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INCAPABILITY INDEX WITH ASYMMETRIC TOLERANCES

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Abstract: Greenwich and Jahr-Schaffrath (1995) introduced an index C_{pp}, a simple transformation of the index C^*_{pm}, which provides an uncontaminated separation between information concerning the process accuracy and process precision. Unfortunately, the index C_{pp} inconsistently measures process capability in many cases and thus reflects process potential and performance inaccurately. In this paper, we consider a generalization, C''_{pp}, to handle processes with asymmetric tolerances. The generalization is shown to be superior to the original index C_{pp}. In addition, we investigate the statistical properties of a natural estimator of C''_{pp} assuming the process is normally distributed.

Key words and phrases: Asymmetric tolerances, estimation, incapability index.

1. Introduction

Numerous process capability indices have been proposed to provide a unitless measure on whether a process is capable of producing items meeting the quality requirement preset by the product designer. The most commonly used measures of process capability indices are C_p and C_{pk} (see Kane (1986)). As noted by Boyles (1991), C_p and C_{pk} are yield-based indices which are independent of the target T, which may fail to account for process centering (the ability to cluster around the target). Chan, Cheng, and Spiring (1988) developed an index called C_{pm} which takes into account the target value. This index is defined as the following:

$$C_{pm} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}} = \frac{d}{3\sqrt{\sigma^2 + (\mu - T)^2}},$$

where μ is the process mean and σ is the process standard deviation, T is the target value, $d = (USL - LSL)/2$, USL is the upper specification limit, and LSL is the lower specification limit.

The index C_{pm} is also generalized to C^*_{pm} to handle processes with asymmetric tolerances. The generalization C^*_{pm} is defined as (Chan, Cheng, and Spiring (1988)):

$$C^*_{pm} = \frac{\min(D_l, D_u)}{3\sqrt{\sigma^2 + (\mu - T)^2}} = \frac{d^*}{3\sqrt{\sigma^2 + (\mu - T)^2}},$$

where $D_l = T - LSL$, $D_u = USL - T$, $d^* = \min(D_l, D_u)$.

Greenwich and Jahr-Schaffrath (1995) considered a simple transformation of the index C_{pm}^* called C_{pp} which was defined as:

$$C_{pp} = (\frac{\mu - T}{D})^2 + (\frac{\sigma}{D})^2,$$

where $D = d^*/3$. Greenwich and Jahr-Schaffrath (1995) defined the inaccuracy index $C_{ia} = (\mu - T)^2/D^2$ and imprecision index $C_{ip} = \sigma^2/D^2$. Thus, $C_{pp} = C_{ia} + C_{ip}$.

For C_{pp}, consider the following example with asymmetric tolerance (LSL, T, USL), where $T = \{3(USL) + (LSL)\}/4$ and $\sigma = d/3$. Then for processes A and B with $\mu_A = T - d/2 = m$ (the midpoint of the specification interval) and $\mu_B = T + d/2 = USL$, both have the index value of $C_{pp} = 13$ and equal degree of clustering around the target (as $|\mu - T| = d/2$ for both processes A and B). However, the expected proportions non-conforming are approximately 0.27% for process A and 50% for process B. Clearly, C_{pp} inconsistently measures process capability in this case and is inappropriate for those with asymmetric tolerances. This problem calls for a need to generalize the index C_{pp} to cover cases with asymmetric tolerances so that appropriate use of the incapability index can be continued.

2. New Incapability Index C''_{pp}

In this section, we consider a new generalization of C_{pp} to handle processes with asymmetric tolerances. We refer to this generalization as C''_{pp}, which may be defined as:

$$C''_{pp} = (\frac{A}{D})^2 + (\frac{\sigma}{D})^2,$$

where $A = \max\{(\mu - T)d/D_u, (T - \mu)d/D_l\}$. Let $(A/D)^2$ be denoted by C''_{ia}. Then $C''_{pp} = C''_{ia} + C_{ip}$. If the tolerances are symmetric $(T = m)$, then $A = |\mu - T|$, C''_{ia} reduces to the index C_{ia} and $C''_{pp} = C_{pp}$.

In developing the generalization we have replaced the term $|\mu - T|$ in C_{pp} by A. This ensures that the new index obtains the minimal value at $\mu = T$ regardless of whether the tolerances are symmetric or asymmetric. For processes with asymmetric tolerances, the corresponding loss function is also asymmetric in T. We take into account the asymmetry of the loss function by adding the factors d/D_u and $-d/D_l$ to $\mu - T$ according to whether μ is greater or less than T. The factors d/D_u and $-d/D_l$ ensure that if processes A and B with $\mu_A > T$ and $\mu_B < T$ satisfy $(\mu_A - T)/D_u = (T - \mu_B)/D_l$, then the index values given to A and B are the same. It is easy to verify that if the process is on target, then $C''_{pp} = C_{ip} = (\sigma/D)^2$ is the minimum value.

Table 1. A comparison between C_{pp} and C''_{pp} (with fixed $\sigma = d/4$).

μ	C_{pp}	C_{ia}	C_{ip}	C''_{pp}	C''_{ia}
LSL	83.25	81.00	2.25	38.25	36.00
$T - 1.45d$	77.94	75.69	2.25	35.89	33.64
$T - 1.40d$	72.81	70.56	2.25	33.61	31.36
$T - 1.35d$	67.86	65.61	2.25	31.41	29.16
$T - 1.30d$	63.09	60.84	2.25	29.29	27.04
$T - 1.25d$	58.50	56.25	2.25	27.25	25.00
$T - 1.20d$	54.09	51.84	2.25	25.29	23.04
$T - 1.15d$	49.86	47.61	2.25	23.41	21.16
$T - 1.10d$	45.81	43.56	2.25	21.61	19.36
$T - 1.05d$	41.94	39.69	2.25	19.89	17.64
$T - 1.00d$	38.25	36.00	2.25	18.25	16.00
$T - 0.95d$	34.74	32.49	2.25	16.69	14.44
$T - 0.90d$	31.41	29.16	2.25	15.21	12.96
$T - 0.85d$	28.26	26.01	2.25	13.81	11.56
$T - 0.80d$	25.29	23.04	2.25	12.49	10.24
$T - 0.75d$	22.50	20.25	2.25	11.25	9.00
$T - 0.70d$	19.89	17.64	2.25	10.09	7.84
$T - 0.65d$	17.46	15.21	2.25	9.01	6.76
$T - 0.60d$	15.21	12.96	2.25	8.01	5.76
$T - 0.55d$	13.14	10.89	2.25	7.09	4.84
$T - 0.50d$	11.25	9.00	2.25	6.25	4.00
$T - 0.45d$	9.54	7.29	2.25	5.49	3.24
$T - 0.40d$	8.01	5.76	2.25	4.81	2.56
$T - 0.35d$	6.66	4.41	2.25	4.21	1.96
$T - 0.30d$	5.49	3.24	2.25	3.69	1.44
$T - 0.25d$	4.50	2.25	2.25	3.25	1.00
$T - 0.20d$	3.69	1.44	2.25	2.89	0.64
$T - 0.15d$	3.06	0.81	2.25	2.61	0.36
$T - 0.10d$	2.61	0.36	2.25	2.41	0.16
$T - 0.05d$	2.34	0.09	2.25	2.29	0.04
T	2.25	0.00	2.25	2.25	0.00
$T + 0.05d$	2.34	0.09	2.25	2.61	0.36
$T + 0.10d$	2.61	0.36	2.25	3.69	1.44
$T + 0.15d$	3.06	0.81	2.25	5.49	3.24
$T + 0.20d$	3.69	1.44	2.25	8.01	5.76
$T + 0.25d$	4.50	2.25	2.25	11.25	9.00
$T + 0.30d$	5.49	3.24	2.25	15.21	12.96
$T + 0.35d$	6.66	4.41	2.25	19.89	17.64
$T + 0.40d$	8.01	5.76	2.25	25.29	23.04
$T + 0.45d$	9.54	7.29	2.25	31.41	29.16
USL	11.25	9.00	2.25	38.25	36.00

3. Comparisons

In this section, the generalization C''_{pp} is compared with the original index C_{pp} in terms of a process characteristic considered by Choi and Owen (1990). We consider the following example with manufacturing specifications $LSL = T - 1.50d$, $USL = T + 0.50d$.

Table 1 displays the values of C_{pp}, C_{ia}, C_{ip}, C''_{pp}, and C''_{ia} for various values of μ, with fixed $\sigma = d/4$. We note that C_{pp} and C_{ia} have the minimum value at the target. But their values at the upper specification limit (say, when the expected proportions non-conforming is 50%) are equal to those at the mid-point m. These indices, being symmetric about the target value, do not take into account the location of the process mean. On the other hand, our new index takes into account the location of the process mean for asymmetric tolerances. Thus, given two processes A and B with $\mu_A > T$ and $\mu_B < T$ satisfying $(\mu_A - T) = (T - \mu_B)$ and $D_l > D_u$, B has significantly higher yield than A, so the (new) incapability index value of A is greater than the index value of B. For example, in Table 1 $C''_{pp} = 38.25$ for $\mu_A = T + 0.5d$ and $C''_{pp} = 6.25$ for $\mu_B = T - 0.5d$. The two process means have equal distance from the target, but B has significantly higher yield than A, so intuitively A should score higher than B. Therefore, we conclude that the proposed new incapability index C''_{pp} is superior to the original index C_{pp}.

4. Estimation of C''_{pp}

We treat the case when the characteristic of the process is normally distributed. Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with mean μ and variance σ^2 measuring the characteristic under investigation. To estimate the new incapability index C''_{pp}, we consider the natural estimator which can be defined as follows:

$$\hat{C}''_{pp} = \frac{(\hat{A})^2}{D^2} + \frac{S_n^2}{D^2},$$

where $\hat{A} = \max\{d(\bar{X} - T)/D_u, d(T - \bar{X})/D_l\}$, the mean μ is estimated by the sample mean, $\bar{X} = (n^{-1})\sum_{i=1}^n X_i$, and the variance σ^2 by the maximum likelihood estimator, $S_n^2 = (n^{-1})\sum_{i=1}^n (X_i - \bar{X})^2$. For the case where the production tolerance is symmetric, $\hat{A}$ may be simplified as $|\bar{X} - T|$. Thus, the estimator $\hat{C}''_{pp}$ reduces to $\hat{C}_{pp} = (n^{-1}D^{-2})\sum_{i=1}^n (X_i - T)^2$, the natural estimator of C_{pp} discussed by Greenwich and Jahr-Schaffrath (1995). Consequently, we may view the estimator $\hat{C}''_{pp}$ as a direct extension of $\hat{C}_{pp}$. To derive the rth moment of $\hat{C}''_{pp}$ we use reasoning inspired by Vännman (1995), who derived the expected value for the estimators of superstructure capability indices. The derivations are given in the appendix from which we have the rth moment of $\hat{C}''_{pp}$ as:

$$E(\hat{C}''_{pp})^r = \Big(\frac{\sigma^2}{nD^2}\Big)^r \frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty}(P_j) \times 2^r \times \Gamma\Big(\frac{n+j}{2}+r\Big)$$
$$\times \Bigg\{\sum_{i=0}^{r}\binom{r}{i}\frac{\Gamma([(1+j)/2]+i)}{\Gamma([(n+j)/2]+i)}\Big(\Big(d_u^2-1\Big)^i+(-1)^j\Big(d_l^2-1\Big)^i\Big)\Bigg\},$$

where $\lambda=\delta^2$, $\delta=\sqrt{n}(\mu-T)/\sigma$, $P_j=(\sqrt{2}\delta)^j/(j!)$, $d_u=d/D_u$, and $d_l=d/D_l$. In particular,

$$E(\hat{C}''_{pp}) = \Big(\frac{(n-1)\sigma^2}{nD^2}\Big)+\Big(\frac{\sigma^2}{nD^2}\Big)\frac{e^{-\lambda/2}}{2\sqrt{\pi}}\sum_{j=0}^{\infty}(P_j)\Gamma\Big(\frac{1+j}{2}\Big)(1+j)\Big(d_u^2+(-1)^j d_l^2\Big),$$

$$\mathrm{Var}\,(\hat{C}''_{pp}) = \Big(\frac{\sigma^4}{n^2D^4}\Big)\frac{e^{-\lambda/2}}{2\sqrt{\pi}}\sum_{j=0}^{\infty}(P_j)\Gamma\Big(\frac{1+j}{2}\Big)(1+j)(3+j)\Big(d_u^4+(-1)^j d_l^4\Big)$$
$$-\Big\{\Big(\frac{\sigma^2}{nD^2}\Big)\frac{e^{-\lambda/2}}{2\sqrt{\pi}}\sum_{j=0}^{\infty}(P_j)\Gamma\Big(\frac{1+j}{2}\Big)(1+j)\Big(d_u^2+(-1)^j d_l^2\Big)\Big\}^2$$
$$+\Big(\frac{2(n-1)\sigma^4}{n^2D^4}\Big).$$

We note that the estimator $\hat{C}''_{pp}$ is biased. The bias of $\hat{C}''_{pp}$ is $B_{pp}=E(\hat{C}''_{pp})-C''_{pp}$, and the mean squared error, which is more relevant to the analysis of process quality, is $\mathrm{MSE}(\hat{C}''_{pp})=\mathrm{Var}(\hat{C}''_{pp})+(B_{pp})^2$. To explore the behavior of the estimator $\hat{C}''_{pp}$, the bias and the mean squared error were calculated using *Maple* (a computer software) for various values of $a=(\mu-T)/\sigma$, $b=\sigma/D$, d_u, d_l, and sample size n. For example, Table 2 displays the bias and MSE of $\hat{C}''_{pp}$ for $a=-1.0(0.5)1.0$, $b=1$, $du=5/4$, $d_l=5/6$, and $n=10(10)50$.

Table 2. The bias and MSE of $\hat{C}''_{pp}$ for $a=-1.0(0.5)1.0$, $b=1$, $du=5/4$, $d_l=5/6$, and $n=10(10)50$.

	$a=1.0$		$a=0.5$		$a=0.0$		$a=-0.5$		$a=-1.0$	
n	bias	MSE	bias	MSE	bias	MSE	bias	MSE	bias	MSE
10	0.056	2.060	0.055	0.549	0.013	0.211	−0.029	0.238	−0.031	0.383
20	0.028	1.399	0.028	0.289	0.006	0.103	−0.015	0.122	−0.015	0.194
30	0.019	1.182	0.019	0.207	0.004	0.068	−0.010	0.082	−0.010	0.130
40	0.014	1.074	0.014	0.166	0.003	0.051	−0.008	0.061	−0.008	0.098
50	0.011	1.010	0.011	0.142	0.003	0.040	−0.006	0.049	−0.006	0.078

The results in Table 2 indicate that as $|a|$ increases, the bias and the mean squared error also increase. Further, as the sample size increases, the bias and the mean squared error decrease. The bias of $\hat{C}''_{pp}$ (vs. n) are plotted in Figure 1 with $a = -1.0(1.0)1.0$ (from bottom to top in the plot).

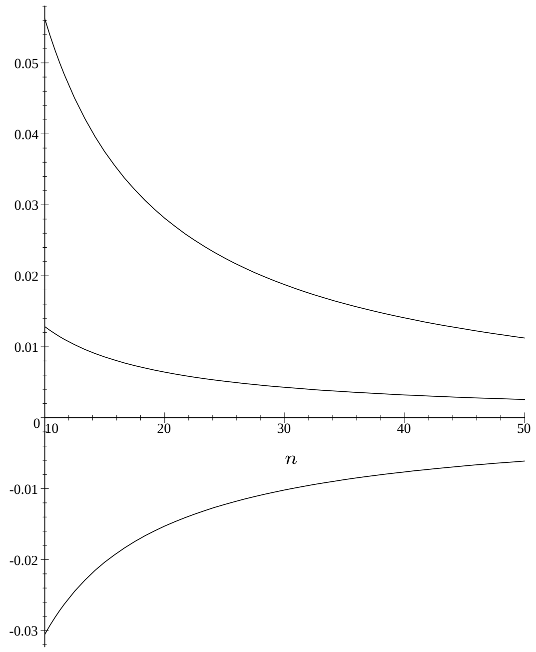

Figure 1. The bias of $\hat{C}''_{pp}$ (vs. n) as plotted with $a = -1.0(1.0)1.0$ (from bottom to top in the plot).

For the case where the production tolerance is symmetric, since $d_u = d_l = 1$, $\hat{C}''_{pp}$ is an unbiased estimator of C''_{pp} ($B_{pp} = 0$). The unbiased estimator depends only on the complete, sufficient statistic $(\bar{X}, S_n^2)$; it follows that $\hat{C}''_{pp}$ is a uniformly minimum variance unbiased estimator of C''_{pp}. In addition, we have the rth moment of $\hat{C}''_{pp}$ as

$$E(\hat{C}''_{pp})^r = E(\hat{C}_{pp})^r = \Big(\frac{\sigma^2}{nD^2}\Big)^r \sum_{j=0}^{\infty} \Big(\frac{e^{-(\lambda/2)}(\lambda/2)^j}{j!}\Big) \times \Big(\frac{2^r\Gamma[(n/2)+j+r]}{\Gamma[(n/2)+j]}\Big).$$

To estimate the new inaccuracy index C''_{ia}, we consider the natural estimator $\hat{C}''_{ia} = (\hat{A})^2/(D)^2$. For the case where the production tolerance is symmetric, $\hat{A}$ may be simplified as $|\bar{X}-T|$ and the estimator $\hat{C}''_{ia}$ reduces to $\hat{C}_{ia} = (\bar{X}-T)^2/D^2$, the natural estimator of C_{ia} discussed by Greenwich and Jahr-Schaffrath (1995). The rth moment about zero for $\hat{C}''_{ia}$ is:

$$E(\hat{C}''_{ia})^r = \Big(\frac{\sigma^2}{nD^2}\Big)^r \frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty}(P_j) \times 2^r \times \Gamma(\frac{1+j}{2}+r)(d_u^{2r}+(-1)^j d_l^{2r}).$$

In particular,

$$E(\hat{C}''_{ia}) = \Big(\frac{\sigma^2}{nD^2}\Big)\frac{e^{-\lambda/2}}{2\sqrt{\pi}}\sum_{j=0}^{\infty}(P_j)\Gamma(\frac{1+j}{2})(1+j)(d_u^2+(-1)^j d_l^2).$$

$$\mathrm{Var}\,(\hat{C}''_{ia}) = \Big(\frac{\sigma^4}{n^2D^4}\Big)\frac{e^{-\lambda/2}}{2\sqrt{\pi}}\sum_{j=0}^{\infty}(P_j)\Gamma(\frac{1+j}{2})(1+j)(3+j)(d_u^4+(-1)^j d_l^4).$$

$$-\left\{\Big(\frac{\sigma^2}{nD^2}\Big)\frac{e^{-\lambda/2}}{2\sqrt{\pi}}\sum_{j=0}^{\infty}(P_j)\Gamma(\frac{1+j}{2})(1+j)(d_u^2+(-1)^j d_l^2)\right\}^2.$$

Table 3 displays the bias and MSE of $\hat{C}''_{ia}$ for $a = -1.0(0.5)1.0$, $b = 1$, $d_u = 5/4$, $d_l = 5/6$, and $n = 10(10)50$.

The results in Table 3 indicate that as $|a|$ increases, the mean squared error also increases. Further, as the sample size increases, the bias and the mean squared error decrease. The bias of $\hat{C}''_{ia}$ (vs. n) are plotted in Figure 2 with $a = -1.0(1.0)1.0$ (from bottom to top in the plot).

In the case where the production tolerance is symmetric, we have:

$$E(\hat{C}''_{ia})^r = E(\hat{C}_{ia})^r = \Big(\frac{\sigma^2}{nD^2}\Big)^r\sum_{j=0}^{\infty}\Big(\frac{e^{-(\lambda/2)}(\lambda/2)^j}{j!}\Big)\times\Big(\frac{2^r\Gamma[(1/2)+j+r]}{\Gamma[(1/2)+j]}\Big).$$

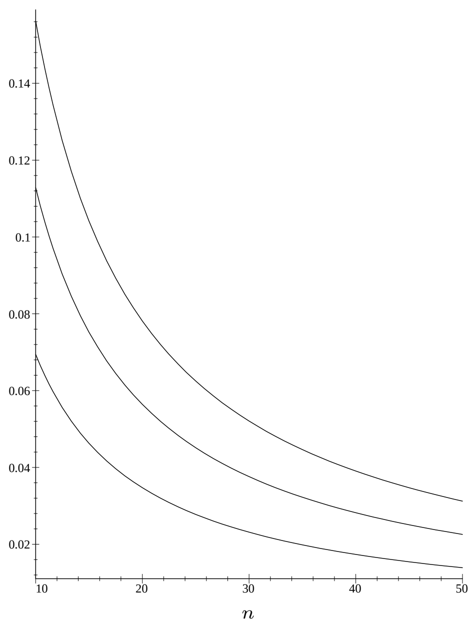

Figure 2. The bias of $\hat{C}''_{ia}$ (vs. n) as plotted with $a = -1.0(1.0)1.0$ (from bottom to top in the plot).

Table 3. The bias and MSE of $\hat{C}''_{ia}$ for $a = -1.0(0.5)1.0$, $b = 1$, $d_u = 5/4$, $d_l = 5/6$ and $n = 10(10)50$.

	$a = 1.0$		$a = 0.5$		$a = 0.0$		$a = -0.5$		$a = -1.0$	
n	bias	MSE	bias	MSE	bias	MSE	bias	MSE	bias	MSE
10	0.156	1.050	0.155	0.294	0.113	0.044	0.071	0.063	0.069	0.207
20	0.078	0.507	0.078	0.134	0.056	0.011	0.035	0.028	0.035	0.100
30	0.052	0.334	0.052	0.087	0.038	0.005	0.023	0.018	0.023	0.066
40	0.039	0.249	0.039	0.064	0.028	0.003	0.017	0.013	0.017	0.049
50	0.031	0.198	0.031	0.051	0.023	0.002	0.014	0.010	0.014	0.039

The index C_{ip} reflects the process imprecision (process variation). Greenwich and Jahr-Schaffrath (1995) considered an unbiased estimator of C_{ip} which can be defined as $\hat{C}_{ip} = S^2/D^2$.

On the assumption of normality, $\hat{C}_{ip}$ is distributed as $[\sigma^2/(n-1)D^2]$ times a chi-square variable with $(n-1)$ degrees of freedom. The rth moment about zero for $\hat{C}_{ip}$ is:

$$E(\hat{C}_{ip})^r = \Big(\frac{\sigma^2}{(n-1)D^2}\Big)^r \times \Big(\frac{2^r\Gamma[r+(n-1)/2]}{\Gamma[(n-1)/2]}\Big).$$

In particular, $E(\hat{C}_{ip}) = C_{ip}$ and $\mathrm{Var}\,(\hat{C}_{ip}) = (2\sigma^4)/[(n-1)D^4]$.

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Appendix

Theorem 1. *Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$, $Y = \max^2\{d_u Z, -d_l Z\}$ where $Z = (n)^{1/2}(\bar{X} - T)/\sigma$ is distributed as $N(\delta, 1)$ and $\delta = (n)^{1/2}(\mu - T)/\sigma$. Then Y has a weighted non-central chi-square distribution with one degree of freedom (d.f.) and non-centrality parameter δ. The probability density function of Y is:*

$$f_Y(y) = \frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} (P_j)\Gamma(\frac{1+j}{2})\Big((d_u^{-2})f_{Y_j}(y_u) + (-1)^j(d_l^{-2})f_{Y_j}(y_l)\Big),$$

where $y_u = (y/d_u^2)$, $y_l = (y/d_l^2)$, and Y_j is distributed as χ^2_{1+j}. For the case where $d_u = d_l = 1$, this reduces to the probability density function of a non-central chi-square distribution with one d.f. and non-centrality parameter δ.

Proof. Based on the notation of theorem 1, the cumulative distribution function of Y is:

$$F_Y(y) = \int_{-\sqrt{y}/d_l}^{\sqrt{y}/d_u} \frac{1}{\sqrt{2\pi}} \exp\Big\{ -\frac{(z-\delta)^2}{2}\Big\} dz.$$

Then

$$f_Y(y) = \frac{e^{-\lambda/2}}{2\sqrt{\pi}} \Big(\frac{(d_u^{-2})}{2\sqrt{y_u}} \times e^{-(y_u/2)} \times e^{(\delta\sqrt{y_u})} + \frac{(d_l^{-2})}{2\sqrt{y_l}} \times e^{-(y_l/2)} \times e^{(\delta\sqrt{y_l})} \Big).$$

Expanding e^y in power series, we have

$$f_Y(y) = \frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} (P_j) \Gamma(\frac{1+j}{2}) \Big((d_u^{-2}) f_{Y_j}(y_u) + (-1)^j (d_l^{-2}) f_{Y_j}(y_l) \Big).$$

Theorem 2. *The rth moment about zero of $\hat{C}''_{pp}$ is:*

$$E(\hat{C}''_{pp})^r = \Big(\frac{\sigma^2}{nD^2}\Big)^r \frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} (P_j) \times 2^r \times \Gamma(\frac{n+j}{2} + r)$$
$$\Big\{ \sum_{i=0}^{r} \binom{r}{i} \frac{\Gamma([(1+j)/2]+i)}{\Gamma([(n+j)/2]+i)} ((d_u^2 - 1)^i + (-1)^j (d_l^2 - 1)^i) \Big\}.$$

Proof. To derive the rth moment of $\hat{C}''_{pp}$, we introduce the notation:
1. $B = \sigma^2/(nD^2)$,
2. $K = (nS_n^2)/\sigma^2$,
3. $Y = \max^2\{d_u Z, -d_l Z\}$.

Assume that the process is normally distributed with mean μ and variance σ^2; then K is distributed as χ^2_{n-1}, Y is distributed as a weighted non-central chi-square with 1 $d.f.$ and non-centrality parameter δ (see Theorem 1). In this notation the estimator $\hat{C}''_{pp}$ can be rewritten as $\hat{C}''_{pp} = B(Y+K)$. Thus, the rth moment of $\hat{C}''_{pp}$ is $E(\hat{C}''_{pp})^r = (B)^r E(Y+K)^r$. Since Y is distributed as a weighted non-central chi-square with 1 $d.f.$ and non-centrality parameter δ, we have:

$$E(\hat{C}''_{pp})^r = (B)^r \frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} (P_j) \Gamma(\frac{1+j}{2}) \{ E[K + (d_u^2) Y_j]^r + (-1)^j E[K + (d_l^2) Y_j]^r \},$$

where Y_j is is distributed as χ^2_{1+j}. Let $e_j = Y_j/(K+Y_j)$ and $W_j = K + Y_j$. Under the assumption of normality e_j and W_j are independent random variables (see, for instance, Johnson and Kotz (1970) or Vännman (1995)), and

e_j is distributed according to $\beta((1+j)/2, (n-1)/2))$. Furthermore, W_j has a chi-square distribution with $(n+j)$ degrees of freedom. Therefore

$$E(K+vY_j)^r = E(W_j)^r E(1+(v-1)e_j)^r,$$
$$E(W_j)^r = \frac{2^r\Gamma((n+j)/2+r)}{\Gamma((n+j)/2)},$$

and

$$E(1+(v-1)e_j)^r = \sum_{i=0}^{r} \binom{r}{i} (v-1)^i \frac{\Gamma((1+j)/2+i)\Gamma((n+1)/2)}{\Gamma((n+j)/2+i)\Gamma((1+j)/2)}.$$

Combining the results, we can obtain the rth moment of $\hat{C}''_{pp}$ as displayed in Theorem 2.

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