Statistica Sinica **5**(1995), 251-260

ON TIME-SEQUENTIAL TEST FOR A CLASS OF DISTRIBUTIONS

Jeng-Fu Liu

Fu Jen Catholic University

Abstract. This paper studies the sequential life testing problem in which the underlying distribution of the length of life can be transformed to an exponential distribution in a certain way. We obtain some asymptotic optimum properties of time-sequential probability ratio tests (TSPRT), and give approximate formulas for the error probabilities and for the expected number of failures of these tests, which improve some of the results in Epstein and Sobel (1955).

Key words and phrases: Asymptotic optimality, intensity, on-test time, time-sequential test, time-sequential probability ratio test.

1. Introduction

This paper studies sequential life tests in which the underlying distribution of the length of life has intensity of the form

$$h_\theta(y) = \frac{1}{\theta}\lambda(y), \tag{1.1}$$

for $y \geq 0$, where $\lambda(t)$ is a known positive continuous function with $\int_0^\infty \lambda(t)dt = \infty$, and θ is a parameter of interest. We are interested in testing the hypothesis $H_0 : \theta = \theta_0$ versus the hypothesis $H_1 : \theta = \theta_1$. The test is carried out by drawing n items at random from the population and placing them on a life test where one may curtail the experiment at the failure times and make a terminal decision so as to reduce the on-test time. We study this time-sequential testing problem with time censoring data.

The preceding time-sequential hypothesis testing problem in the exponential case was studied by Epstein and Sobel (1955), who proposed a continuous analogue of the SPRT of Wald for the problem and used a Wald approximation to obtain some formulas for the operating characteristic curve, expected number of failures, and the expected waiting time. Time-sequential testing problems were also studied by Dvoretzky, Kiefer and Wolfowitz (1953), Kiefer and Wolfowitz (1956) and Bhat (1988) in the context of stochastic processes. They argued that, in their models, the time-sequential probability ratio test (TSPRT) enjoys optimum properties similar to those of the classical SPRT of Wald (cf. Wald and Wolfowitz (1948)).

In this paper, we obtain an asymptotic optimum property of TSPRT for (1.1), and apply the result of Siegmund (1975) to get approximate formulas for the error probabilities and for the expected number of failures of the TSPRT.

To establish the optimal properties, we follow the classical approach of Wald and Wolfowitz (1948) to introduce a Bayes auxiliary problem. Instead of working on this Bayes problem as an optimal (time-sequential) stopping time problem on its own, we transform and reduce it to a Wald-Wolfowitz type SPRT problem with an upper bound n on the total number of observations. Although the finite horizon introduces non-stationarity into the problem, we can apply the optimal stopping theory of Chow, Robbins and Siegmund (1971, Chap. 5) to show that its Bayes solution is a generalized time-sequential probability ratio test (GTSPRT) in the sense that the stopping time is defined by random boundaries. With this result, we then proceed to get the asymptotic optimum property of the TSPRT with constant boundaries.

The plan of this paper is as follows. Section 2 describes the time-sequential testing problem and proposes a test. Section 3 transforms the model (1.1) to the exponential case, reformulate the problem as a classical Wald-Wolfowitz problem with finite horizon and finds the optimal GTSPRT. Section 4 establishes an asymptotic optimum property of the TSPRT by results in Section 3. Section 5 gives approximate formulas for the error probabilities and for the expected number of failures, which are an improvement of some of the results in Epstein and Sobel (1955). A simulation study of the error probabilities and the expected number of failures of the TSPRT is given to illustrate the previous discussion.

Time-sequential tests have been used worldwide by governments and industries in connection with quality control and procurement activities. We refer the readers to Basu (1991) for a survey and other related references.

2. Time-Sequential Tests

Let $Y_1, \ldots, Y_n$ be a sequence of i.i.d. non-negative random variables with intensity function (1.1) relative to the Lebesgue measure on the real line, where θ is the parameter of interest. Let $\theta_0 < \theta_1$ be two positive numbers. We are interested in testing the hypothesis $H_0 : \theta = \theta_0$ versus the hypothesis $H_1 : \theta = \theta_1$ in the life testing context. We would like to curtail experimentation at the failures $Y_{n1}, \ldots, Y_{nn}$ and make a terminal decision so as to reduce the on-test time, where $Y_{n1}, \ldots, Y_{nn}$ are the order statistics of $Y_1, \ldots, Y_n$.

Let $Y_{n0} = 0$, $W_0^{(n)} = 0$, $\mathcal{F}_k^{(n)} = \sigma\{Y_{n1}, \ldots, Y_{nk}\}$, $Y_k^{(n)} = (Y_{n1}, \ldots, Y_{nk})$, $Z_k^{(n)} =$

$(n-k+1)(Y_{nk}-Y_{nk-1})$, and $W_k^{(n)} = Z_1^{(n)}+\cdots+Z_k^{(n)}$, for $k = 1,\ldots,n$. Let $L_i(Y_k^{(n)})$ be the likelihood of the experiment at $Y_k^{(n)}$ under the hypothesis $H_i, i = 0, 1$, and let $\mathcal{L}_k^{(n)} = L_1(Y_k^{(n)})/L_0(Y_k^{(n)})$ be the likelihood ratio at $Y_k^{(n)}$.

A time-sequential test of the simple hypothesis $H_0 : \theta = \theta_0$ versus the simple alternative $H_1 : \theta = \theta_1$ is given by an $\mathcal{F}_k^{(n)}$-stopping time τ_n, together with a decision rule δ_n associated with τ_n. The statistical problem is to find a time-sequential test (τ_n, δ_n) so that with given bounds on the error probabilities, $E_i(W_{\tau_n}^{(n)})$ is minimal for $i = 0, 1$. Here E_i is the expectation taken when H_i is true.

The time-sequential test we propose, called the time-sequential probability ratio test (TSPRT), is of the form

$$\hat{\tau}_n = \inf\{1 \le k \le n : \mathcal{L}_k^{(n)} \notin (u, v)\}, \tag{2.1}$$

where $u < 1 < v$ are positive and $\inf\{\} = n$; and the terminal decision rule $\hat{\delta}_n$ rejects H_0 if and only if either $\hat{\tau}_n = k < n$ and $\mathcal{L}_k^{(n)} \ge v$ or $\hat{\tau}_n = n$ and $\mathcal{L}_n^{(n)} \ge 1$.

We show that the TSPRT is asymptotically optimal in the sense of asymptotically minimizing the expected on-test time both under H_0 and under H_1 among all tests having no larger error probabilities.

To establish the optimal properties, we follow Wald and Wolfowitz (1948) to consider the following Bayes auxiliary problem. Let π (and $(1-\pi)$) be the prior probability that H_0 (and H_1) is true. Let $a > 0$ (and $b > 0$) be the terminal loss of accepting H_1 when H_0 (and accepting H_0 when H_1) is true. We assume that the cost per unit time on test is $c > 0$. Then the risk of a time-sequential test (τ_n, δ_n) is given by

$$\gamma_n(\pi, \tau_n, \delta_n, a, b, c) = \pi[a\alpha_0 + cE_0(W_{\tau_n}^{(n)})] + (1-\pi)[b\alpha_1 + cE_1(W_{\tau_n}^{(n)})], \tag{2.2}$$

where α_0 and α_1 are respectively the type 1 and type 2 error probabilities of (τ_n, δ_n). The Bayes procedure is to find a test (τ_n^*, δ_n^*) which minimizes (2.2) for given π, a, b and c. Note that the Bayes problem can be studied as an optimal stopping problem.

3. The Bayes Problem

We introduce the following transformation to simplify the discussion. Let

$$T_k = \rho(Y_k), \qquad k = 1, \ldots, n,$$

where $\rho(t) = \int_0^t \lambda(y)dy$, for $t \ge 0$. Then the random variables $T_1, \ldots, T_n$ are i.i.d. exponential with intensity $1/\theta$, and $T_{nk} = \rho(Y_{nk})$ for $k = 1, \ldots, n$, where $T_{n1} \le \cdots \le T_{nn}$ are the order statistics of $T_1, \ldots, T_n$. It can be shown that the tests based on $(Y_{n1}, \ldots, Y_{nk})$ can be transformed to be tests based on $(T_{n1}, \ldots, T_{nk})$ in a unique way, and the likelihood ratio $L_1(T_k^{(n)})/L_0(T_k^{(n)})$ at $T_k^{(n)}$ is equal to that at $Y_k^{(n)}$ in a natural way, where $T_k^{(n)} = (T_{n1}, \ldots, T_{nk})$. With this understanding, it is just natural to consider the transformed data.

Note that the on-test time based on $(T_{n1}, \ldots, T_{nk})$ becomes $\sum_{i=1}^k \rho(Y_{ni}) + (n-k)\rho(Y_{nk})$, which can be expressed as a sum of k i.i.d. random variables having intensity $1/\theta$.

With this transformation, we may assume without loss of generality that Y_i is an exponential distribution with intensity $1/\theta$, i.e. $\lambda(y) = 1$ in Model (1.1). In this situation, the random variables $Z_1^{(n)}, \ldots, Z_n^{(n)}$ defined in Section 2 are also i.i.d. with intensity $1/\theta$. Then for any stopping time τ_n, $E_i(W_{\tau_n}^{(n)}) = \theta_i E_i(\tau_n)$, $i = 0, 1$, by Wald's Lemma. Hence, instead of formulating the Bayes problem in terms of the $W_{\tau_n}^{(n)}$ and the posterior probability $\pi_{\tau_n}^{(n)}$ in favor of H_0, we can reformulate it in terms of τ_n and $\pi_{\tau_n}^{(n)}$.

This reformulation simplifies the problem and brings it closer to the classical Wald-Wolfowitz theorem on the Bayes character of the sequential probability ratio test with the following differences; namely, (i) while the classical theorem assumes cost c per observation, the present setting assumes cost $c\theta_0$ under H_0 and $c\theta_1$ under H_1; and (ii) the present problem has a finite horizon n while the classical problem does not impose an upper bound on the stopping rules.

For (i), we can use a rescaling to transform the problem to one having a sampling cost c' per observation under both H_0 and H_1 as follows. Let $\pi' = \pi\theta_0/[\pi\theta_0 + (1-\pi)\theta_1]$, $a' = \pi a/\pi'$, $b' = (1-\pi)b/(1-\pi')$ and $c' = c[\pi\theta_0 + (1-\pi)\theta_1]$, then

$$\gamma_n(\pi, \tau_n, \delta_n, a, b, c) = \pi'[a'\alpha_0 + c'E_0(\tau_n)] + (1-\pi')[b'\alpha_1 + c'E_1(\tau_n)], \tag{3.1}$$

which will be denoted by $\gamma_n'(\pi', \tau_n, \delta_n, a', b', c')$. Note also the one to one correspondence between π', a', b', c' and π, a, b, c respectively.

For (ii), the finite horizon introduces non-stationarity into the problem. Applying the optimal stopping theorem of Chow, Robbins and Siegmund (1971, Chap. 5) for the non-stationary Markov case, we get the GSPRT instead of the SPRT as the optimal solution. In fact, a careful analysis by the method of backward induction gives the following neat formula for the boundaries defining the test statistic.

Theorem 3.1. *If $a > c\theta_0$ and $b > c\theta_1$, then there exist two monotone sequences $\{A_k\}_{k=0}^\infty$ and $\{B_k\}_{k=0}^\infty$ of positive numbers, such that $A_k \searrow A$ and $B_k \nearrow B$ for some positive numbers A and B; and the Bayes procedure (τ_n^*, δ_n^*) with respect to π, a, b and c is given by*

$$\tau_n^* = \inf\{1 \le k \le n : \mathcal{L}_k^{(n)} \notin (\frac{\pi(1-B_{n-k})}{(1-\pi)B_{n-k}}, \frac{\pi(1-A_{n-k})}{(1-\pi)A_{n-k}})\},$$

and rejects H_0 *if and only if* $\mathcal{L}_{\tau_n^*}^{(n)} \geq \pi(1 - A_{n-\tau_n^*})/[(1-\pi)A_{n-\tau_n^*}]$.

Note that the two monotone sequences $\{A_k\}$ and $\{B_k\}$ in Theorem 3.1 are determined uniquely by a, b and c. The proof of Theorem 3.1 is omitted because it is tedious and it follows basically the classical pattern. If $a \leq c\theta_0$ or $b \leq c\theta_1$, then the Bayes procedure takes no observation.

4. The Asymptotic Optimum Properties

Since A_k is decreasing to A and B_k is increasing to B in Theorem 3.1, it is natural to consider the time-sequential test $(\hat{\tau}_n, \hat{\delta}_n)$ defined as (2.1). The main result in this section is to establish the asymptotic optimum properties of $(\hat{\tau}_n, \hat{\delta}_n)$.

Theorem 4.1. *The time-sequential test* $(\hat{\tau}_n, \hat{\delta}_n)$ *defined as* (2.1) *is asymptotic Bayes in the sense that given any* ξ *with* $0 < \xi < 1$, *there exist numbers* $0 < w^* < 1$ *and* $c^* > 0$ *such that*

$$\lim_{n\to\infty} \gamma_n'(\xi, \hat{\tau}_n, \hat{\delta}_n, 1-w^*, w^*, c^*) = \lim_{n\to\infty} [\inf_{(\tau_n,\delta_n)\in\Delta_n} \gamma_n'(\xi, \tau_n, \delta_n, 1-w^*, w^*, c^*)], \quad (4.1)$$

where Δ_n *is the set of all time-sequential tests.*

Proof. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. exponential random variables with density $g_\theta(x) = (1/\theta)\exp[(-1/\theta)x]$. Assume that (τ^*, δ^*) is the Wald SPRT of the hypothesis H_0 versus the hypothesis H_1 with boundary (u, v); that is, the stopping rule

$$\tau^* = \inf\{k \geq 1 : \mathcal{L}_k \notin (u, v)\}, \quad (4.2)$$

and H_0 is rejected if and only if $\mathcal{L}_{\tau^*} \geq v$, where $\mathcal{L}_k = \prod_{i=1}^k [g_{\theta_1}(X_i)/g_{\theta_0}(X_i)]$ is the likelihood ratio based on $X_1, \ldots, X_k$. Let $\tau^* \wedge n$ be the minimum of τ^* and n. Since $\hat{\tau}_n$ has the same distribution as $\tau^* \wedge n$, which converges almost surely to τ^*, it follows that, for any given constants π', a', b' and c',

$$\lim_{n\to\infty} \gamma_n'(\pi', \hat{\tau}_n, \hat{\delta}_n, a', b', c') = \pi'[a'\alpha + c'E_0(\tau^*)] + (1-\pi')[b'\beta + c'E_1(\tau^*)], \quad (4.3)$$

where α and β are respectively the type 1 and type 2 error probabilities of (τ^*, δ^*). Thus, the theorem follows from (4.3) and the Bayesian property of the Wald SPRT. This completes the proof.

Theorem 4.2. *Let* $(\hat{\tau}_n, \hat{\delta}_n)$ *be the* TSPRT *defined as* (2.1), *and let* $\alpha_0 \equiv \alpha_0(\hat{\tau}_n, \hat{\delta}_n)$ *and* $\alpha_1 \equiv \alpha_1(\hat{\tau}_n, \hat{\delta}_n)$ *be respectively the type 1 and type 2 error probabilities of* $(\hat{\tau}_n, \hat{\delta}_n)$. *Assume* (τ_n, δ_n) *is any time-sequential test of* H_0 *versus* H_1, *and let* $\alpha_0' \equiv \alpha_0'(\tau_n, \delta_n)$ *and* $\alpha_1' \equiv \alpha_1'(\tau_n, \delta_n)$ *be respectively the type 1 and type 2 error probabilities of* (τ_n, δ_n). *If*

$$\lim_{n\to\infty} [\alpha_i(\hat{\tau}_n, \hat{\delta}_n) - \alpha_i'(\tau_n, \delta_n)] \geq 0, \quad i = 0, 1, \quad (4.4)$$

then we have

$$\limsup_{n\to\infty}[E_i(\hat{\tau}_n) - E_i(\tau_n)] \le 0, \quad i = 0, 1. \tag{4.5}$$

Proof. Given ξ with $0 < \xi < 1$, it follows from Theorem 4.1 that there exist numbers $0 < w^* < 1$ and $c^* > 0$ such that (4.1) holds. Hence, using (4.1) and (3.1), we have

$$\begin{aligned}&\lim_{n\to\infty}\{\xi[(1-w^*)\alpha_0 + c^*E_0(\hat{\tau}_n)] + (1-\xi)[w^*\alpha_1 + c^*E_1(\hat{\tau}_n)]\}\\ \le\ &\liminf_{n\to\infty}\{\xi[(1-w^*)\alpha_0' + c^*E_0(\tau_n)] + (1-\xi)[w^*\alpha_1' + c^*E_1(\tau_n)]\}.\end{aligned} \tag{4.6}$$

By (4.4) and (4.6), we obtain

$$\lim_{n\to\infty}[\xi E_0(\hat{\tau}_n) + (1-\xi)E_1(\hat{\tau}_n)] \le \liminf_{n\to\infty}[\xi E_0(\tau_n) + (1-\xi)E_1(\tau_n)].$$

Thus, (4.5) holds, since ξ is arbitrary in $(0,1)$. This completes the proof.

Note that it follows from (4.5) and Wald's identity that

$$\limsup_{n\to\infty}[E_i(W_{\hat{\tau}_n}) - E_i(W_{\tau_n})] \le 0, \quad i = 0, 1,$$

where $W_k = \sum_{i=1}^k \rho(Y_{ni}) + (n-k)\rho(Y_{nk})$.

5. The Error Probabilities and the Expected Number of Failures

Epstein and Sobel (1955) use the Wald approximation to get approximate formulas for the operating characteristic curve, the expected number of failures, and the expected waiting time. In this section, we make use of the results of Siegmund (1975) to improve the formulas for the error probabilities and the expected number of failures of the TSPRT.

Let $(\hat{\tau}_n, \hat{\delta}_n)$ be the TSPRT defined as (2.1), and let $\alpha_0(\hat{\tau}_n, \hat{\delta}_n)$ and $\alpha_1(\hat{\tau}_n, \hat{\delta}_n)$ be respectively the type 1 and type 2 error probabilities of $(\hat{\tau}_n, \hat{\delta}_n)$. Assume that (τ^*, δ^*) is the Wald SPRT defined as (4.2).

Since $\hat{\tau}_n$ and $\tau^* \wedge n$ have the same distribution, we get $\alpha_0(\hat{\tau}_n, \hat{\delta}_n) \to \alpha$, $\alpha_1(\hat{\tau}_n, \hat{\delta}_n) \to \beta$ and $E_i(\hat{\tau}_n) \to E_i(\tau^*)$, $i = 0, 1$, as $n \to \infty$, where α and β are respectively the type 1 and type 2 error probabilities of (τ^*, δ^*). This together with the approximation of Siegmund (1975) (see also Woodroofe (1982, Chap. 3, Sec. 1)) gives the following results:

Theorem 5.1. *Let* $\mu_0 = \log(\theta_0/\theta_1) + 1 - (\theta_0/\theta_1)$ *and* $\mu_1 = \log(\theta_0/\theta_1) - 1 + (\theta_1/\theta_0)$. *Then*

$$\alpha = \frac{\theta_0}{\theta_1 v} + o(\frac{1}{v}) \quad \text{and} \quad \beta = \frac{-\theta_0\mu_1 u}{\theta_1\mu_0} + o(u),$$

as $u \to 0^+$ *and* $v \to \infty$.

Theorem 5.2.

$$E_1(\tau^*) = \frac{1}{\mu_1}(\log v + \frac{\theta_1}{\theta_0} - 1) + o(1)$$

and

$$E_0(\tau^*) = \frac{-1}{\mu_0}[-\log u + \frac{{\mu_0}^2 + (1-\theta_0/\theta_1)^2}{-2\mu_0} - \sum_{k=1}^{\infty} \frac{1}{k}\rho_k] + o(1),$$

as $u \to 0^+$, $v \to \infty$ *with* $u \log v \to 0$ *and* $v^{-1}\log(1/u) \to 0$, *where*

$$\rho_k = k\mu_0 + \int_0^{\frac{k\theta_0\theta_1\log(\theta_1/\theta_0)}{\theta_1-\theta_0}} [k\log\frac{\theta_1}{\theta_0} - (\frac{1}{\theta_0} - \frac{1}{\theta 1})t]\frac{1}{\Gamma(k){\theta_0}^k} t^{k-1} e^{-\frac{1}{\theta_0}t}dt.$$

It follows from the preceeding theorems that, if $1/u$, v and n are sufficiently large, we have the following approximate formulas:

$$\alpha_0(\hat{\tau}_n, \hat{\delta}_n) \approx \frac{\theta_0}{\theta_1 v}, \tag{5.1}$$

$$\alpha_1(\hat{\tau}_n, \hat{\delta}_n) \approx \frac{-\theta_0\mu_1 u}{\theta_1\mu_0}, \tag{5.2}$$

$$E_1(\hat{\tau}_n) \approx \frac{1}{\mu_1}(\log v + \frac{\theta_1}{\theta_0} - 1), \tag{5.3}$$

$$E_0(\hat{\tau}_n) \approx \frac{-1}{\mu_0}[-\log u + \frac{{\mu_0}^2 + (1-\theta_0/\theta_1)^2}{-2\mu_0} - \sum_{k=1}^{\infty}\frac{1}{k}\rho_k]. \tag{5.4}$$

It is not difficult to see the right-hand sides of (5.1), (5.2), (5.3) and (5.4) are functions of $h = \theta_1/\theta_0$ for given u and v. Based on (5.1) $-$ (5.4), we give in Table 1-1 approximate values of $E_i(\hat{\tau}_n), i = 0, 1$, for four values of h, and for the four number pairs (α, β) which can be formed from the numbers 0.01 and 0.05. Table 1-2 gives the approximate values of expected number of failures based on Epstein and Sobel (1955). It is clear that these two approximations can be quite different. In fact, from Wald's approximations,

$$\alpha \approx (\frac{1}{u} - 1)/(\frac{v}{u} - 1), \tag{5.5}$$

$$\beta \approx (v-1)/(\frac{v}{u} - 1). \tag{5.6}$$

Since $(1/u-1)/(v/u-1) \approx 1/v$ and $(v-1)/(v/u-1) \approx u$ as $u \to 0^+$ and $v \to \infty$, we know the approximations (5.5) and (5.6) are asymptotically incorrect by a constant factor, as shown in Theorem 5.1.

A simulation study for the comparisons between (5.1)$-$(5.2) and (5.5)$-$(5.6) has been carried out. We give, in Table 2-1 and Table 2-2, simulation results for the error probabilities and the expected number of failures of $\hat{\tau}_n$, which are respectively defined by (5.1) – (5.2) and (5.5) – (5.6) for $\alpha = \beta = 0.01$, four values of h and several different values of n.

For example, in Table 2-1, if $\hat{\tau}_n$ is defined by (5.1) and (5.2) with $h = 2, n = 120$, and $\alpha = \beta = 0.01$, then the simulation values of error probability and expected number of failures of $\hat{\tau}_n$ are 0.009 and 23.91 under H_0 (and 0.012 and 15.86 under H_1). The Wald's approximation values of error probability and the expected number of failures of $\hat{\tau}_n$, based on Epstein and Sobel (1955), are 0.020 and 21.80 under H_0 (and 0.012 and 12.42 under H_1). The approximation values of error probability and the expected number of failures of $\hat{\tau}_n$, based on Siegmund (1975), used in this paper, are 0.010 and 23.93 under H_0 (and 0.010 and 16.01 under H_1). It is clear that the approximation based on (5.1) – (5.4) is better than that of Epstein and Sobel (1955).

Acknowledgements

This work is based on part of my Ph.D. dissertation at National Central University, Taiwan, R.O.C. I wish to thank my advisor Professor I-Shou Chang for his guidence and helpful comments. The author thanks the referee, whose comments helped to clarify several important points.

Table 1-1. Approximate values of $E_i(\hat{\tau}_n)$, $i = 0, 1$, based on $(5.1) - (5.4)$, for various values of α, β and $h = \theta_1/\theta_0$.

	h	3/2		2		5/2		3	
H	$\alpha \diagdown \beta$	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05
H_0	0.01	64.33	42.02	23.93	15.60	14.65	9.56	10.74	7.02
	0.05	64.33	42.02	23.93	15.60	14.65	9.56	10.74	7.02
H_1	0.01	49.71	49.71	16.01	16.01	8.89	8.89	6.11	6.11
	0.05	32.69	32.69	10.76	10.76	6.13	6.13	4.32	4.32

Table 1-2. Approximate values of $E_i(\hat{\tau}_n)$, $i = 0, 1$, based on Epstein and Sobel (1955), for various values of α, β and $h = \theta_1/\theta_0$.

	h	3/2		2		5/2		3	
H	$\alpha \diagdown \beta$	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05
H_0	0.01	62.43	40.35	23.31	15.07	14.24	9.20	10.43	6.74
	0.05	57.91	36.74	21.63	13.72	13.21	8.38	9.67	6.14
H_1	0.01	47.64	44.18	14.68	13.61	7.71	7.16	5.00	4.63
	0.05	30.79	28.03	9.48	8.64	4.99	4.54	3.23	2.94

Table 2-1. Simulation values of error probabilities and expected number of failures of $(\hat{\tau}_n, \hat{\hat{\delta}}_n)$, which are defined by $(5.1) - (5.2)$ with $\alpha = \beta = 0.01$ for various values of $h = \theta_1/\theta_0$ and for several different values of n.

	h	3/2		2		5/2		3	
		α_i	$E_i(\hat{\tau}_n)$	α_i	$E_i(\hat{\tau}_n)$	α_i	$E_i(\hat{\tau}_n)$	α_i	$E_i(\hat{\tau}_n)$
H_0	$n = 40$	0.080	37.21	0.018	22.34	0.012	14.83	0.014	10.40
	$n = 80$	0.036	55.23	0.010	24.35	0.015	13.96	0.011	10.69
	$n = 120$	0.012	61.78	0.009	23.91	0.014	14.57	0.008	10.58
H_1	$n = 40$	0.122	32.50	0.018	15.01	0.013	8.65	0.007	6.08
	$n = 80$	0.052	43.73	0.014	16.45	0.011	8.70	0.005	5.80
	$n = 120$	0.024	48.09	0.012	15.86	0.008	8.94	0.009	5.99
E-S	H_0	0.015	60.19	0.020	21.80	0.025	12.98	0.030	9.29
	H_1	0.011	43.39	0.012	12.42	0.013	6.14	0.014	3.77
Sieg.	H_0	0.010	64.33	0.010	23.93	0.010	14.65	0.010	10.74
	H_1	0.010	49.71	0.010	16.01	0.010	8.89	0.010	6.11

Table 2-2. Simulation values of error probabilities and expected number of failures of $(\hat{\tau}_n, \hat{\delta}_n)$, which are defined by (5.5) − (5.6) with $\alpha = \beta = 0.01$ for various values of $h = \theta_1/\theta_0$ and for several different values of n.

	h	3/2		2		5/2		3	
		α_i	$E_i(\hat{\tau}_n)$	α_i	$E_i(\hat{\tau}_n)$	α_i	$E_i(\hat{\tau}_n)$	α_i	$E_i(\hat{\tau}_n)$
H_0	$n = 40$	0.090	38.19	0.014	23.30	0.009	15.44	0.007	11.55
	$n = 80$	0.038	55.35	0.003	25.02	0.004	15.29	0.001	11.44
	$n = 120$	0.014	62.92	0.003	24.72	0.004	15.68	0.003	11.41
H_1	$n = 40$	0.113	33.54	0.019	16.99	0.009	10.32	0.009	7.02
	$n = 80$	0.048	47.02	0.008	17.49	0.012	10.68	0.005	7.33
	$n = 120$	0.018	52.80	0.013	18.15	0.010	10.38	0.009	7.32
E-S	H_0	0.010	62.43	0.010	23.31	0.010	14.24	0.010	10.43
	H_1	0.010	47.64	0.010	14.68	0.010	7.71	0.010	5.00
Sieg.	H_0	0.007	66.07	0.005	25.08	0.004	15.58	0.003	11.56
	H_1	0.009	53.90	0.008	18.23	0.007	10.44	0.007	7.32

References

Basu, A. P. (1991). Sequential methods in reliability and life testing. Handbook of Sequential Analysis (Edited by B. K. Ghosh and P. K. Sen), 581-592, Marcel Dekker, New York.

Bhat, B. R. (1988). Optimal properties of SPRT for some stochastic processes. Contemporary Mathematics **80**, 285-299.

Chow, Y. S., Robbins, H. and Siegmund, D. (1971). Great Expectations: The Theory of Optimal Stopping. Houghton Mifflin, Boston.

Dvoretzky, A., Kiefer, J. and Wolfowitz, J. (1953). Sequential decision problems for processes with continuous time parameter. Testing hypotheses. Ann. Math. Statist. **24**, 254-264.

Epstein, B. and Sobel, M. (1955). Sequential life tests in the exponential case. Ann. Math. Statist. **26**, 82-93.

Kiefer, J. and Wolfowitz, J. (1956). Sequential tests of hypotheses about mean occurrence time of a continuous parameter Poisson process. Naval. Res. Logis. Quart. **3**, 205-219.

Siegmund, D. (1975). Error probabilities and average sample number of the sequential probability ratio test. J. Roy. Statist. Soc. Ser.B **37**, 394-401.

Wald, A. and Wolfowitz, J. (1948). Optimum character of the sequential probability ratio test. Ann. Math. Statist. **19**, 326-339.

Woodroofe, M. (1982). Nonlinear Renewal Theory in Sequential Analysis. Society for Industrial and Applied Mathematics, Philadelphia.

Department of Statistics, Fu Jen Catholic University, Taipei 24205, Taiwan.

(Received July 1992; accepted May 1994)