Statistica Sinica **14**(2004), 591-601

CRITERION-ROBUST OPTIMAL DESIGNS FOR MODEL DISCRIMINATION AND PARAMETER ESTIMATION: MULTIVARIATE POLYNOMIAL REGRESSION CASE

Min-Hsiao Tsai and Mei-Mei Zen

National Cheng-Kung University

Abstract: Consider the problem of discriminating between two polynomial regression models on the q-cube $[-1,1]^q$, $q \geq 2$, and estimating parameters in the models. To find designs which are efficient for both model discrimination and parameter estimation, Zen and Tsai (2002) proposed a multiple-objective optimality criterion for the univariate case. In this work, taking the same M_γ-criterion which uses weight γ $(0 \leq \gamma \leq 1)$ for model discrimination and $1-\gamma$ for parameter estimation, the corresponding M_γ-optimal product design is investigated. Based on the maximin principle on the M_γ-efficiency of any $M_{\gamma'}$-optimal product design, a criterion-robust optimal product design is proposed.

Key words and phrases: Efficiency, M_γ-criterion, multiple-objective, product design.

1. Introduction

The study of optimal designs for multivariate polynomial regression models has received considerable attention; see Kiefer (1961), Kôno (1962), Farrell, Kiefer and Walbran (1967), Lim and Studden (1988), Rafajlowicz and Myszka (1988, 1992), Dette (1994), Wong (1994), Gaffke and Heiligers (1995) and Dette and Röder (1996) for details. Most of the works were done under the assumption that the degree of the underlying model is given. But, in many practical situations, experimenters are uncertain about the degree. Dette and Röder (1997) considered multifactor experiments and proposed a Φ_p-optimality criterion (weighted p-mean of D-efficiencies) to construct optimal designs for model discrimination. They claimed that the optimal discrimination designs also have high efficiency for statistical analysis in the identified model. In this study, we seek nonsequential experimental designs which take both model discrimination and parameter estimation into consideration.

Consider the multivariate polynomial regression model

$$\begin{aligned} E(Y_{\boldsymbol{x}}) &= \theta_0 + \sum_{i=1}^{q} \theta_i x_i + \sum_{1 \leq i_1 \leq i_2 \leq q} \theta_{i_1,i_2} x_{i_1} x_{i_2} + \cdots + \sum_{1 \leq i_1 \leq \cdots \leq i_l \leq q} \theta_{i_1,\ldots,i_l} \prod_{j=1}^{l} x_{i_j} \\ &= \boldsymbol{\theta}_l^\top \boldsymbol{f}_l(\boldsymbol{x}), \end{aligned}$$

where $\boldsymbol{\theta}_l$ is the vector of parameters, $\boldsymbol{f}_l(\boldsymbol{x})$ is the vector of the $N_{q,l} = (q+l)!/(q!\,l!)$ monomials $\prod_{i=1}^{q} x_i^{m_i}$ with $\sum_{i=1}^{q} m_i \leq l$, and $l \in \mathbb{N}$ denotes the degree of the model with $x \in \mathcal{X} = [-1,1]^q$. We tackle the problem of discriminating between

$$\text{Model } A : l = k-1 \quad \text{and} \quad \text{Model } B : l = k,$$

and consider parameter estimation simultaneously. For the univariate case, Tsai and Zen (2003) have extended their results with l in Model A replaced by $k-r$, $r \geq 1$, as considered by Studden (1982a) and Song and Wong (1999). In this study, an approximate design η is a probability measure on the q-cube $[-1,1]^q$ and its performance depends on the information matrix

$$M_l(\eta) = \int_{\mathcal{X}} \boldsymbol{f}_l(\boldsymbol{x}) \boldsymbol{f}_l^{\top}(\boldsymbol{x}) d\eta(\boldsymbol{x}). \tag{1.1}$$

Discriminating between Model A and Model B is equivalent to testing the hypothesis $H_0 : \theta_{1,\ldots,1,1} = \theta_{1,\ldots,1,2} = \cdots = \theta_{q,\ldots,q,q} = 0$. Note that the number of parameters under H_0 equals $N_{q,k} - N_{q,k-1} = N_{q-1,k}$. As in Lim and Studden (1988), the D_s-optimal design η_s maximizes the objective function

$$\phi_s(\eta) = \left(\frac{|M_k(\eta)|}{|M_{k-1}(\eta)|} \right)^{1/s}, \tag{1.2}$$

where $s = N_{q-1,k}$ and $|\cdot|$ denotes the determinant. To make inferences in Model A or B, applying the usual D-criterion, the corresponding optimal design, η_A or η_B, maximizes the objective function

$$\psi_l(\eta) = |M_l(\eta)|^{1/N_{q,l}}. \tag{1.3}$$

To serve the three objectives described in Pukelsheim and Rosenberger (1993), Zen and Tsai (2002) proposed a weighted optimality criterion and derived criterion-robust optimal designs. Herein, we use the same criterion for multivariate polynomial regression models. Considering all three objectives simultaneously, a multiple-objective optimal design is defined to maximize the weighted geometric average of $\phi_s(\eta)$ in (1.2) and $\psi_{k-1}(\eta)$, $\psi_k(\eta)$ in (1.3). Let

$$\begin{aligned}\psi_M(\eta) &= \psi_{k-1}^{\alpha}(\eta)\, \psi_k^{\beta}(\eta)\, \phi_s^{\gamma}(\eta) \\ &= |M_{k-1}(\eta)|^{\frac{\alpha}{N_{q,k-1}} - \frac{\gamma}{N_{q-1,k}}} \, |M_k(\eta)|^{\frac{\beta}{N_{q,k}} + \frac{\gamma}{N_{q-1,k}}}, \end{aligned} \tag{1.4}$$

where $0 \leq \alpha, \beta, \gamma \leq 1$ and $\alpha + \beta + \gamma = 1$. Since there is no information about which model is appropriate before model discrimination, it seems reasonable to put $\alpha = \beta$ and reduce (1.4) to

$$\begin{aligned}\psi_{M_\gamma}(\eta) &= \Big[\psi_{k-1}(\eta)\, \psi_k(\eta)\Big]^{\frac{1-\gamma}{2}} \, [\phi_s(\eta)]^{\gamma} \\ &= |M_{k-1}(\eta)|^{\frac{1-\gamma}{2N_{q,k-1}} - \frac{\gamma}{N_{q-1,k}}} \, |M_k(\eta)|^{\frac{1-\gamma}{2N_{q,k}} + \frac{\gamma}{N_{q-1,k}}}. \end{aligned} \tag{1.5}$$

Thus the design η_{M_γ} which maximizes (1.5) is called the M_γ-optimal design. Note that (1.4) can be usefully generalized to a wider setup with different values of α and β. For instance, $\alpha = 2\beta$ suggests that the 'smaller' model is more likely to be the true model; the techniques are similar. To gain further insights into properties of our designs, we give some numerical results in Section 4.

As pointed out by Lim and Studden (1988), the D- and D_s-optimal designs for multivariate polynomial regression models can be obtained numerically only for small q and l, and the algorithms fail to converge if q is too large. Thus they restricted the optimization to the class of all product designs and claim that there is not much loss in efficiency. We also restrict the optimization to the class of all product designs on $[-1,1]^q$; see Dette and Röder (1997) for details.

The paper is organized as follows. In Section 2, an M_γ-optimal product design is expressed by the canonical moments of its common factor. Since different selection criteria result in different optimal designs, an appropriate selection criterion is important for the problem itself. In Section 3, we derive the minimum M_γ-efficiency of arbitrary $M_{\gamma'}$-optimal product designs and investigate the behavior of these minimum values. It turns out that the maximum value of the minimum M_γ-efficiency of any $M_{\gamma'}$-optimal product design occurs at $\gamma' = \gamma^*$. Based on the maximin principle, a criterion-robust optimal product design $\eta_{M_{\gamma^*}}$ is derived for any degree k. For practical use, the support points and weights of the common factor of $\eta_{M_{\gamma^*}}$ with minimum M_γ-efficiency will be given for $k \le 5$. In Section 4, we make a comparison of efficiencies among some special optimal designs. Also, a comparison with the optimal discrimination designs proposed by Dette and Röder (1997) is made.

2. The M_γ-Optimal Product Design

In this section, we derive the M_γ-optimal product designs. Let $\eta = \xi_1 \times \cdots \times \xi_q$ be a product design on $[-1,1]^q$ and let $p_1^{(j)}, p_2^{(j)}, \ldots$ denote the canonical moments of ξ_j. The theory of canonical moments was first introduced by Skibinsky (1967), then widely applied by Studden (1980, 1982a, 1982b, 1989) to determine optimal designs for polynomial regression models. For more details, the readers can refer to Skibinsky (1986) and Dette and Studden (1997). From Lemma 5.1 in Lim and Studden (1988), direct computation gives that the determinant in (1.1) can be expressed as

$$|M_l(\eta)| = \prod_{j=1}^{q} \prod_{i=1}^{l} \left(2^2 q_{2i-2}^{(j)}\, p_{2i-1}^{(j)}\, q_{2i-1}^{(j)}\, p_{2i}^{(j)}\right)^{N_{q,l-i}}, \tag{2.1}$$

where $q_0^{(j)} = 1$ and $q_i^{(j)} = 1 - p_i^{(j)}$. Substituting (2.1) into (1.3) and (1.5), we have the following results.

Lemma 2.1. *For any product design* η*, the objective functions in* (1.3) *and* (1.5) *can be expressed in terms of the canonical moments of each factor* ξ_j *as*

$$\psi_l(\eta) = \prod_{j=1}^{q} \left[p_{2l}^{(j)} \prod_{i=1}^{l} \left(2^2 p_{2i-1}^{(j)} q_{2i-1}^{(j)}\right)^{N_{q,l-i}} \prod_{i=1}^{l-1} \left(p_{2i}^{(j)}\right)^{N_{q,l-i}} \left(q_{2i}^{(j)}\right)^{N_{q,l-(i+1)}} \right]^{\frac{1}{N_{q,l}}},$$

$$\begin{aligned}\psi_{M_\gamma}(\eta) = \prod_{j=1}^{q} \Bigg[\left(p_{2k}^{(j)}\right)^{B_{q,k}(\gamma)} & \prod_{i=1}^{k} \left(2^2 p_{2i-1}^{(j)} q_{2i-1}^{(j)}\right)^{B_{q,k}(\gamma)N_{q,k-i}+A_{q,k}(\gamma)N_{q,k-(i+1)}} \\ & \times \prod_{i=1}^{k-1} \left(p_{2i}^{(j)}\right)^{B_{q,k}(\gamma)N_{q,k-i}+A_{q,k}(\gamma)N_{q,k-(i+1)}} \\ & \quad \times \left(q_{2i}^{(j)}\right)^{B_{q,k}(\gamma)N_{q,k-(i+1)}+A_{q,k}(\gamma)N_{q,k-(i+2)}} \Bigg],\end{aligned}$$

where $B_{q,k}(\gamma) = \frac{q+(q+2k)\gamma}{2(q+k)N_{q-1,k}}$, $A_{q,k}(\gamma) = \frac{q-(q+2k)\gamma}{2(q+k-1)N_{q-1,k-1}}$, *and* $N_{q,-1} = 0$, $\forall q \in \mathbb{N}$.

Lemma 2.2. *For any* $q \in \mathbb{N}$ *and* $k \geq 2$*, we have*

(i) *the canonical moments of each factor of the* D*-optimal product design, for model with degree* l*, satisfy* $p_{2i+1} = 1/2$, $p_{2i} = [q+(l-i)]/[q+2(l-i)]$, $i \leq l-1$, *and* $p_{2l} = 1$*;*

(ii) *the canonical moments of each factor of the* M_γ*-optimal product design satisfy* $p_{2i+1} = 1/2$,

$$p_{2i} = \frac{B_{q,k}(\gamma)N_{q,k-i} + A_{q,k}(\gamma)N_{q,k-(i+1)}}{B_{q,k}(\gamma)[N_{q,k-i}+N_{q,k-(i+1)}] + A_{q,k}(\gamma)[N_{q,k-(i+1)}+N_{q,k-(i+2)}]}, \quad (2.2)$$

$i \leq k-1$, *and* $p_{2k} = 1$.

The proof of (i) was given in Lim and Studden (1988); part (ii) follows directly from Dette (1994) and (1.5). The termination of canonical moments at 1 shows that each factor is uniquely determined with a finite number of support points; see Skibinsky (1986). Moreover, all canonical moments of odd order in Theorem 2.2 are 1/2, which means that the corresponding product design is symmetric about the original point; see Lau (1983).

Proposition 2.3. p_{2i} *in* (2.2) *is strictly decreasing in* γ, $i \leq k-1$.

Proof. Direct differentiation gives

$$\frac{d}{d\gamma}p_{2i} = \frac{(N_{q,k-(i+1)}^2 - N_{q,k-i}N_{q,k-(i+2)})(B_{q,k}(\gamma)A'_{q,k}(\gamma) - A_{q,k}(\gamma)B'_{q,k}(\gamma))}{(B_{q,k}(\gamma)[N_{q,k-i}+N_{q,k-(i+1)}] + A_{q,k}(\gamma)[N_{q,k-(i+1)}+N_{q,k-(i+2)}])^2}.$$

The assertion then follows from

$$N_{q,k-(i+1)}^2 - N_{q,k-i}N_{q,k-(i+2)} = \frac{q\,N_{q,k-(i+1)}^2}{(k-i)(q+k-i-1)} > 0,$$
$$B_{q,k}(\gamma)A'_{q,k}(\gamma) - A_{q,k}(\gamma)B'_{q,k}(\gamma) = \frac{-(q+2k)}{2qN_{q,k}N_{q,k-1}} < 0.$$

To explicitly determine the optimal product designs in Theorem 2.2, it is essential to find the support points and weights of the common factor ξ, which can be found in Studden (1982a).

3. Criterion-Robust Optimal Product Design

3.1. The efficiency of the M_γ-optimal product design

As discussed in Zen and Tsai (2002), the performance of any $M_{\gamma'}$-optimal product design $\eta_{M_{\gamma'}}$ under the M_γ-criterion is evaluated by the M_γ-efficiency defined by

$$M_\gamma\text{-eff}(\eta_{M_{\gamma'}}) = \frac{\psi_{M_\gamma}(\eta_{M_{\gamma'}})}{\psi_{M_\gamma}(\eta_{M_\gamma})}\,,\ \gamma,\ \gamma' \in [0,1].$$

Lemma 2.1 and Theorem 2.2 give the following result directly.

Theorem 3.1. *For any fixed $\gamma' \in [0,1]$, $q \in \mathbb{N}$ and $k \geq 2$, let $\eta_{M_{\gamma'}}$ be the $M_{\gamma'}$-optimal product design. Then the M_γ-efficiency of $\eta_{M_{\gamma'}}$ for any $\gamma \in [0,1]$ can be expressed as*

$$M_\gamma\text{-eff}(\eta_{M_{\gamma'}}) = \left[\prod_{i=1}^{k-1}\left(\frac{p_{2i}^{[\gamma']}}{p_{2i}^{[\gamma]}}\right)^{B_{q,k}(\gamma)N_{q,k-i}+A_{q,k}(\gamma)N_{q,k-(i+1)}} \times \left(\frac{q_{2i}^{[\gamma']}}{q_{2i}^{[\gamma]}}\right)^{B_{q,k}(\gamma)N_{q,k-(i+1)}+A_{q,k}(\gamma)N_{q,k-(i+2)}}\right]^q, \tag{3.1}$$

where $p_{2i}^{[\cdot]}$ is as in (2.2) and $q_{2i}^{[\cdot]} = 1 - p_{2i}^{[\cdot]}$.

3.2. Minimum M_γ-efficiency of $M_{\gamma'}$-optimal product design

Using arguments similar to those of Zen and Tsai (2002), we have the following results; the details can be found in Tsai and Zen (2002).

Lemma 3.2. *For any $q \in \mathbb{N}$ and $k \geq 2$,*

(i) *given $\gamma' \in [0,1]$, M_γ-eff$(\eta_{M_{\gamma'}})$ is strictly increasing on $[0,\gamma')$ and decreasing on $(\gamma',1]$;*

(ii) *given* $\gamma \in [0,1]$, M_γ-eff$(\eta_{M_{\gamma'}})$ *is strictly increasing on* $[0,\gamma)$ *and decreasing on* $(\gamma,1]$.

Lemma 3.2 (i) shows that, for any given γ', the minimum value of M_γ-eff$(\eta_{M_{\gamma'}})$ will be attained at either $\gamma = 0$ or 1. To determine which of these applies, let $h(\gamma') = M_1\text{-eff}(\eta_{M_{\gamma'}})/M_0\text{-eff}(\eta_{M_{\gamma'}})$ and plug in (3.1) to get

$$h(\gamma') = \left[\prod_{i=1}^{k-1}\left(\frac{p_{2i}^{[\gamma']}}{p_{2i}^{[0]}}\right)^{iN_{q-1,k-i}}\left(\frac{q_{2i}^{[\gamma']}}{q_{2i}^{[0]}}\right)^{(i+1)N_{q-1,k-(i+1)}}\right]^{(q+2k)/(2kN_{q,k})} \times \left[\prod_{i=1}^{k-1}\left(\frac{p_{2i}^{[0]}}{p_{2i}^{[1]}}\right)^{N_{q-1,k-i}}\left(\frac{q_{2i}^{[0]}}{q_{2i}^{[1]}}\right)^{N_{q-1,k-(i+1)}}\right]^{q/N_{q-1,k}}. \tag{3.2}$$

It can be shown that $h(\gamma')$ is strictly increasing in γ', and we have the following result.

Theorem 3.3. *For any fixed* $\gamma' \in [0,\, 1]$, $q \in \mathbb{N}$ *and* $k \geq 2$, *we have*

$$\min_{0\le\gamma\le1}\left\{M_\gamma\text{-eff}(\eta_{M_{\gamma'}})\right\} = \begin{cases} M_0\text{-eff}(\eta_{M_{\gamma'}})\ ,\ \textit{if } \gamma' \geq \gamma^*, \\ M_1\text{-eff}(\eta_{M_{\gamma'}})\ ,\ \textit{if } \gamma' < \gamma^*, \end{cases}$$

where M_0-eff$(\eta_{M_{\gamma'}})$ *and* M_1-eff$(\eta_{M_{\gamma'}})$ *are as in* (3.1) *with* $\gamma = 0$ *and* 1, *respectively, and* γ^* *is the root of* $h(\gamma') = 1$ *in* (3.2).

To find a criterion-robust optimal design, the maximum value of $\min_{0\le\gamma\le1}\{M_\gamma\text{-eff}(\eta_{M_{\gamma'}})\}$ plays an important role. From Lemma 3.2 (ii), we have the fact that M_0-eff$(\eta_{M_{\gamma'}})$ is strictly decreasing in γ' and M_1-eff$(\eta_{M_{\gamma'}})$ is strictly increasing in γ', which gives the main result.

Theorem 3.4. *For any* $q \in \mathbb{N}$ *and* $k \geq 2$, $\min_{0\le\gamma\le1}\{M_\gamma\text{-eff}(\eta_{M_{\gamma'}})\}$ *first increases, then decreases in* γ', *and the maximum value of* $\min_{0\le\gamma\le1}\{M_\gamma\text{-eff}(\eta_{M_{\gamma'}})\}$ *is attained at* $\gamma' = \gamma^* = \arg\{\max_{0\le\gamma'\le1}\min_{0\le\gamma\le1} M_\gamma\text{-eff}(\eta_{M_{\gamma'}})\}$, *where* γ^* *is the root of* $h(\gamma') = 1$ *in* (3.2).

To describe the main theorem graphically, Figure 1 shows the plots of $\min_{0\le\gamma\le1}\{M_\gamma\text{-eff}(\eta_{M_{\gamma'}})\}$ versus γ' for some q and k; the corresponding values of γ^* are 0.39649 and 0.39736, respectively. For practical use, Table 1 gives some numerical results on γ^* and the corresponding minimum efficiencies of $\eta_{M_{\gamma^*}}$, $\eta_{M_{1/3}}$ and $\eta_{M_{1/2}}$ for various q and k, where $\eta_{M_{1/3}}$ and $\eta_{M_{1/2}}$ are chosen due to the fact that γ^* varies between 1/3 to 1/2. The minimum efficiency, $\min_{0\le\gamma\le1}\{M_\gamma\text{-eff}(\eta_{M_{\gamma^*}})\}$, is increasing in k and greater than 0.9748 for $q \geq 2$. That means that for any M_γ-criterion, $\eta_{M_{\gamma^*}}$ is very robust in the sense of having high minimum efficiency. Therefore, it can serve as a criterion-robust optimal design for the problem of model discrimination and parameter estimation.

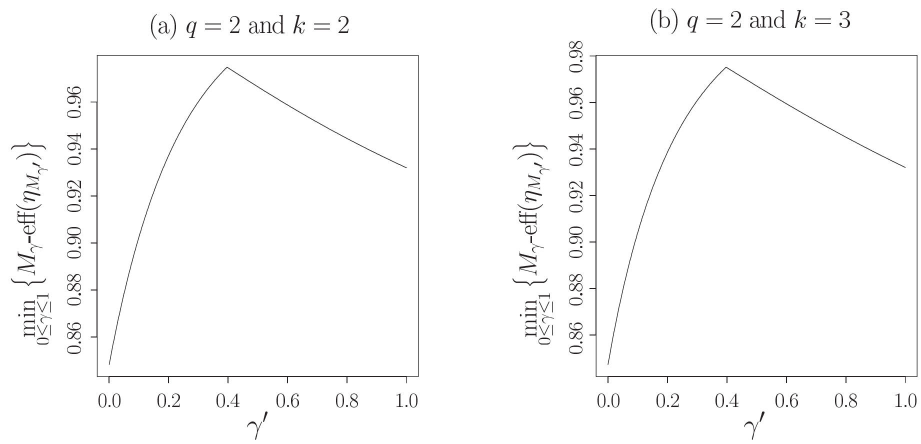

Figure 1. Plots of $\min_{0\leq\gamma\leq 1}\{M_\gamma\text{-eff}(\eta_{M_{\gamma'}})\}$ vs. γ'.

Table 1. The values of γ^* and minimum efficiencies for small q and k.

q	k	γ^*	$\min_{0\leq\gamma\leq 1} M_\gamma\text{-eff}(\eta_{M_{\gamma^*}})$	$\min_{0\leq\gamma\leq 1} M_\gamma\text{-eff}(\eta_{M_{1/3}})$	$\min_{0\leq\gamma\leq 1} M_\gamma\text{-eff}(\eta_{M_{1/2}})$
1	2	0.3678	0.9526	0.9433	0.9346
	3	0.3566	0.9522	0.9460	0.9336
	4	0.3457	0.9551	0.9520	0.9372
	5	0.3362	0.9581	0.9575	0.9411
2	2	0.3965	0.9748	0.9659	0.9665
	3	0.3974	0.9752	0.9664	0.9671
	4	0.3961	0.9774	0.9696	0.9701
	5	0.3944	0.9796	0.9728	0.9729
3	2	0.4097	0.9839	0.9771	0.9791
	3	0.4170	0.9841	0.9769	0.9797
	4	0.4204	0.9855	0.9787	0.9816
	5	0.4223	0.9869	0.9806	0.9834
4	2	0.4174	0.9886	0.9832	0.9853
	3	0.4283	0.9886	0.9827	0.9857
	4	0.4340	0.9894	0.9838	0.9870
	5	0.4374	0.9903	0.9850	0.9882

Note that the value of γ^* depends on both q and k and requires numerical solution. To describe how the value γ^* varies, Figure 2 shows the plot of γ^* versus k under different q. It is noted that γ^* is decreasing in k for $q = 1$ and 2, and increasing in k for $q = 3$ and 4. Moreover, for any fixed k, γ^* is increasing in q.

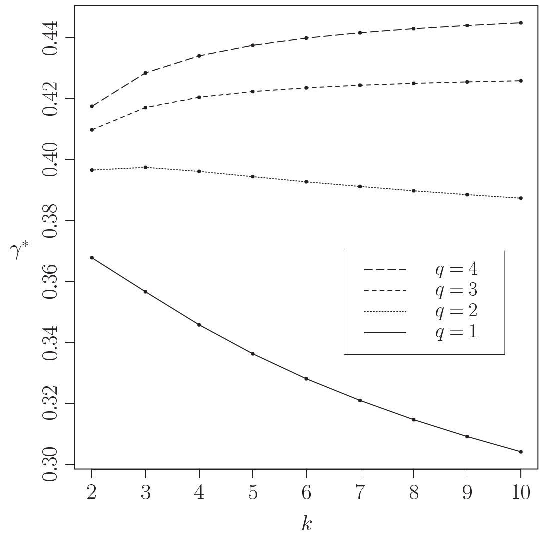

Figure 2. The plot of γ^* vs. k.

Since different selection criteria result in different optimal designs, under the M_{γ^*}-criterion, from Theorem 2.2 (ii) we can derive the canonical moments of the common factor of $\eta_{M_{\gamma^*}}$ precisely. Table 2 gives the support points and weights of the common factor ξ_{γ^*}.

Table 2. The support points and weights of the factor ξ_{γ^*}, $\eta_{M_{\gamma^*}} = \xi_{\gamma^*} \times \cdots \times \xi_{\gamma^*}$.

k	$q = 2$	$q = 3$	$q = 4$
2	$\begin{pmatrix} \pm 1 & 0 \\ 0.3693 & 0.2614 \end{pmatrix}$	$\begin{pmatrix} \pm 1 & 0 \\ 0.4011 & 0.1978 \end{pmatrix}$	$\begin{pmatrix} \pm 1 & 0 \\ 0.4203 & 0.1594 \end{pmatrix}$
3	$\begin{pmatrix} \pm 1 & \pm 0.4236 \\ 0.2866 & 0.2134 \end{pmatrix}$	$\begin{pmatrix} \pm 1 & \pm 0.3847 \\ 0.3280 & 0.1720 \end{pmatrix}$	$\begin{pmatrix} \pm 1 & \pm 0.3554 \\ 0.3558 & 0.1442 \end{pmatrix}$
4	$\begin{pmatrix} \pm 1 & \pm 0.6330 & 0 \\ 0.2336 & 0.1769 & 0.1790 \end{pmatrix}$	$\begin{pmatrix} \pm 1 & \pm 0.5897 & 0 \\ 0.2767 & 0.1485 & 0.1496 \end{pmatrix}$	$\begin{pmatrix} \pm 1 & \pm 0.5549 & 0 \\ 0.3076 & 0.1281 & 0.1286 \end{pmatrix}$
5	$\begin{pmatrix} \pm 1 & \pm 0.7477 & \pm 0.2773 \\ 0.1971 & 0.1503 & 0.1526 \end{pmatrix}$	$\begin{pmatrix} \pm 1 & \pm 0.7087 & \pm 0.2578 \\ 0.2391 & 0.1298 & 0.1311 \end{pmatrix}$	$\begin{pmatrix} \pm 1 & \pm 0.6758 & \pm 0.2422 \\ 0.2708 & 0.1142 & 0.1150 \end{pmatrix}$

4. Comparison with Some Special Designs

In this section, to make a comparison of the M_{γ^*}-optimal product design $\eta_{M_{\gamma^*}}$ with some special optimal designs, we investigate the efficiencies of designs η_s, η_A, η_B, η_{M_0}, $\eta_{M_{1/3}}$, $\eta_{M_{1/2}}$ and $\eta_{M_{\gamma^*}}$ defined in Section 1. Figure 3 shows the plots of the M_{γ^*}-, D_s-, D_B- and D_A-efficiencies of the these seven designs versus various k with $q = 2$.

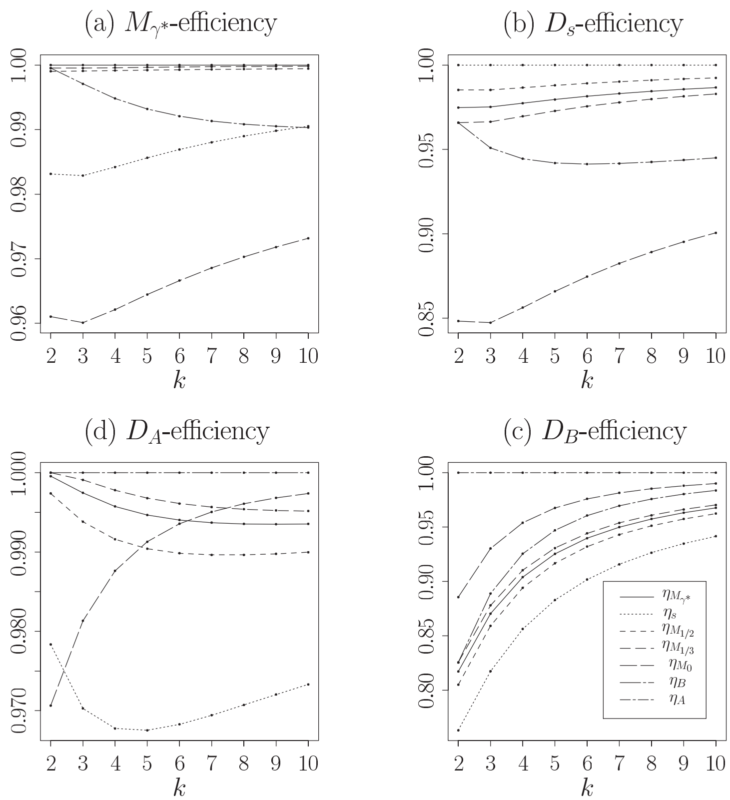

Figure 3. The M_{γ^*}-, D_s-, D_B- and D_A-efficiencies of designs $\eta_{M_{\gamma^*}}$, η_s, $\eta_{M_{1/2}}$, $\eta_{M_{1/3}}$, η_{M_0}, η_B and η_A.

To go further, we consider the Φ_p-optimal discrimination designs with $p = 0$ mentioned in Tables 3 and 4 in Dette and Röder (1997) and denoted by η_{DR}. Note that η_{DR} treats the regression models of degree $1, \ldots, k$ uniformly, and is said to have high efficiency for parameter estimation. Let $\eta_{M_{\gamma^*}}$ denote the M_{γ^*}-optimal product design defined in Section 3 with $\alpha = \beta$ in (1.4), and $\eta_{M_{\gamma^{**}}}$ denote the corresponding optimal product design with $\alpha = 2\beta$. Table 3 gives the D_s-efficiency (for model discrimination) and D_A-, D_B-efficiencies (for parameter estimation) of η_s, η_B, η_A, $\eta_{M_{\gamma^*}}$, $\eta_{M_{\gamma^{**}}}$ and η_{DR} for small q and k. It can be seen that $\eta_{M_{\gamma^*}}$, $\eta_{M_{\gamma^{**}}}$ and η_{DR} typically perform well with respect to three criteria even though they are not optimal individually. Moreover, the proposed designs $\eta_{M_{\gamma^*}}$ and $\eta_{M_{\gamma^{**}}}$ have higher D_s- and D_B-efficiencies than η_{DR}, with a lower D_A-efficiency. It is noted that our new design and the D_B-optimal design behave similarly because the exponent of the lower order determinant is close to 0.

Table 3. The efficiencies of some special designs.

		$q=2$			$q=3$		
k	Design	D_s-eff	D_B-eff	D_A-eff	D_s-eff	D_B-eff	D_A-eff
2	η_s	1	0.97835	0.76314	1	0.98955	0.80593
	η_B	0.96585	1	0.82548	0.98534	1	0.84590
	η_A	0	0	1	0	0	1
	$\eta_{M_{\gamma^*}}$	0.97481	0.99955	0.81716	0.98394	0.99998	0.84766
	$\eta_{M_{\gamma^{**}}}$	0.97197	0.99980	0.81996	0.98209	0.99988	0.84985
	η_{DR}	0.93728	0.99711	0.84574	0.95629	0.99431	0.87220
3	η_s	1	0.97028	0.81740	1	0.98388	0.85912
	η_B	0.95091	1	0.88889	0.97687	1	0.90852
	η_A	0	0	1	0	0	1
	$\eta_{M_{\gamma^*}}$	0.97519	0.99746	0.87038	0.98412	0.99952	0.90096
	$\eta_{M_{\gamma^{**}}}$	0.97272	0.99803	0.87268	0.98267	0.99970	0.90262
	η_{DR}	0.91058	0.99535	0.90786	0.92992	0.98986	0.93514
4	η_s	1	0.96776	0.85627	1	0.98121	0.89310
	η_B	0.94447	1	0.92548	0.97252	1	0.94274
	η_A	0	0	1	0	0	1
	$\eta_{M_{\gamma^*}}$	0.97741	0.99575	0.90395	0.98549	0.99859	0.93112
	$\eta_{M_{\gamma^{**}}}$	0.97534	0.99639	0.90579	0.98437	0.99886	0.93235
	η_{DR}	0.89978	0.99436	0.94017	0.91578	0.98683	0.96360

Acknowledgement

The authors would like to thank the referees for their helpful comments; this research was partially supported by NSC90-2118-M006-009 from the National Science Council, R.O.C.

References

Dette, H. (1994). Robust designs for multivariate polynomial regression on the d-cube. *J. Statist. Plann. Inference* **38**, 105-124.

Dette, H. and Röder, I. (1996). Optimal product designs for multivariate regression with missing terms. *Scand. J. Statist.* **23**, 195-208.

Dette, H. and Röder, I. (1997). Optimal discrimination designs for multifactor experiments. *Ann. Statist.* **25**, 1161-1175.

Dette, H. and Studden, W. J. (1997). *The Theory of Canonical Moments with Applications in Statistics, Probability, and Analysis.* Wiley, New York.

Farrell, R. H., Kiefer, J. and Walbran, J. (1967). Optimum multivariate designs. *Proc. Fifth Berkeley Symp. Math. Statist. Probab.* **1**, 113-138. Univ. California Press.

Gaffke, N. and Heiligers, B. (1995). Computing optimal approximate invariant designs for cubic regression on multidimensional balls and cubes. *J. Statist. Plann. Inference* **47**, 347-376.

Kiefer, J. (1961). Optimum designs in regression problems, II. *Ann. Math. Statist.* **32**, 298-325.

Kôno, K. (1962). Optimal designs for quadratic regression on the k-cube. *Mem. Fac. Sci. Kyushu Univ. Ser. A* **16**, 114-122.

Lau, T. S. (1983). Theory of canonical moments and its application in polynomial regression, I and II. Technical Reports 83-23 and 83-24, Department of Statistics, Purdue University.

Lim, Y. B. and Studden, W. J. (1988). Efficient D_s-optimal designs for multivariate polynomial regression on the q-cube. *Ann. Statist.* **16**, 1225-1240.

Pukelsheim, F. and Rosenberger, J. L. (1993). Experimental designs for model discrimination. *J. Amer. Statist. Assoc.* **88**, 642-649.

Rafajlowicz, E. and Myszka, W. (1988). Optimum experimental design for a regression on a hypercube - a generalization of Hoel's result. *Ann. Inst. Statist. Math.* **40**, 821-828.

Rafajlowicz, E. and Myszka, W. (1992). When product type optimal design is optimal? Brief survey and new results. *Metrika* **39**, 321-333.

Skibinsky, M. (1967). The range of the $(n+1)$-th moment for distributions on $[0,1]$. *J. Appl. Probab.* **4**, 543-552.

Skibinsky, M. (1986). Principal representations and canonical moment sequences for distribution on an interval. *J. Math. Anal. Appl.* **120**, 95-120.

Song, D. and Wong, W. K. (1999). On the construction of G_{rm}-optimal designs. *Statist. Sinica* **9**, 263-272.

Studden, W. J. (1980). D_s-optimal designs for polynomial regression using continued fractions. *Ann. Statist.* **8**, 1132-1141.

Studden, W. J. (1982a). Some robust-type D-optimal designs in polynomial regression. *J. Amer. Statist. Assoc.* **77**, 916-921.

Studden, W. J. (1982b). Optimal designs for weighted polynomial regression using canonical moments. In *Statistical Decision Theory and Related Topics III* **2** (Edited by S. S. Gupta and J. O. Berger), 335-350. Academic, New York.

Studden, W. J. (1989). Note on some Φ_p-optimal designs for polynomial regression. *Ann. Statist.* **17**, 618-623.

Tsai, M. H. and Zen, M. M. (2002). Criterion-robust optimal designs for model discrimination and parameter estimation: multivariate polynomial regression case. Technical Report No. 45, Department of Statistics, National Cheng-Kung University.

Tsai, M. H. and Zen, M. M. (2003). Criterion-robust optimal designs for model discrimination and parameter estimation when the degree difference of two polynomial regression is arbitray. Technical Report No. 49, Department of Statistics, National Cheng-Kung University.

Wong, W. K. (1994). G-optimal designs for multi-factor experiments. *J. Statist. Plann. Inference* **40**, 127-133.

Zen, M. M. and Tsai, M. H. (2002). Some criterion-robust optimal designs for the dual problem of model discrimination and parameter estimation. *Sankhyā Ser. B* **64**, 322-338.

Department of Statistics, National Cheng-Kung University, Tainan 70101, Taiwan, R.O.C.

E-mail: mhtsai@ibm.stat.ncku.edu.tw

Department of Statistics, National Cheng-Kung University, Tainan 70101, Taiwan, R.O.C.

E-mail: mmzen@ibm.stat.ncku.edu.tw

(Received October 2002; accepted March 2003)