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OPTIMALITY ASPECTS OF AGRAWAL'S DESIGNS: PART II

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Abstract. We report optimality aspects of row-column designs for 7 treatments, 7 rows, and 7 columns with three treatment replications within the class of designs with row-column, row-treatment, and column-treatment incidence matrices generated by binary circulants. In particular, we find a 3-way BIBD which doubles the efficiencies of the Agrawal (1966a,b) design for all three factors, and which is optimal w.r.t. Kiefer's Φ_p-criteria within this class of designs. Also it turns out to be universally optimal within a large subclass of designs.

Key words and phrases: BIBD, 3-way BIBD, circulants, Loewner partial ordering, Φ_p-optimality, row-column designs, Schur optimality, universal optimality.

1. Introduction

Agrawal (1966a,b) provided a series of 3-way BIBDs for the parameters $R = C = \nu = 4\lambda + 3$ a prime power, and $r = 2\lambda + 1$, (i.e, row-column designs with the same completely symmetric C-matrices for all of the three factors, treatments, rows, and columns; see Hedayat and Raghavarao (1975)). Shah and Sinha (1990) investigated optimality aspects of these designs. For the Agrawal design with 7 levels for each factor (i.e., $\lambda = 1$) they came up with a competitor which fares better than the former one with respect to all three factors.

This paper is a follow-up of the Shah & Sinha paper and may be regarded as a rejoinder to the same for the specific Agrawal setup with $\lambda = 1$. While we were examining the prospect of optimality of Shah-Sinha designs, many interesting features of this problem surfaced and we propose to present them here. In particular, we discovered another set of 3-way BIBDs for which every member doubles the efficiency of the Agrawal design for all three factors. These designs also fare better than the one found by Shah and Sinha (1990) w.r.t. many optimality criteria: The designs presented here are universally optimal within the subclass of binary circulant designs such that at least one of the incidence matrices yields a BIBD structure, and, more interestingly, they are optimal in the sense of Kiefer's Φ_p-criteria (see e.g. Kiefer (1975)) for all $p \geq 0$ within the set of *all* binary designs for three treatment replications and incidence matrices generated by *arbitrary* circulants. We refer to Shah and Sinha (1989) for a discussion on optimality criteria along with available results on row-column designs.

2. Preliminaries

We consider in the Agrawal setup, with $\lambda = 1$, optimality aspects of row-column designs with corresponding incidence matrices N_{rc}, N_{rt}, and N_{ct} for row-column, row-treatment, and column-treatment, respectively, generated by binary circulants. In this setup, Agrawal (1966a,b) constructed a 3-way BIBD (henceforth denoted by d^A) with all three incidence matrices generated by the same circulant,

$$N_{rc}^A = N_{rt}^A = N_{ct}^A = ((0\ 1\ 1\ 0\ 1\ 0\ 0)),$$

and C-matrices for treatment, row, and column effects given by

$$C_t^A = C_r^A = C_c^A = ((6\ \ -1\ \ -1\ \ -1\ \ -1\ \ -1\ \ -1))/7.$$

A possible layout of d^A, given here for convenience, is as follows.

$$d^A \simeq \begin{bmatrix} - & 2 & 4 & - & 1 & - & - \\ - & - & 3 & 5 & - & 2 & - \\ - & - & - & 4 & 6 & - & 3 \\ 4 & - & - & - & 5 & 7 & - \\ - & 5 & - & - & - & 6 & 1 \\ 2 & - & 6 & - & - & - & 7 \\ 1 & 3 & - & 7 & - & - & - \end{bmatrix}.$$

In an attempt to investigate optimality aspects of Agrawal's design, Shah and Sinha (1990) analyzed the trace of the C-matrix for varietal comparisons, and came up with a competitor (to be denoted by d^{SS}), which has row-column and row-treatment pattern $N_{rc}^{SS} = N_{rt}^{SS} = ((0\ 1\ 1\ 0\ 1\ 0\ 0))$ as above, while the column-treatment pattern is changed to $N_{ct}^{SS} = ((1\ 0\ 1\ 0\ 0\ 1\ 0))$. The resulting C-matrix for treatments is found to be the circulant

$$C_t^{SS} = ((10\ \ 0\ \ -2\ \ -3\ \ -3\ \ -2\ \ 0))/7,$$

which strongly dominates C_t^A. Moreover, Shah and Sinha also observed that d^{SS} strongly dominates d^A for row and column comparisons as well. (Here, strong dominance refers to the Loewner partial ordering for the associated C-matrices; discussions on the role of that ordering in the context of design optimality can be found in the book of Pukelsheim (1993), for example.)

We decided to examine by an exhaustive *computer search* the class of all designs with row-column, row-treatment, and column-treatment incidence matrices generated by circulants composed of the numbers 0 and 1 with repetitions 4 and 3, respectively. This gives 35 possibilities for each incidence matrix, but only 5 of them are distinct. Actually, without any loss, we can assign fixed combinations of 5×5 combinations to two of the three factor combinations; for finding feasible designs, however, we have to examine all 35 possibilities for the third factor combination, (see below). This results into 875 combinations of incidence matrices N_{rc}, N_{rt} and N_{ct} to be checked.

3. Optimality Results

We found that out of the 875 only 80 combinations provide feasible designs. Actually, any feasible combination $d \simeq (N_{rc}, N_{rt}, N_{ct})$ represents the 49 designs

$$\{ (P^i N_{rc}, P^j N_{rt}, P^{j-i} N_{ct}) : 0 \le i, j \le 6 \},$$

where P is the circulant $((0\ 1\ 0\ 0\ 0\ 0\ 0))$. For, feasibility of d implies feasibility of both designs

$$(PN_{rc}, PN_{rt}, N_{ct}) \qquad \text{and} \qquad (N_{rc}, PN_{rt}, PN_{ct}).$$

We define two designs $d \simeq (N_{rc}, N_{rt}, N_{ct})$ and $\tilde{d} \simeq (\tilde{N}_{rc}, \tilde{N}_{rt}, \tilde{N}_{ct})$ to be equivalent if there exist integers $0 \le i, j \le 6$ such that

$$\tilde{N}_{rc} = P^i N_{rc}, \quad \tilde{N}_{rt} = P^j N_{rt}, \quad \text{and} \quad \tilde{N}_{ct} = P^{j-i} N_{ct}.$$

Note that the C-matrices for treatments, rows, and columns of d are given by

$$\begin{aligned}
C_t &= 3^{-1}[(9I - N_{tr}N_{rt}) - (3N_{tc} - N_{tr}N_{rc})(9I - N_{cr}N_{rc})^-(3N_{ct} - N_{cr}N_{rt})], \\
C_r &= 3^{-1}[(9I - N_{rt}N_{tr}) - (3N_{rc} - N_{rt}N_{tc})(9I - N_{ct}N_{tc})^-(3N_{cr} - N_{ct}N_{tr})], \quad \text{and} \\
C_c &= 3^{-1}[(9I - N_{cr}N_{rc}) - (3N_{ct} - N_{cr}N_{rt})(9I - N_{tr}N_{rt})^-(3N_{tc} - N_{tr}N_{rc})].
\end{aligned}$$

From these representations (and observing that $PN = NP$ for all circular matrices N) it is easily seen that for equivalent designs the respective C-matrices for treatment, row, and column effects coincide. Note that two of the three incidence matrices of designs from the same equivalence class vary independently over the set of all possible incidence matrices.

The 80 feasible combinations we found can be broadly classified according to the corresponding C-matrices for treatments, rows, and columns. When viewing those C-matrices as equivalent which are obtained by interchanging certain rows and columns from a given one, then only the following 11 different C-matrices are obtained.

Table 1. C-matrices associated with feasible row-column designs.

i	$7\,C_i$
1	$((12.634\ \ -4.268\ \ -2.561\ \ 0.512\ \ 0.512\ \ -2.561\ \ -4.268))$
2	$((12\ \ -2\ \ -2\ \ -2\ \ -2\ \ -2\ \ -2))$
3	$((11.268\ \ -5.976\ \ -1.537\ \ 1.878\ \ 1.878\ \ -1.537\ \ -5.976))$
4	$((10.585\ \ -2.902\ \ -2.220\ \ -0.171\ \ -0.171\ \ -2.220\ \ -2.902))$
5	$((10.\ \ -2\ \ -3\ \ 0\ \ -0\ \ -3\ \ -2))$
6	$((8.537\ \ -3.073\ \ -3.244\ \ -2.049\ \ 2.049\ \ -3.244\ \ -3.073))$
7	$((8.195\ \ -2.390\ \ 1.366\ \ -3.073\ \ -3.073\ \ 1.366\ \ -2.390))$
8	$((8\ \ -4\ \ -2\ \ 0\ \ 0\ \ -2\ \ -4))$
9	$((7.512\ \ -3.415\ \ 0.683\ \ -1.024\ \ -1.024\ \ 0.683\ \ -3.415))$
10	$((6.828\ \ 0.341\ \ -1.195\ \ -2.561\ \ -2.561\ \ -1.195\ \ 0.341))$
11	$((6\ \ -1\ \ -1\ \ -1\ \ -1\ \ -1\ \ -1))$

The combinations of C-matrices for the three factors coming along with feasible designs are listed below. Here we use the notation (i, j, k) to indicate that for some feasible design the corresponding triplet of C-matrices (C_t, C_r, C_c) is either equal to (C_i, C_j, C_k) or to a permutation of the latter. For example, each of the combinations $(C_t, C_r, C_c) = (C_3, C_6, C_6)$, $(C_t, C_r, C_c) = (C_6, C_3, C_6)$, and $(C_t, C_r, C_c) = (C_6, C_6, C_3)$ was obtained three times, yielding the frequency 9 for the triplet (3,6,6) in the table. We remark that each of the 11 matrices C_i, $1 \le i \le 11$, from above appeared as the C-matrix for each of the factors.

Table 2. Combinations of C-matrices associated with feasible designs.

type	C-matrices	frequencies
I	(1,1,1)	3
II	(2,2,2)	2
III	(3,6,6)	9
IV	(4,5,5)	18
V	(7,8,9)	36
VI	(10,10,10)	6
VII	(11,11,11)	6

The designs d^A and d^{SS} are of type VII and type IV, respectively. Note that there are actually 49×6 designs of Agrawal, and 49×18 designs of Shah-Sinha type.

The designs of type I might be of particular interest, since the associated C-matrices possess the largest trace. The corresponding designs are generated by equal, *non*-BIBD row-column, row-treatment, and column-treatment incidence matrices, i.e.,

$$\begin{aligned} N_{rc} &= N_{rt} = N_{ct} = ((1\ 1\ 1\ 0\ 0\ 0\ 0))\,, \\ N_{rc} &= N_{rt} = N_{ct} = ((1\ 1\ 0\ 0\ 1\ 0\ 0))\,, \qquad \text{and} \\ N_{rc} &= N_{rt} = N_{ct} = ((1\ 0\ 1\ 0\ 1\ 0\ 0))\,. \end{aligned}$$

Table 2 shows that there are 98 3-way BIBD's of type II (located in two equivalence classes) having $C_2 = ((12\ -2\ -2\ -2\ -2\ -2\ -2))/7$ as common C-matrix for all three factors, which is twice the common C-matrix of Agrawal's 3-way BIBD. Two particular designs $d^\ell \simeq (N^\ell_{rc}, N^\ell_{rt}, N^\ell_{ct})$, $\ell = 1, 2$, representing the two associated equivalence classes are given by

$N^1_{rc} = ((1\ 1\ 0\ 1\ 0\ 0\ 0))\,,\ N^1_{rt} = ((1\ 1\ 0\ 0\ 0\ 1\ 0))\,,\ N^1_{ct} = ((1\ 0\ 0\ 0\ 1\ 1\ 0))\,,$
and
$N^2_{rc} = ((1\ 1\ 0\ 0\ 0\ 1\ 0))\,,\ N^2_{rt} = ((1\ 1\ 0\ 1\ 0\ 0\ 0))\,,\ N^2_{ct} = ((1\ 0\ 1\ 1\ 0\ 0\ 0))\,.$

Possible layouts of d^1 and d^2 are as follows.

$$d^1 \simeq \begin{bmatrix} 1 & 6 & - & 2 & - & - & - \\ - & 2 & 7 & - & 3 & - & - \\ - & - & 3 & 1 & - & 4 & - \\ - & - & - & 4 & 2 & - & 5 \\ 6 & - & - & - & 5 & 3 & - \\ - & 7 & - & - & - & 6 & 4 \\ 5 & - & 1 & - & - & - & 7 \end{bmatrix}, \quad d^2 \simeq \begin{bmatrix} 1 & 4 & - & - & - & 2 & - \\ - & 2 & 5 & - & - & - & 3 \\ 4 & - & 3 & 6 & - & - & - \\ - & 5 & - & 4 & 7 & - & - \\ - & - & 6 & - & 5 & 1 & - \\ - & - & - & 7 & - & 6 & 2 \\ 3 & - & - & - & 1 & - & 7 \end{bmatrix}.$$

Recall that the complete class of designs of type II is given by

$$\{\, d \simeq (P^i N^\ell_{rc}, P^j N^\ell_{rt}, P^{j-i} N^\ell_{ct}) : \ell = 1, 2,\ 0 \le i, j \le 6 \,\}\,.$$

Since C_2 is completely symmetric, we obtain by inspecting the traces of C_i, $1 \le i \le 11$, that all designs of type II are universally optimal for all three factors among all designs except those of type I. It may be noted that the designs of type I are based on non-BIBD structures of incidence matrices for all the three factor combinations.

Because $\mathrm{trace}(C_1) > \mathrm{trace}(C_2)$ the designs of type II fail to be Schur optimal within the set of all designs; actually, among all designs there does not exist a Schur optimal one. However, the positive eigenvalues of C_1 are $19.787/7\,(2)$, $16.904/7\,(2)$, $7.528/7\,(2)$ (the numbers in parenthesis denote the multiplicities), and the constant positive eigenvalue of C_2 is 2. Now straightforward analysis shows that for $p \ge 0$ the Φ_p-value of C_2 is smaller than that of C_1, and therefore all designs of type II are optimal among all designs w.r.t. Kiefer's Φ_p-criteria for all $p \geq 0$. These include the well known D-, A-, and E-criteria.

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