Statistica Sinica **17**(2007), 725-733

STOPPING AT THE LAST RENEWAL UP TO A TERMINAL TIME

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Abstract: Given a renewal process $\{S_n\}_{n\geq 1}$ and a fixed terminal time $T > 0$, we want to find a strategy that maximizes the probability of stopping, without recall, at the last renewal up to T. We give a simple criterion to guarantee that the optimal strategy is of threshold type. This simple criterion is applied to several cases, including a problem of optimal stopping on patterns discussed in Bruss and Louchard (2003).

Key words and phrases: Last renewal, monotone case, optimal stopping rule, renewal process.

1. Introduction

Suppose we hold some stocks and we want to sell them within a fixed time interval I. It is desirable to sell the stocks at a point in time when the price is the largest seen by then (we say that a record price appears). Therefore, when a record price appears, we may sell the stocks or hold them to wait for the next record price (at the risk of seeing no new record price). Best of all, we could sell the stocks at the point of time when the last record price appears (within the time interval I). Problems of selecting the last event up to some fixed time in a stochastic process have attracted several author's attention, for example Bruss (2000), Bruss and Paindaveine (2000), Hsiau and Yang (2002) and Bruss and Louchard (2003). These papers considered problems of selecting the last event in independent or Markovian Bernoulli sequences with finite horizon. In their results, the optimal strategies are of threshold type except for a curious solution of nonthreshold type in Hsiau and Yang (2002). Here we consider the same kind of problem in a renewal process.

Let $X_1, X_2, \ldots$ be a sequence of nonnegative independent random variables with a common distribution F satisfying $F(0) < 1$. Let $S_n = \sum_{i=1}^n X_i$ for $n \geq 1$. In the terminology of renewal theory, we call S_n the time of the nth renewal and $X_1, X_2, \ldots$ the interarrival times. Let $T > 0$ be a fixed terminal time. We want to find an optimal strategy that maximizes the probability of stopping, without recall, at the last renewal up to T in observing the sequence $S_1, S_2, \ldots$.

In Chow, Robbins and Siegmund (1971), many useful techniques for solving this kind of problem were developed. Using their terminology, our problem can be formulated as follows. For each $n \geq 1$, let $\mathcal{F}_n = \sigma(S_1, \ldots, S_n)$ be the σ-field generated by $S_1, \ldots, S_n$ and C the class of all finite stopping times adapted to $\{\mathcal{F}_n\}_{n \geq 1}$. We want to find a stopping time $\tau_T \in C$ such that $P(S_{\tau_T} \leq T < S_{\tau_T+1}) = \sup_{\tau \in C} P(S_\tau \leq T < S_{\tau+1})$.

For each $n \geq 1$, define the reward function

$$Y_n = E(I_{\{S_n \leq T < S_{n+1}\}} | \mathcal{F}_n), \tag{1.1}$$

the conditional probability of S_n being the last renewal up to T, given the observation $S_1, \ldots, S_n$. Here we note that Y_n is $\mathcal{F}_n$-measurable and, for any stopping time $\tau \in C$, we have

$$\begin{aligned}
E(Y_\tau) &= \sum_{n=1}^{\infty} E(Y_\tau | \tau = n) \cdot P(\tau = n) \\
&= \sum_{n=1}^{\infty} E(Y_n | \tau = n) \cdot P(\tau = n) \\
&= \sum_{n=1}^{\infty} E(I_{\{S_n \leq T < S_{n+1}\}} | \tau = n) \cdot P(\tau = n) \\
&= \sum_{n=1}^{\infty} E(I_{\{S_\tau \leq T < S_{\tau+1}\}} | \tau = n) \cdot P(\tau = n) \\
&= E(I_{\{S_\tau \leq T < S_{\tau+1}\}}) \\
&= P(S_\tau \leq T < S_{\tau+1}).
\end{aligned}$$

Therefore, the original problem is equivalent to finding a stopping time $\tau_T \in C$ such that $E(Y_{\tau_T}) = \sup_{\tau \in C} E(Y_\tau)$.

2. Monotone Cases and the First Threshold

Generally speaking, it is not easy to describe the optimal stopping rule explicitly. However, if we are in the so-called monotone case, a notion due to Chow and Robbins (1961), then the optimal stopping rule can be described in a better way. To begin put

$$A_n = \{E(Y_{n+1} | \mathcal{F}_n) \leq Y_n\}, \qquad n = 1, 2, \ldots. \tag{2.1}$$

We say that we are in the monotone case if

$$A_1 \subset A_2 \subset \ldots \quad \text{and} \quad P(\cup_{n=1}^{\infty} A_n) = 1. \tag{2.2}$$

When (2.2) holds, we have the important stopping rule

$$\tilde{\sigma} = \text{ first } n \geq 1 \text{ such that } Y_n \geq E(Y_{n+1}|\mathcal{F}_n). \tag{2.3}$$

Theorem A. *In the monotone case, if*

$$\liminf_n \int_{\{\tilde{\sigma}>n\}} Y_n^+ = 0 \tag{2.4}$$

holds, then $E(X_{\tilde{\sigma}}) \geq E(X_\tau)$ *for all* $\tau \in C$ *for which*

$$\liminf_n \int_{\{\tau>n\}} Y_n^- = 0. \tag{2.5}$$

In our problem, since $Y_n \geq 0$, (2.5) clearly holds for all $\tau \in C$. Moreover, since $|Y_n| \leq 1$ and $Y_n \to 0$ a.s. by the fact $S_n \to \infty$ a.s., we have $\lim_{n\to\infty} \int |Y_n| = 0$ by the Dominated Convergence Theorem, and this implies (2.4). Thus if we are in the monotone case, the stopping rule $\tilde{\sigma}$ at (2.3) is indeed an optimal stopping rule. Next we want to find the value of T such that we are in the monotone case.

In view of (1.1), for $0 \leq x \leq T$, on $\{S_n = x\}$ we have

$$\begin{aligned}
Y_n &= E(I_{\{S_n \leq T < S_{n+1}\}}|S_n = x) \\
&= E(I_{\{X_{n+1} > T-x\}}|S_n = x) \\
&= P(X_{n+1} > T - x) = 1 - F(T - x), \\
E(Y_{n+1}|\mathcal{F}_n) &= E(I_{\{S_{n+1} \leq T < S_{n+2}\}} \mid S_n = x) \\
&= P(X_{n+1} \leq T - x < X_{n+1} + X_{n+2}) \\
&= P(S_1 \leq T - x < S_2) \\
&= P(S_1 \leq T - x) - P(S_2 \leq T - x) \\
&= F(T - x) - F_2(T - x),
\end{aligned}$$

where $F_2(t) \equiv F * F(t)$, the convolution of two F's. For $x > T$, on $\{S_n = x\}$ we have $Y_n = E(Y_{n+1}|\mathcal{F}_n) = 0$. Therefore, in view of (2.1) and (2.2), we are in the monotone case if $1-F(T-x) \geq F(T-x)-F_2(T-x)$ holds for all $0 \leq x \leq T$. Let $H(t) = 1 - 2F(t) + F_2(t)$ for $t \geq 0$, then the above criterion is equivalent to that $H(t) \geq 0$ for all $0 \leq t < T$. Since $F(t)$ and $F_2(t)$ both are right continuous, so is $H(t)$. Moreover, $H(0) = 1-2F(0)+F_2(0) = 1-2F(0)+F^2(0) = [1-F(0)]^2 > 0$. Thus there exists a largest $T_1 > 0$ (T_1 may be ∞) such that $H(t) \geq 0$ holds for all $0 \leq t < T_1$. We call T_1 the *first threshold*.

From the above discussion, we see that if $T < T_1$, then we are in the monotone case and the stopping rule $\tilde{\sigma}$ defined by (2.3) is an optimal stopping rule, which tells us to stop at the first renewal, S_1. For $T \geq T_1$, we may not be in the monotone case, but we also can say something important. In fact, if $S_1, \ldots, S_n$ have been observed and $T - S_n < T_1$, then the optimal strategy is just to stop at S_n. This is because, given $S_1, \ldots, S_n$ satisfying $T - S_n < T_1$, the optimal strategy just depends on $T - S_n$, and at the moment we are in the monotone case. From now on, the statement "we are on the time t" means that $S_1, \ldots, S_n$ have been observed and $T - S_n = t$. With this convention, we can summarize the above discussion as follows.

Theorem 2.1. *If we are on the time t with $t < T_1$, then the optimal strategy is to stop at the present renewal.*

In the following, we need the notation $T_s = \sup\{\ t \mid F(t) < 1\ \}$. Note that if $T_s < \infty$ then $F(T_s) = 1$, since F is a right continuous function. Moreover, in this case it is clear that $F_2(T_s) < 1$ and thus $H(T_s) = 1 - 2F(T_s) + F_2(T_s) = -1 + F_2(T_s) < 0$, which implies that $T_s \geq T_1$ by the definition of T_1. Hence $T_s \geq T_1$ always holds whether $T_s < \infty$ or $T_s = \infty$.

Though the first threshold exists, the optimal stopping rule τ_T may not be of threshold type (see Example 3.1.). However, if $1 - F(t) < F(t) - F_2(t)$, i.e., $H(t) < 0$ holds whenever $T_s > t > T_1$, then the optimal stopping rule possesses the property: If we are on the time t with $t > T_1$, then the next renewal needs to be observed. This property is verified if we can find a better strategy than stopping at the present renewal. In fact, if $\infty > t \geq T_s$, observing the next renewal is better than stopping at the present one because the present renewal is not the last one up to T (by the definition of T_s). If $T_s > t > T_1$, stopping at the next renewal has the expected reward $F(t) - F_2(t)$, which is larger than the expected reward $1 - F(t)$ gained by stopping at the present renewal. As for the case that $t = T_1$, the optimal stopping rule depends on the value of $H(T_1)$. If $H(T_1) = 0$, then the optimal stopping rule is to stop at the present renewal when $t = T_1$ (see the Remark below Theorem 2.2); if $H(T_1) < 0$, then the next renewal has to be observed when $t = T_1$. Here we note that, by the definition of T_1, $H(T_1) > 0$ cannot occur since $H(t)$ is right continuous.

Theorem 2.2. *Let T_1 be the first threshold. Assume that $T_s = T_1$ or that $H(t) < 0$ for all $T_s > t > T_1$. If $H(T_1) = 0$, then the optimal stopping rule τ_T can be described as follows: Observe $S_1, S_2, \ldots$ sequentially until the first S_n satisfying $T - S_n \leq T_1$, then stop at this S_n; if $H(T_1) < 0$, then τ_T is the same as above except that $T - S_n \leq T_1$ should be replaced by $T - S_n < T_1$.*

Remark. If $H(T_1) = 0$, then $Y_n = E(Y_{n+1}|\mathcal{F}_n)$ on $\{T - S_n = T_1\}$. Therefore, if $H(T_1) = 0$ and we are on the time $t = T_1$, both stopping at the present renewal and stopping at the next renewal have the same expected reward.

In general, it is not easy to verify the assumption in Theorem 2.2, that is, either $T_s = T_1$ or $H(t) < 0$ for all $T_s > t > T_1$. Fortunately, we have a simple criterion to guarantee this assumption and such a criterion can be applied to many well-known interarrival time distributions (see Section 3). Before stating this criterion, we note that $1 - F(t) > 0$ for $T_s > t > 0$, by the definition of T_s.

Theorem 2.3. *If $(F(t) - F_2(t))/(1 - F(t))$ is nondecreasing in t for $T_s > t > 0$, then $T_1 = \sup\{\ T_s > t > 0 \mid (F(t) - F_2(t))/(1 - F(t)) \le 1\ \}$, and either $T_s = T_1$ or $H(t) < 0$ holds for all $T_s > t > T_1$.*

Proof. Since $T_s \ge T_1$, and $1 - F(t) > 0$ for $T_s > t > 0$, it is easy to see, by the definition of T_1 and the assumption of nondecreasing property of $(F(t) - F_2(t))/(1 - F(t))$, that

$$\begin{aligned} T_1 &= \sup\{\ T_s > t > 0 \mid 1 - F(x) \ge F(x) - F_2(x) \ \text{ for all } \ 0 \le x \le t\ \} \\ &= \sup\{\ T_s > t > 0 \mid \frac{F(x) - F_2(x)}{1 - F(x)} \le 1 \ \text{ for all } \ 0 \le x \le t\ \} \\ &= \sup\{\ T_s > t > 0 \mid \frac{F(t) - F_2(t)}{1 - F(t)} \le 1\ \}. \end{aligned}$$

If $T_s = T_1$, there is nothing to prove. If $T_s > T_1$, then we have $(F(T_1) - F_2(T_1))/(1 - F(T_1)) \ge 1$ since F and F_2 are right continuous and so is $(F - F_2)/(1 - F)$. However, $(F(t) - F_2(t))/(1 - F(t))$ is nondecreasing for $T_s > t > 0$, thus $(F(t) - F_2(t))/(1 - F(t)) > 1$ for all $T_s > t > T_1$, by the above third equality, and this implies that $H(t) < 0$ for all $T_s > t > T_1$.

3. Some Applications and an Example

In view of Theorems 2.2 and 2.3, we know that if $(F(t) - F_2(t))/(1 - F(t))$ is nondecreasing in t for $T_s > t > 0$, then the optimal stopping rule τ_T is of threshold type. In the following, we give several applications of this simple criterion.

Application 1. Let $0 < p < 1$, and consider the renewal process with interarrival times that are Geometric with parameter p, $F(t) = p\sum_{n=1}^{[t]}(1-p)^{n-1} = 1 - (1-p)^{[t]}$, $t \ge 0$, where $[t]$ denotes the greatest integer not greater than t. Then $T_s = \infty$ and $F_2(t) = F * F(t) = p^2\sum_{n=2}^{[t]}(n-1)(1-p)^{n-2} = 1 - [t](1-p)^{[t]-1} + ([t]-1)(1-p)^{[t]}$. Thus, for $t \ge 0$,

$$\frac{F(t) - F_2(t)}{1 - F(t)} = \frac{[t]((1-p)^{[t]-1} - (1-p)^{[t]})}{(1-p)^{[t]}} = \frac{p[t]}{1-p}.$$

It is clear that $p[t]/(1-p)$ is nondecreasing in t for $\infty > t > 0$. By Theorem 2.3,

$$T_1 = \sup\{\ \infty > t > 0 \mid \frac{F(t) - F_2(t)}{1 - F(t)} \le 1\}$$

$$
\begin{aligned}
&= \sup\{\ \infty > t > 0 \mid \frac{p[t]}{1-p} \le 1\} \\
&= \sup\{\ \infty > t > 0 \mid [t] \le \frac{1-p}{p}\} \\
&= [\frac{1-p}{p}] + 1 = [\frac{1}{p}]
\end{aligned}
$$

and we also have that $H(t) < 0$ for $\infty > t > [\frac{1}{p}]$. Since

$$
\frac{F([\frac{1}{p}]) - F_2([\frac{1}{p}])}{1 - F([\frac{1}{p}])} = \frac{p[\frac{1}{p}]}{1-p} > 1,
$$

we have $H([\frac{1}{p}]) < 0$. Thus, by Theorem 2.2, the optimal strategy is to observe $S_1, S_2, \dots$ until the first S_n satisfying $T - S_n < T_1$, then stop at this S_n.

Application 2. Let $\lambda > 0$ and consider the renewal process with interarrival times that are exponential with parameter λ, $F(t) = 1 - e^{-\lambda t}$, $t \ge 0$. Then $T_s = \infty$ and $F_2(t) = F * F(t) = 1 - e^{-\lambda t} - \lambda t e^{-\lambda t}$, $t \ge 0$. Thus

$$
\frac{F(t) - F_2(t)}{1 - F(t)} = \frac{\lambda t e^{-\lambda t}}{e^{-\lambda t}} = \lambda t.
$$

Since λt is increasing for $\infty > t > 0$, we have by Theorem 2.3,

$$
T_1 = \sup\{\ t \mid \frac{F(t) - F_2(t)}{1 - F(t)} \le 1\ \} = \sup\{\ t \mid \lambda t \le 1\ \} = \frac{1}{\lambda}
$$

and $H(t) < 0$ for $\infty > t > 1/\lambda$. Since $H(1/\lambda) = 0$, we have, by Theorem 2.2, the optimal strategy is to observe $S_1, S_2, \dots$ until the first S_n satisfying $T - S_n \le 1/\lambda$, then stop at this S_n.

Remark. The arguments used in the above two examples also work well (but require some modification) for renewal processes with interarrival times that are Gamma, Negative binomial, Binomial and Poisson. In other words, for such renewal processes the optimal strategies are all of threshold type. Here we remind the readers that in the above two examples the thresholds happen to be about $E(X_1)$.

The last application is to a problem discussed in Bruss and Louchard (2003). Our version is a modified one.

Application 3. Given an alphabet and a fixed uncorrelated pattern $H = H_1 H_2 \cdots H_l$, we observe sequentially the outcome $Y_1, \dots, Y_n$ of $n > l$ draws from the alphabet. Whenever H is achieved, we are allowed either to stop or to continue observing. What strategy maximizes the probability of stopping on the very last appearance of H up to Y_n? Here $Y_1, \ldots, Y_n$ are assumed to be independent with the same distribution over the alphabet. The uncorrelated pattern $H = H_1H_2 \cdots H_l$ means that either $l = 1$ or $l \geq 2$ with the property that, for each k, $1 \leq k \leq l$, $H_1H_2 \cdots H_{l-k+1}$ does not coincide with $H_kH_{k+1} \cdots H_l$.

To embed the above problem to our model, we take X_1 to be the first i that makes $Y_{i-l+1}Y_{i-l+2} \cdots Y_i$ coincide with H, that is, X_1 is the time when the pattern H first appears. Similarly, let X_2 be the interarrival time between the first and the second appearances of the pattern H; $X_3, X_4, \ldots$ are defined sequentially in the same way. Since $Y_1, Y_2, \ldots$ is an i.i.d. sequence and the pattern H is uncorrelated, $X_1, X_2, \ldots$ is also a positive i.i.d. sequence. Now the original problem can be reformulated as: Find an optimal strategy to maximize the probability of stopping at the last renewal in $S_1, S_2, \ldots$ up to $T = n$.

Let $P_i = P(X_1 = i)$, $i \geq 0$, and F be the distribution of X_1. It is clear that $P_0 = P_1 = \cdots = P_{l-1} = 0$. Let $P_l = p < 1$. Since $H = H_1H_2 \cdots H_l$ is uncorrelated, for $i > l$,

$$\begin{aligned}
P_i &= P(H \text{ does not appear in } Y_1Y_2 \cdots Y_{i-l} \text{ and } Y_{i-l+1}Y_{i-l+2} \cdots Y_i = H) \\
&= P(H \text{ does not appear in } Y_1Y_2 \cdots Y_{i-l}) \cdot P(Y_{i-l+1}Y_{i-l+2} \cdots Y_i = H) \\
&= [1 - F(i-l)] \cdot P_l = p\ [1 - F(i-l)]. \qquad (3.1)
\end{aligned}$$

Next, let $F_2 = F * F$. Then $F_2(i) = \sum_{j=0}^{i} P_jF(i-j)$ for $i \geq 0$, and thus

$$\begin{aligned}
F_2(i+1) - F_2(i) &= \sum_{j=0}^{i+1} P_j \cdot F(i+1-j) - \sum_{j=0}^{i} P_j \cdot F(i-j) \\
&= \sum_{j=0}^{i} P_j[F(i+1-j) - F(i-j)] + P_{i+1}F(0) \\
&= \sum_{j=0}^{i} P_j \cdot P_{i+1-j} \qquad (\text{ since } F(0) = P_0 = 0) \\
&= \sum_{j=0}^{i+1-l} P_j \cdot P_{i+1-j} \qquad (\text{ since } P_0 = P_1 = \cdots = P_{l-1} = 0) \\
&= \sum_{j=0}^{i+1-l} P_j \cdot p \cdot [1 - F(i+1-j-l)] \qquad (\text{Using (3.1) }) \\
&= p\ \{\sum_{j=0}^{i+1-l} P_j - \sum_{j=0}^{i+1-l} P_j \cdot F(i+1-j-l)\} \\
&= p\ [F(i+1-l) - F_2(i+1-l)]. \qquad (3.2)
\end{aligned}$$

Now we are ready to show that $(F(i)-F_2(i))/(1-F(i))$ is nondecreasing in i. Since

$$\begin{aligned}&\frac{F(i+1)-F_2(i+1)}{1-F(i+1)}-\frac{F(i)-F_2(i)}{1-F(i)}\\&\quad=\frac{[F(i+1)-F(i)][1-F_2(i)]-[F_2(i+1)-F_2(i)][1-F(i)]}{[1-F(i)][1-F(i+1)]},\end{aligned}$$

we see that $(F(i+1)-F_2(i+1))/(1-F(i+1))\ge(F(i)-F_2(i))/(1-F(i))$ if and only if

$$[F(i+1)-F(i)][1-F_2(i)]\ge[F_2(i+1)-F_2(i)][1-F(i)]. \tag{3.3}$$

Using (3.1) and (3.2), (3.3) becomes

$$\begin{aligned}&p\,[1-F(i+1-l)][1-F_2(i)]\ge p\,[F(i+1-l)-F_2(i+1-l)][1-F(i)]\\&\iff\frac{1-F_2(i)}{1-F(i)}\ge\frac{F(i+1-l)-F_2(i+1-l)}{1-F(i+1-l)}\\&\iff 1+\frac{F(i)-F_2(i)}{1-F(i)}\ge\frac{F(i+1-l)-F_2(i+1-l)}{1-F(i+1-l)}.\end{aligned}$$

But $(F(i)-F_2(i))/(1-F(i))=0$ for $i=1,\dots,l-1$, and $(F(l)-F_2(l))/(1-F(l))=p/(1-p)>0$. By induction, $(F(i+1)-F_2(i+1))/(1-F(i+1))\ge(F(i)-F_2(i))/(1-F(i))$ for all $i\ge0$. Hence, the optimal strategy is of threshold type.

Remark. In Bruss and Louchard (2003), $P(O_i=1)$ and $P(O_i=0)$ correspond to our $F(i)-F_2(i)$ and $1-F(i)$, respectively. Their Lemma 3.6 says, implicitly that $(F(i)-F_2(i))/(1-F(i))$ is nondecreasing in i; in fact, both their proof and ours use the same argument with small changes.

Finally, we construct a simple example in which the optimal strategy is not of threshold type.

Example 3.1. Let $P(X_1=i)=p_i$ with $P_1=3/5$, $P_2=1/60$, $P_3=1-P_1-P_2=23/60$. It is clear that if we are on the time $t\ge3$, then the optimal strategy is to observe the next renewal. If we are on the time $t=0$, the optimal strategy is, no doubt, to stop at the present renewal. If we are on the time $t=1$, the expected rewards for stopping at the present renewal and the next renewal are $1-P_1=2/5$ and $P_1=3/5$, respectively; thus, the optimal strategy is to stop at the next renewal. If we are on the time $t=2$, then stopping at the present renewal gets the expected reward $1-P_1-P_2=23/60$, while observing the next renewal and then adopting the optimal strategy will get the expected reward $P_1\cdot P_1+P_2=113/300<23/60$; thus, the optimal strategy is to stop at the present renewal. Hence, the optimal strategy is not of threshold type.

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(Received November 2004; accepted October 2005)