c for $n = 3k+2$ and $c \leq (5-k)/(2k+1)$.

6. Optimal Designs for Subsets of the Coefficients

For any symmetric design ξ, let $M_{(i)}(\xi)$ and $M_{(ij)}(\xi)$ denote the information matrices for β_i and $(\beta_i, \beta_j)'$, respectively, where $i = 0,1,2$, and $(i,j) = (0,1),(0,2),(1,2)$. The optimal design for β_i is to maximize (scalar) $M_{(i)}(\xi)$, $i = 0,1,2$. The ϕ_p-optimal design for $(\beta_i, \beta_j)'$ is to maximize $\phi_p(M_{(ij)})$, $0 \leq i < j \leq 2$, where $p \in [-\infty, 1]$. Let $K'\beta$ denote a subset of β_0, β_1, and β_2, where K is a $3 \times s$ matrix of rank s. Then the information matrix for $K'\beta$ is given by $C_K(M(\xi)) = (K'M^-(\xi)K)^{-1}$, where $M(\xi)$ satisfies $\mathcal{C}(K) \subset \mathcal{C}(M(\xi))$. Theorem 3 gives optimal designs for β_i, $i = 0,1,2$. Theorems 4, 5, and 6 give explicit D-, A-, and E-optimal designs for $(\beta_0,\beta_1)'$, $(\beta_0,\beta_2)'$, and $(\beta_1,\beta_2)'$, respectively. Note that $\mu \in [c,1]$ and Lemma 1 shows that $\nu \in [\mu^2, \nu_m(\mu)]$. If $\mu = c$ or 1, then $\nu = \mu^2$. Thus, for $\nu > \mu^2$, $(\mu,\nu) \in S$ where $S \equiv \{(\mu,\nu) : \mu \in (c,1), \nu \in (\mu^2, \nu_m(\mu)]\}$. This fact will be used in the Appendix in the proofs of Theorems 5(b) (D-optimality) and 6(b) (A- and E-optimality). Proofs of the other results in this section are similar in spirit and are omitted.

Theorem 3. (a) *For $\alpha \in [\pi, 2\pi]$, any symmetric design ξ with $\mu = 0$ is optimal for β_0. For $\alpha \in (0,\pi)$, the design $\xi_m((1+c)^{-1})$ is optimal for β_0.*
(b) *The design $\xi_m(1/2)$ is optimal for β_1.*
(c) *The design $\xi_u(-\min(\pi/2, \alpha/2), \min(\pi/2, \alpha/2))$ is optimal for β_2.*

Theorem 4. *The designs $\xi_m(1/2)$, $\xi_m(w_{a01})$, and $\xi_m(w_{e01})$ are D-, A-, and E-optimal for $(\beta_0,\beta_1)'$, respectively, where $w_{a01} = \{1 + (1/2 + c^2/2)^{1/2}\}^{-1}$ and $w_{e01} = (3+c)(5+2c+c^2)^{-1}$.*

Theorem 5. (a) *For $\alpha \in [\pi, 2\pi]$, the design $\xi_u(-\pi/2, \pi/2)$ is ϕ_p-optimal for $(\beta_0,\beta_2)'$, for all $p \in [-\infty, 1]$.*
(b) *For $\alpha \in (0,\pi)$, the designs $\xi_m(w_{d02})$, $\xi_m(w_{a02})$, and $\xi_m((1+c)^{-1})$ are D-, A-, and E-optimal for $(\beta_0,\beta_2)'$, respectively, where $w_{d02} = \{1 - c^2/4 + (c/4)(8 + c^2)^{1/2}\}^{-1}$ and $w_{a02} = \{1 + c(1/2 + c/2)^{1/2}\}^{-1}$.*

Theorem 6. (a) *For $\alpha \in [4\pi/3, 2\pi]$, any design ξ with $M(\xi)$ given by (4) with $(\mu,\nu) = (0, 1/2)$ is ϕ_p-optimal for $(\beta_1,\beta_2)'$, for all $p \in [-\infty, 1]$.*

(b) *For* $\alpha \in (0, 4\pi/3)$, *the designs* $\xi_m(2/3)$, $\xi_m(w_{a12})$, *and* $\xi_m(w_{e12})$ *are D-, A-, and E-optimal for* $(\beta_1, \beta_2)'$, *respectively, where* $w_{a12} = \{1 + (1/2 + c/2)^{1/2}\}^{-1}$, $w_{e12} = 1/2$ *if* $\alpha \in (0, \pi - \arccos(1/3)]$, *and* $w_{e12} = (-2c)/(1-c)$ *if* $\alpha \in (\pi - \arccos(1/3), 4\pi/3)$.

Remark 2. For $\mathcal{X} = [\alpha_1, \alpha_2]$, where $-\pi \leq \alpha_1 < \alpha_2 \leq \pi$, let $\xi_m(w)$ and α_0 be defined in Remark 1. Then Theorems 3(a) and 6 still hold, with $\alpha = \alpha_2 - \alpha_1$, $\xi_u(-\pi/2, \pi/2)$ replaced by $\xi_u(-\pi/2 + \alpha_0, \pi/2 + \alpha_0)$, and $\xi_u(-2\pi/3, 0, 2\pi/3)$ replaced by $\xi_u(-2\pi/3 + \alpha_0, \alpha_0, 2\pi/3 + \alpha_0)$.

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Appendix

Proof of Lemma 1. The proof of part (a) is straightforward. Let $v(x) = (\cos x - c)/(1-c)$ for $x \in \mathcal{X}$, and $m_i = \int_{\mathcal{X}} (v(x))^i d\xi(x)$ for $i = 1, 2$. Then for part (b), since $v(x) \in [0, 1]$, $m_2 = (\nu - 2c\mu + c^2)(1-c)^{-2} \leq m_1 = (\mu - c)/(1-c)$, that is, $\nu \leq (1+c)\mu - c = \nu_m(\mu)$. This proves part (b) since clearly $\nu \geq \mu^2$. For part (c), note that $m_2 = (\nu_m(\mu) - 2c\mu + c^2)(1-c)^{-2} = (\mu - c)/(1-c) = m_1$, that is, $\int_{\mathcal{X}} v(x)(1 - v(x))d\xi(x) = 0$. This implies $v(x) = 0$ or 1 for all x with $\xi(x) > 0$, because $v(x) \in [0, 1]$. Then the support points of ξ are $\pm\alpha/2$ and 0, and the symmetric design ξ is $\xi_m((1-\mu)/(1-c))$.

For part (d), let $1 - w(1-t) = \mu$ and $1 - w(1-t^2) = \nu$. If $\mu = 1$, then $\nu = 1$ and we take $w = t = 0$. If $\mu < 1$, we have $t = (\mu - \nu)/(1-\mu)$ and $w = (1-\mu)/(1-t)$. It remains to show that $t \in [c, 1)$ and $w \in [0, 1]$. This follows from the fact that $\mu^2 \leq \nu \leq \nu_m(\mu) = (1+c)\mu - c$.

Proof of Theorem 1. By Lemma 1(d), it suffices to consider designs $\xi(w, x)$, and hence the information matrix $M(\xi(w, x))$ given by (5). Since $|M(\xi(w, x))| = \{1 - w + wt^2 - (1 - w + wt)^2\}(w - wt^2) = w^2(1-w)(1-t)^3(1+t)$, $M(\xi(w, x))$ is nonsingular iff $w \in (0, 1)$ and $t \in [c, 1)$. Thus it suffices to consider $M(\xi(w, x))$

for $w \in (0,1)$ and $t \in [c,1)$. Note that $c = \cos(\alpha/2) \in (-1/2, 1)$, since $\alpha \in (0, 4\pi/3)$.

(a) D-optimality would maximize $f(w,t) \equiv |M(\xi(w,x))|$ for $w \in (0,1)$ and $t \in [c,1)$. Note that $w^2(1-w)$ is maximized at $w = 2/3$, for $w \in (0,1)$. Since $\{(1-t)^3(1+t)\}' = -2(1-t)^2(1+2t) < 0$ for $-1/2 < c \le t < 1$, then $(1-t)^3(1+t)$ is maximized at $t = c$ for $t \in [c,1)$. Thus, $f(w,t)$ is maximized at $w = 2/3$ and $t = c$ (i.e., $x = \alpha/2$).

(b) A-optimality would minimize the trace of $\{M(\xi(w,x))\}^{-1}$, that is, minimize

$$g(w,t) = \mathrm{tr}(\{M(\xi(w,x))\}^{-1}) = \frac{2-w+wt^2}{w(1-w)(1-t)^2} + \frac{1}{w-wt^2} = \frac{3+t-(2-t^2-t^3)w}{w(1-w)(1-t)^2(1+t)}$$

for $w \in (0,1)$ and $t \in [c,1)$. Note that, $g(0+,t) = g(1-,t) = +\infty$, and that

$$\frac{\partial}{\partial w} g(w,t) = \frac{(t^3+t^2-2)w^2 + 2(t+3)w - (t+3)}{w^2(1-w)^2(1-t)^2(1+t)} = 0$$

has a unique root $w = w(t) = (3+t)^{1/2}\{(3+t)^{1/2} + (1+t+t^2+t^3)^{1/2}\}^{-1}$ for $w \in (0,1)$. Thus, for given $t \in [c,1)$, $g(w,t)$ is minimized at $w = w(t)$ for $w \in (0,1)$.

Further observe that

$$\frac{\partial}{\partial s} g(w(t),s)|_{s=t} = \frac{2(1+t)^3 w(t) + 2(t^2+5t+2)(1-w(t))}{(1-t)^3(1+t)^2 w(t)(1-w(t))} > 0. \tag{8}$$

Clearly (8) holds for $t^2+5t+2 \ge 0$. If $t^2+5t+2 < 0$, then $(1+t)^6(3+t) - (t^2+5t+2)^2(1+t+t^2+t^3) = (1+t)(1+2t)(1-t)^2\{(1+t)-(t^2+5t+2)\} > 0$, and again (8) holds. By (8), there exists $\delta(t) > 0$ such that, for $s \in [c,1)$ and $0 \le t-s < \delta(t)$, we have $g(w(t),t) \ge g(w(t),s)$, and thus $g(w(t),t) \ge g(w(t),s) \ge g(w(s),s)$. This shows that $g(w(t),t)$ increases on $t \in [c,1)$ and is thus minimized at $t = c$, which proves part (b).

(c) Since $\lambda_1 \ge \lambda_2$, E-optimality would maximize $\min(\lambda_2, \lambda_3)$, where $\lambda_2 = 1 - (1/2)\{w - wt^2 + (h(w,t))^{1/2}\}$ and $\lambda_3 = w - wt^2$, with $h(w,t) = (w-wt^2)^2 + 4(1-w+wt)^2$. Consider first maximizing $\lambda_2 = \lambda_2(w,t)$. Note that $\lambda_2(0+,t) = \lambda_2(1-,t) = 0$, and that

$$\frac{\partial}{\partial w}\lambda_2(w,t) = \frac{t-1}{2}\left\{(1+t) - \frac{w(t^3+t^2+3t-5)+4}{\{h(w,t)\}^{1/2}}\right\} = 0$$

has a unique root $w = w_m(t) = (3+t)(5+2t+t^2)^{-1}$ for $w \in (0,1)$. This follows by observing that $w(t^3+t^2+3t-5)+4 = 1+t > 0$ and that $(1+t)^2 h(w,t) - \{w(t^3+t^2+3t-5)+4\}^2 = 4(1-t)\{1-w(1-t)\}\{w(5+2t+t^2)-(3+t)\}$. Thus, for given $t \in [c, 1)$, $\lambda_2(w, t)$ is maximized at $w = w_m(t)$ for $w \in (0, 1)$. Note that $h(w_m(t), t) = 1 + t$ and thus $\lambda_2(w_m(t), t) = (1 - t)^2\{4 + (1 + t)^2\}^{-1}$, maximized at $t = c$ for $t \in [c, 1)$. Thus, λ_2 is maximized at $w = w_m(c)$ and $t = c$.

Consider now maximizing $\min(\lambda_2, \lambda_3)$. Note that $\lambda_2(w_m(c), c) = (1-c)^2\{4+(1+c)^2\}^{-1}$ and $\lambda_3(w_m(c), c) = (3+c)(1-c^2)\{4+(1+c)^2\}^{-1}$. Thus $\lambda_2(w_m(c), c) > \lambda_3(w_m(c), c)$ iff $c^2 + 5c + 2 < 0$ iff $c \in (-1/2, c_*)$, where $c_* = (\sqrt{17} - 5)/2$. Hence, for $c \in [c_*, 1)$,

$$\min(\lambda_2, \lambda_3) \le \lambda_2 \le \lambda_2(w_m(c), c) = \min\{\lambda_2(w_m(c), c), \lambda_3(w_m(c), c)\}.$$

That is, $\min(\lambda_2, \lambda_3)$ is maximized at $w = w_m(c)$ and $t = c$.

For $c \in (-1/2, c_*)$, if $t \in [c_*, 1)$ the above result shows that $\min(\lambda_2, \lambda_3)$ is maximized at $w = w_m(c_*)$ and $t = c_*$. Thus, it suffices to consider maximizing $\min(\lambda_2, \lambda_3)$ for $t \in [c, c_*]$ and $w \in (0, 1)$. For given $t \in [c, c_*]$, λ_2 increases from 0 to $\lambda_2(w_m(t), t)$ and then decreases to 0, while λ_2 linearly increases from 0 to $1 - t^2$ for $w \in [0, 1]$. The above paragraph shows that $\lambda_2(w_m(t), t) > \lambda_3(w_m(t), t)$. Thus, given $t \in [c, c_*]$, $\min(\lambda_2, \lambda_3)$ is maximized when $\lambda_2(w, t) = \lambda_3(w, t) > 0$, that is, when $2 - 3w(1 - t^2) = \{h(w, t)\}^{1/2}$, or when $w = w_{eq}(t) = (1 + 3t)(1 - t)^{-1}(1 + 4t + 2t^2)^{-1}$ for $w \in (0, 1)$. Then $f(t) \equiv \lambda_2(w_{eq}(t), t) = \lambda_3(w_{eq}(t), t) = (1 + 3t)(1 + t)\left(1 + 4t + 2t^2\right)^{-1}$, which is maximized at $t = c$ since $f'(t) = 2t(1 + 2t)(1 + 4t + 2t^2)^{-2} < 0$, for $t \in [c, c_*]$. Thus, for $c \in (-1/2, c_*)$, $\min(\lambda_2, \lambda_3)$ is maximized at $t = c$ and $w = w_{eq}(c)$. This proves part (c).

Lemma 2. *Suppose that* $g(\mu, \nu) \ge 0$ *for* $(\mu, \nu) \in S$, $g(\mu, \nu)$ *increases on* $(\mu^2, \nu_m(\mu)]$ *for given* $\mu \in (c, 1)$, *and* $g_1(\mu) \equiv g(\mu, \nu_m(\mu))$ *is differentiable on* $(c, 1)$. *If* $g_1(c+) = g_1(1-) = 0$ *and* $g_1'(\mu) = 0$ *has a unique solution* $\mu_0 \in (c, 1)$, *then* $\max_{(\mu,\nu)\in S} g(\mu, \nu) = g_1(\mu_0)$.

The proof of Lemma 2 is straightforward and is omitted.

Proof of Theorem 5(b) (D-optimality). Maximize $g(\mu, \nu) = (1-\mu^2/\nu)(1-\nu)$ on S. Note that $g_1(\mu) = g(\mu, \nu_m(\mu)) = (1 + c)(\mu - c)(1 - \mu)^2(\mu + c\mu - c)^{-1}$. Now $g_1'(\mu) = 0$ gives $2(1 + c)\mu^2 - c(4 + c)\mu + c^2 = 0$, which has a unique solution $\mu_0 = c(4 + c + (8 + c^2)^{1/2})(4 + 4c)^{-1}$ for $\mu \in (c, 1)$. The D-optimality of $\xi_m(w_{d02})$ then follows from Lemmas 2 and 1(c).

Proof of Theorem 6(b) (A- and E-optimality). A-optimality would maximize $g(\mu, \nu) = (1 - \nu)(\nu - \mu^2)/(1 - \mu^2)$ on S. Note that $(1 + \mu^2)/2 < \nu_m(\mu)$ iff $c < 0$ and $\mu \in (1 + 2c, 1)$. In this case, we have $g(\mu, \nu) \le g(\mu, (1 + \mu^2)/2) = (1-\mu^2)/4 \le (1-(1+2c)^2)/4 = g(1+2c, \nu_m(1+2c))$, since $1+2c > 0$. Thus, it suffices to consider maximizing $g(\mu, \nu)$ for $\mu \le 1+2c$. Then $g(\mu, \nu) \le g(\mu, \nu_m(\mu)) = (1 + c)(\mu - c)(1 - \mu)/(1 + \mu)$, which is maximized at $\mu_* = -1 + (2 + 2c)^{1/2}$ for $\mu \in (c, 1)$ and $\mu \le 1 + 2c$.

For E-optimality, by Lemma 1(d), it suffices to maximize $g(w,t)=\min(w(1-w)(1-t)^2, w(1-t^2))$ for $w \in (0,1)$ and $t \in [c,1)$. First note that $w(1-w)(1-t)^2$ is maximized at $w = 1/2$ and $t = c$. Since $(1/2)^2(1-c)^2 > (1/2)(1-c^2)$ iff $c < -1/3$, then for $c \in [-1/3,1)$, $g(w,t)$ is maximized at $w = 1/2$ and $t = c$. For $c \in (-1/2,-1/3)$, if $t \in [-1/3,1)$, then $g(w,t)$ is maximized at $w = 1/2$ and $t = -1/3$. Thus it suffices to consider maximizing $g(w,t)$ for $t \in [c,-1/3]$ and $w \in (0,1)$. For given $t \in [c,-1/3]$, $g(w,t)$ is maximized when $w(1-w)(1-t)^2 = w(1-t^2)$, for $w \in (0,1)$, which gives $w = w(t) = (-2t)/(1+t)$. Then $g(t,w(t)) = (-2t)(1-t)$ which is maximized when $t = c$. This completes the proof.

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