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ALMOST SURE CONVERGENCE OF WEIGHTED SUMS

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Abstract. Suppose that $e_1, e_2, \ldots$ are i.i.d. random variables with $Ee_1 = 0$ and $E|e_1|^r < \infty$ for some r with $1 \le r < 2$. In this paper, we obtain sufficient conditions for almost sure convergence of the weighted sum $\sum_{i=1}^n a(i)e_i/A(n)$ and derive that these conditions are necessary in some sense.

Key words and phrases: Almost sure convergence, linear regression, consistency.

1. Introduction and Main Results

Jamison et al. (1965) proved the following result: Suppose that $e_1, e_2, \ldots$ are i.i.d. random variables with $Ee_1 = 0$, $a(1), a(2), \ldots$ are positive constants such that $A(n) = \sum_{i=1}^n a(i) \to \infty$ as $n \to \infty$. Denote by $N(K)$ the number of subscripts i such that $A(i)/a(i) \le K$. If

$$N(K) = O(K), \tag{1.1}$$

then

$$\sum_{i=1}^n a(i)e_i/A(n) \to 0, \qquad \text{a.s..} \tag{1.2}$$

Conversely, if (1.1) is not true, then there exist i.i.d. random variables $e_1, e_2, \ldots$ with $Ee_1 = 0$ such that (1.2) is false.

Three questions may be raised concerning this result.

a). What happens when $\{A(n)\}$ is replaced by some other constant sequence satisfying monotone increasingness? (In the following, we still use the notation $\{A(n)\}$ for the sequence without confusion.)

b). What happens when the sets $\{i : i \ge 1, a(i) > 0\}$ and $\{i : i \ge 1, a(i) < 0\}$ are infinite?

c). What happens when e_i possesses moments of order higher than one ?

These questions have already been considered by some authors for various special series $\{A(n)\}$. See for example Kolmos and Revesz (1964) and Azuma (1967). Zhu (1989), in an unpublished doctoral dissertation, considered the case where $A(n) = \sum_{i=1}^n a^2(i)$ in connection with the strong consistency of LS estimates of linear regression coefficients.

In this paper we consider the above questions under the assumption that $E|e_i|^r < \infty$ for some $r \in [1, 2)$. Write $u(i) = a(i)/A(i)$ and let $([n, 1], \ldots, [n, n])$ be a permutation of $(1, \ldots, n)$ such that

$$|u([n,1])| \geq \cdots \geq |u([n,n])|,$$
$$[n,i] < [n,j], \quad \text{if} \quad i < j \text{ and } |u(i)| = |u(j)|.$$

Let $I(\cdot)$ be the indicator function and define

$$V_1(n,j) = \sum_{i=1}^{n} a(i) I(|u(i)| \geq |u([n,j])|), \qquad \text{for } 1 \leq j \leq n,$$
$$\text{and} \qquad V_1(n) = \max_{1 \leq j \leq n} |V_1(n,j)|.$$

Now we can formulate the main results of this paper:

Theorem 1. *Suppose that $e_1, e_2, \ldots$ are i.i.d. random variables with $Ee_1 = 0$, $a(i) \neq 0$ for $i \geq 1$ and $0 < A(1) \leq A(2) \leq \cdots \to \infty$. Then* (1.2) *holds if* (1.1) *and*

$$V_1(n) = O(1) \tag{1.3}$$

hold simultaneously. Conversely, if at least one of (1.1) *and* (1.3) *is not true, then there exists an i.i.d. sequence $\{e_i\}$ with $Ee_1 = 0$ such that* (1.2) *does not hold.*

The result becomes simpler when e_i possesses a higher-order moment:

Theorem 2. *Suppose that $e_1, e_2, \ldots$ are i.i.d. random variables with $Ee_1 = 0$ and $E|e_1|^r < \infty$ for some $r \in (1, 2)$, $\{a(i), i \geq 1\}$ and $\{A(i), i \geq 1\}$ are constant sequences satisfying the conditions specified in Theorem* 1. *Then* (1.2) *holds true when $N(K) = O(K^r)$. Conversely, if $N(K)$ is not of the order $O(K^r)$, then there exists an i.i.d. sequence $\{e_i\}$ with $Ee_1 = 0$ and $E|e_1|^r < \infty$ such that* (1.2) *is false.*

It is interesting to note that for $r \in (1, 2)$ the additional condition (1.3) is not needed.

2. Some Remarks

In this section, we give some comments on the results stated in Section 1.

1. If $a(i) > 0$ for all i, condition (1.3) becomes

$$\sum_{i=1}^{n} a(i) = O(A(n)). \tag{2.1}$$

This gives an answer of the question a) in Section 1: $A(n)$ must satisfy the condition that $\sum_{i=1}^n a(i) = O(A(n))$. Therefore, in the case where $a(i) > 0$ for all i, $\sum_{i=1}^n a(i) = A(n)$ is essentially the only possible choice to ensure the truth of (1.2).

2. Theorem 2 can not be extended to $r \geq 2$. True, the condition $N(K) = O(K^r)$ remains necessary even for $r \geq 2$, but is no longer sufficient. A simple counter-example is that $e_1, e_2, \ldots$ are i.i.d. with common distribution $N(0,1)$, $a(i) = 1$ for $i \geq 1$ and $A(n) = n^{1/r}$.

3. What happens when the random variables $e_1, e_2, \ldots$ are assumed to be independent but not necessarily identically distributed? Let F_i be the distribution of e_i. It is easy to see that if there are only finitely many different distributions in $\{F_i\}$, then the conclusion of Theorem 2 remains valid. The same is true for Theorem 1 if $a(i)$ is positive for all i. Even this simplest extension becomes invalid for Theorem 1 without this additional restriction. Here is a counter-example: Put $A(i) = i$, and

$$a(i) = \begin{cases} (-1)^i \sqrt{i}, & \text{for } 2^m - [2^{m/2}] \leq i \leq 2^m,\ m = 2, 3, \ldots, \\ 1, & \text{other } i \geq 1. \end{cases}$$

It is easy to verify that $\{a(i), A(i) : i \geq 1\}$ so defined satisfies (1.1) and (1.3), but $\{|a(i)|, A(i) : i \geq 1\}$ does not satisfy (1.3). According to Theorem 1, there exist i.i.d. variables $\tilde{e}_1, \tilde{e}_2, \ldots$ with $E\tilde{e}_1 = 0$ such that $\sum_{i=1}^n |a(i)| \tilde{e}_i / A(n)$ does not converge almost surely to zero. Put $e_i = \operatorname{sgn}(a(i)) \tilde{e}_i$ for $i \geq 1$, then $\{e_1, e_2, \ldots\}$ is an independent series with $Ee_i = 0$ and there are only two different members in the distribution series $\{F_i\}$, but (1.2) is false.

4. Even if $a(i) > 0$ for all i, the condition (1.3) does not follow from (1.1). This convinces us that this condition is, indeed, essential. The following is a counter-example. Take a subsequence of positive integers $n_1 < n_2 < \cdots$ such that

$$k^{-1} (\log k)^{-2} \log n_k \to \infty, \quad \text{as} \quad k \to \infty. \tag{2.2}$$

Define $a(i) = 1$ for $1 \leq i \leq n_1$, and for $k \geq 2$,

$$a(i) = (\log k)^2 / (i - n_1 - \cdots - n_{k-1}) \quad \text{for } n_1 + \cdots + n_{k-1} < i \leq n_1 + \cdots + n_k.$$

Let $A(n) = a^2(1) + \cdots + a^2(n)$. Obviously $A(n) \to \infty$ and there exist positive constants c_1 and c_2 such that

$$c_1 k (\log k)^4 \leq A(n) \leq c_2 k (\log k)^4, \quad k \geq 2. \tag{2.3}$$

Denote by $M_k(K)$ the number of subscripts i such that $n_1 + \cdots + n_{k-1} < i \leq n_1 + \cdots + n_k$ and $a(i)/A(i) > K^{-1}$. Then by (2.3) it is easily seen that

$M_k(K) \le Kc_1^{-1}k^{-1}(\log k)^{-2}$. Therefore

$$N(K) \le n_1 + Kc_1^{-1}\sum_{k=2}^{\infty} k^{-1}(\log k)^{-2} = O(K)$$

and (1.1) is satisfied. On the other hand, in view of (2.2) and (2.3), we have

$$\begin{aligned} V(n_1+\cdots+n_k) &= \sum_{i=1}^{n_1+\cdots+n_k} a(i)/A(n_1+\cdots+n_k) \\ &\ge c_2^{-1}k^{-1}(\log k)^{-4}(\log k)^2\sum_{i=1}^{n_k} i^{-1} \\ &\ge c_2^{-1}k^{-1}(\log k)^{-2}\log n_k \to \infty, \quad \text{as} \quad k\to\infty. \end{aligned}$$

So (1.3) fails.

3. Proof of the Theorems — Sufficiency

Suppose that the conditions of Theorem 1 or Theorem 2 are satisfied. Put

$$e_i' = e_i I(|e_i| < |u(i)|^{-1}), i \ge 1, \quad \text{and} \quad Z'(n) = \sum_{i=1}^{n} a(i)e_i'/A(n).$$

Then using an argument similar to that used in Jamison et al. (1965), we prove that

$$P\Big(\lim_{n\to\infty} Z(n) = 0\Big) = P\Big(\lim_{n\to\infty} Z'(n) = 0\Big), \tag{3.1}$$

and

$$Z'(n) - EZ'(n) \to 0, \quad \text{a.s..} \tag{3.2}$$

We mention that the condition (1.3) is not used in the proof of (3.1) and (3.2). This fact is important in the proof of necessity of (1.3).

From (3.1) and (3.2), it follows that in order to prove (1.2) we need only to show that

$$\lim_{n\to\infty} EZ'(n) = \lim_{n\to\infty}\sum_{i=1}^{n} a(i)Ee_i'/A(n) = 0. \tag{3.3}$$

Define

$$\begin{aligned} V_r(n,j) &= \sum_{i=1}^{j} a([n,i])|u([n,j])|^{r-1}/A(n) \quad \text{for} \quad 1\le j\le n, \\ \text{and} \qquad V_r(n) &= \max_{1\le j\le n}|V_r(n,j)| \quad \text{for} \quad 1<r<2. \end{aligned} \tag{3.4}$$

The condition $N(K) = O(K^r)$ entails $u(i)\to 0$ as $i\to\infty$, so we can define

$$u_i = \text{ the } i\text{-th largest among } \{|u(k)| : k \ge 1\},$$

we then have $u_1 \geq u_2 \geq \cdots > 0$.

We need the following lemma.

Lemma 1. *Suppose that $N(K) = O(K^r)$ holds for given $r \in (1, 2)$, then*

$$V_r(n) = O(1).$$

Proof. There exists a positive integer R such that the number of elements in the set $\{i : i \geq 1, |u(i)|^r > 1/m\}$ does not exceed mR, $m = 1, 2, \ldots$. Hence

$$|u_i|^r \leq m^{-1}, \quad i > mR, \quad \text{for} \quad m = 1, 2, \ldots.$$

Since $|u([n, i])| \leq |u_i|$ for $1 \leq i \leq n$, we have

$$|u([n, i])|^r \leq m^{-1}, \quad mR < i \leq n.$$

Therefore $i|u([n, i])|^r \leq i/m \leq (m + 1)R/m \leq 2R$, when $mR < i \leq (m + 1)R$, $i \leq n, m = 1, 2, \ldots$. Further, $i|u([n, i])|^r \leq Ru_1^r$ when $i \leq R$. Hence there exists a constant c such that for $1 \leq i \leq n$

$$|u([n, i])|^r \leq ci^{-1/r}, \tag{3.5}$$

which entails, on noting that $0 < A([n, i]) \leq A(n)$,

$$|a([n, i])|/A(n) \leq |a([n, i])|/A([n, i]) = |u([n, i])| \leq ci^{-1/r}, \quad \text{for} \quad 1 \leq i \leq n.$$

Combining this and (3.5), and noting that $r > 1$, we get for $1 \leq j \leq n$

$$|V_r(n, j)| \leq c^{r-1} j^{-(r-1)/r} \sum_{i=1}^{j} ci^{-1/r} \leq c_0 j^{1-1/r} j^{-(r-1)/r} = c_0, \tag{3.6}$$

for some constant c_0. The lemma is proved.

We note that (3.6) breaks down if $r = 1$. This is the reason why, for $r = 1$, the additional condition (1.3) is needed.

Now turn to the proof of (3.3). Denote by F the distribution of e_1, and define

$$p(i, r) = \int_{|u([n,i])|^{-1} \leq |x| < |u([n,i+1])|^{-1}} |x|^r dF, \quad \text{for} \quad 1 \leq i \leq n - 1,$$
$$p(n, r) = \int_{|x| \geq |u([n,n])|^{-1}} |x|^r dF,$$

and

$$\tilde{p}(i,r) = \int_{|u([n,i])|^{-1} \le |x| < |u([n,i+1])|^{-1}} x^r dF, \quad \text{for} \quad 1 \le i \le n-1,$$
$$\tilde{p}(n,r) = \int_{|x| \ge |u([n,n])|^{-1}} x^r dF.$$

Since $Ee_1 = 0$, we have

$$\begin{aligned} EZ'(n) &= -A(n)^{-1} \sum_{i=1}^{n} a(i) \int_{|x| \ge |u(i)|^{-1}} x dF \\ &= -A(n)^{-1} \sum_{i=1}^{n} a([n,i]) \int_{|x| \ge |u([n,i])|^{-1}} x dF \\ &= -A(n)^{-1} \sum_{i=1}^{n} \Big(a([n,i]) \sum_{j=i}^{n} \tilde{p}(j,1) \Big) \\ &= -A(n)^{-1} \sum_{j=1}^{n} \Big(\tilde{p}(j,1) \sum_{i=1}^{j} a([n,i]) \Big). \end{aligned} \tag{3.7}$$

For $r > 1$, combining the fact that $|\tilde{p}(j,1)| \le |u([n,j])|^{r-1} p(j,r)$ and the definition of $V_r(n,j)$, we get

$$|EZ'(n)| \le \sum_{j=1}^{n} |V_r(n,j)| p(j,r) = \Big(\sum_{j=1}^{h-1} + \sum_{j=h}^{n} \Big) |V_r(n,j)| p(j,r) \equiv J_{n1} + J_{n2}, \tag{3.8}$$

where h is a fixed integer with $2 \le h \le n$. It is easily seen that (3.8) remains true for $r = 1$ with $V_1(n,j)$ defined in Section 1. Indeed, if $|u([n,j])| = |u([n,j+1])|$, then $p(j,1) = 0$, otherwise we have $A^{-1}(n) \sum_{i=1}^{j} a([n,i]) = V_1(n,j)$. So we always have $A^{-1}(n) \sum_{i=1}^{j} a([n,i]) p(j,1) = V_1(n,j) p(j,1)$ and (3.8) follows from (3.7).

Since $|u([n,i])| \le u_1$, $|a([n,i])| = |u([n,i])| A([n,i]) \le u_1 A([n,i])$, we have

$$|V_r(n,j)| \le j \ u_1^r A(\max_{1 \le i \le j} [n,i]) / A(n).$$

Hence

$$J_{n1} \le (h-1)^2 u_1^r A(\max_{1 \le i \le h-1} [n,i]) / A(n).$$

Since $u(i) \ne 0$ for $i \ge 1$ and $u(i) \to 0$ as $i \to \infty$, there exists a positive integer H such that $[n,i] \le H$ for $1 \le i \le h-1$ and n sufficiently large. Therefore,

$$J_{n1} \le (h-1)^2 u_1^r A(H)/A(n)$$

which entails $J_{n1} \to 0$ as $n \to \infty$ for fixed h, since $A(n) \to \infty$ as $n \to \infty$. Further, using (1.3) for $r = 1$ or Lemma 1 if $1 < r < 2$, we see that $|V_r(n,j)| \le c$ for and some constant c, $1 \le j \le n$. Hence

$$J_{n2} \le c \sum_{j=h}^{n} p(j,r) = c \int_{|x| \ge |u([n,h])|^{-1}} |x|^r dF.$$

Since $E|e_1|^r < \infty$ and $|u([n,h])|$ can be made arbitrarily small by taking h large enough and $n \geq h$, J_{n2} can be made arbitrarily small by taking h sufficiently large and $n \geq h$. This, together with $J_{n1} \to 0$ proved earlier, and (3.8), gives (3.3). The proof of sufficiency is concluded.

Remark. Note the slight difference between $V_1(n)$ defined in Section 1 and $V_r(n)|_{r=1}$ where $V_r(n)$ is defined by (3.4). True, if in condition (1.3) $V_1(n)$ is replaced by $V_r(n)|_{r=1}$, then the proof of sufficiency still works, but it can be shown that (1.3) is no longer necessary.

4. Proof of the Theorems — The Converse

According to those stated in the theorems, we establish counterexamples when (1.1) or (1.3) does not hold.

For the case $r = 1$, $a(i) > 0$ and $A(n) = \sum_{i=1}^n a(i) \to \infty$ as $n \to \infty$. Jamison et al. (1965) has proved that if (1.1) does not hold, then there exists an i.i.d. sequence $\{e_1, e_2, \ldots\}$ with $Ee_1 = 0$ such that (1.2) is not true. Their proof, with some minor modifications, can be used here to deal with the condition $N(K) = O(K^r), 1 < r < 2$. So, the remaining task concerns only the condition (1.3) in case $r = 1$, and we may assume that (1.1) holds. Thus, as argued in Section 3, (1.2) is equivalent to $Z'(n) \to 0$ a.s., and the latter in turn is equivalent to (3.3) (remember the fact we mentioned earlier that in proving (3.1) and (3.2), no use is made on the condition (1.3)).

Now suppose that (1.3) does not hold. Define

$$\tilde{V}_1(n,j) = A(n)^{-1} \sum_{i=1}^n a(i) I(|u(i)| \leq |u([n,j])|), \quad \tilde{V}_1(n) = \max_{1 \leq j \leq n} |\tilde{V}_1(n,j)|.$$

Then $\{|\tilde{V}_1(n,j)| : n \geq 1\}$ must be unbounded, for otherwise (1.3) will be true. Hence, turning to some subsequence of positive integers if necessary, we may assume that there exists a sequence of $\{j(n), n \geq 1\}$ with $1 \leq j(n) \leq n$ such that $|\tilde{V}_1(n, j(n))| \to \infty$. We may require that $j(n)$ satisfies the additional conditions that $j(n) \to \infty$ and $|\tilde{V}_1([n, j(n)])| < |\tilde{V}_1([n,1])|$. Find $\varepsilon(n) \downarrow 0$ such that

$$\varepsilon(n)|\tilde{V}_1(n, j(n))| \to \infty. \tag{4.1}$$

Denote by $j' \equiv j'(n)$ the largest integer $i \leq n$ satisfying $|u([n,i])| > |u([n,j(n)])|$, and define

$$W_1(n) = (|u([n,j'])| + |u([n,j(n)])|)/2, \quad W_2(n) = \min_{1 \leq i \leq n} |u(i)|.$$

Find a subsequence of positive integers $m_1 < m_2 < \cdots$ satisfying the following conditions:

(A) $\sum_{k=1}^{\infty} \varepsilon(m_k) < \infty$,
(B) $W_2(m_{k-1}) > W_1(m_k)$,
(C) $\sum_{t \in Q_k} |a(t)|/A(m_k) < 1$,
where

$$Q_k = \{t : 1 \le t \le m_k, |u(t)|^{-1}| \ge \min_{1 \le i \le k-1} \min\{1/W_1(m_i), 1/W_2(m_i)\}\}.$$

The existence of such subsequence follows from $\varepsilon(n) \to 0$, $u(n) \to 0$, $A(n) \to \infty$ and $a(i) \ne 0$ for all i (hence $W_2(n) > 0$). As $W_1(n) \to 0$ and $W_2(n) \to 0$, from (A) it follows that

$$0 < g^{-1} \equiv \sum_{k=1}^{\infty} \varepsilon(m_k)(W_1(m_k) + W_2(m_k)) < \infty. \tag{4.2}$$

Now construct a probability distribution F_0:

$$F_0(\{1/W_1(m_k)\}) = gW_1(m_k)\varepsilon(m_k), F_0(\{-1/W_2(m_k)\}) = gW_2(m_k)\varepsilon(m_k), k \ge 1.$$

We have

$$\int |x| dF_0 = 2g \sum_{k=1}^{\infty} \varepsilon(m_k) < \infty, \quad \int x dF_0 = 0.$$

The latter assertion follows by observing that since $\{m_k\}$ satisfies (B), so the support of F_0 within the interval $[-1/W_2(m_k), 1/W_2(m_k)]$ is $\{1/W_1(m_i), -1/W_2(m_i), 1 \le i \le k\}$, so $\int_{|x| \le 1/W_2(m_k)} x dF_0 = 0$. Letting $k \to \infty$, we get $\int x dF_0 = 0$.

Now we proceed to show that if the common distribution of e_i is F_0, (3.3) will not hold. For this purpose we divide the terms in the expression

$$EZ'(m_k) = A(m_k)^{-1} \sum_{i=1}^{m_k} a(i) \int_{|x| < |u(i)|^{-1}} x dF_0$$

into three groups:
1^o $|u(i)| \le |u([m_k, j(m_k)])|$.

According to (B) and the definitions of $W_1(n)$ and $W_2(n)$, it is easily seen that for such i the support of F_0 within the interval $(-|u(i)|^{-1}, |u(i)|^{-1})$ is the set $\{1/W_1(m_i), -1/W_2(m_i), 1 \le i \le k-1; 1/W_1(m_k)\}$. Hence

$$\int_{|x| < |u(i)|^{-1}} x dF_0 = g\varepsilon(m_k).$$

2^o $i \in H_k \equiv \{i : |u(i)| > |u([m_k, j(m_k)])|\} \cap Q_k$.

Put

$$J = \sum_{j \in H_k} a(j) \int_{|x| < |u(j)|^{-1}} x dF_0 / A(m_k).$$

Then by (C) we have

$$J \leq \int |x| dF_0 < \infty. \tag{4.3}$$

3^o Other $i \leq m_k$.

For such i we have $|u(i)| > |u([m_k, j(m_k)])|$ and $i \notin Q_k$. So $|u(i)| > W_1(m_k)$ in view of the definition of $W_1(n)$. Hence the points $1/W_1(m_t)$ and $-1/W_2(m_t)$ do not fall into the interval $(-|u(i)|^{-1}, |u(i)|^{-1})$ when $t \geq k$. As $i \notin Q_k$, the interval $(-|u(i)|^{-1}, |u(i)|^{-1})$ contains the points $1/W_1(m_j)$ and $-1/W_2(m_j), 1 \leq j \leq k-1$. From this and the definition of F_0 we have $\int_{|x|<|x(i)|^{-1}} x dF_0 = 0$.

Summing up the above three cases and recalling the definition of $\tilde{V}_1(n, j)$, we have

$$EZ'(m_k) = g\varepsilon(m_k)\tilde{V}_1(m_k, j(m_k)) + J. \tag{4.4}$$

From (4.1), (4.3) and (4.4), we see that (3.3) is not true. As argued earlier, this implies that (1.2) is false. This completes the proof of the necessity of (1.3).

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