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SOME OPTIMAL NESTED ROW-COLUMN DESIGNS

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Abstract: We consider experiments run in blocks with two way heterogeneity in each block. The goal is to find "optimal" designs for estimating treatment effects. In this paper, a general method for constructing universally optimal nested row-column designs within the class of treatment connected designs is given. Also, a class of nested row-column designs, which has a completely symmetric information matrix but does not have maximum trace among all possible designs, is shown to have Φ_a-optimality for some a.

Key words and phrases: Nested row-column designs, row-column designs, universal optimality, generalized Youden designs.

1. Introduction

We consider a particular case of a 4-factor additive linear model. The four factors are a treatment factor and three blocking factors in which two of the three blocking factors (row and column factors) are nested in the third (block factor). Since rows and columns are nested within blocks, these designs are called nested row-column designs (NRC). An NRC has v treatments and b blocks with $p \times q$ experimental units in each. The units in each block are arranged in two directions, i.e., two 'nested' blocking factors are used; one factor consists of p levels, and the other consists of q levels. When $b = 1$, a block design with nested rows and columns is an ordinary row-column design.

NRC's are important for controlling heterogeneity in two directions. Many designs have been used to control two sources of extraneous variation by the experimenter, such as Latin squares, Youden squares, generalized Youden designs, and row-column designs. However, Latin squares, Youden squares, and generalized Youden designs exist only for a limited number of parameter combinations, and, therefore, have limited practical use. Also, row-column designs may have row-column interaction problems when many rows and columns are used. Compared to row-column designs, NRC's are more general. Given the number of experimental units, NRC's have fewer rows and columns within each block.

Row-column interactions in NRC's are likely not as severe as in row-column designs. Therefore, NRC's are especially useful for eliminating heterogeneity in two directions when row-column interactions are present.

Nested designs in the general set-up were first studied by Srivastava (1978). Since then, a number of authors have studied various aspects of constructing these designs in the row-column design setting. For constructing balanced incomplete NRC's, Singh & Dey (1979), and Agrawal & Prasad (1982a, 1983) utilize the method of differences. Later, Cheng (1986) obtains a balanced incomplete NRC by combining a balanced incomplete block design with a balanced NRC. For partially balanced incomplete NRC's, Agrawal & Prasad (1982c) show that the method of differences can also be applied to construct partially balanced incomplete NRC's. Various product techniques are also used by Agrawal & Prasad (1982b, 1984) for constructing partially balanced incomplete NRC's. Other NRC's based on cyclic and generalized cyclic methods of construction have been obtained by Jarrett & Hall (1982) and Ipinyomi & John (1985).

The optimality of nested row-column designs has been studied recently. Independently, Chang (1989) and Bagchi, Mukhopadhyay and Sinha (1990) have obtained some general results and construction methods for optimal NRC's in the fixed effects models. One may expect that binary designs would perform well; however, the optimal designs they found are all non-binary. In addition, Bagchi, Mukhopadhyay and Sinha (1990) have also developed methods of constructing optimal NRC's and studied optimality results for mixed effects models.

In this paper, after introducing the model and notation in Section 2, we propose a general method for constructing universally optimal NRC's within the class of treatment connected designs. This is discussed in Section 3. In Section 4, we show that designs which do not have maximum trace of their information matrix can be Φ_a-optimal (defined later), for some a, under certain conditions.

Throughout this paper, without loss of generality, we assume that the number of rows is less than or equal to the number of column. That is $p \leq q$.

2. Model and Notation

The model we consider is a fixed effects model. Let Y_{itkh} be the observation obtained from the ith treatment in the kth row and the hth column in the tth block. Then the model for NRC's is

$$Y_{itkh} = \mu + \tau_i + \beta_t + \rho_{k(t)} + \gamma_{h(t)} + \varepsilon_{itkh},$$

$$(i = 1, 2, \ldots, v; \quad t = 1, 2, \ldots, b; \quad k = 1, 2, \ldots, p; \quad h = 1, 2, \ldots, q)$$

where μ is the overall mean, τ_i is the effect of the ith treatment, β_t is the effect of the tth block, $\rho_{k(t)}$ is the effect of the kth row in the tth block, $\gamma_{h(t)}$ is the effect of the hth column in the tth block, and the ε's are uncorrelated random errors with zero expectation and a common variance σ^2.

For a specified design d, let $N_d = ((r_{dit}))$ be the $v \times b$ matrix of elements r_{dit}, where r_{dit} is the number of times the ith treatment appears in the tth block. Let r_{di} be the number of replications of the ith treatment and r^δ be the diagonal matrix with diagonal entries $r_{d1}, \ldots, r_{dv}$. Also, let N_{d1t} be the $v \times p$ incidence matrix of treatments versus rows in the tth block, N_{d2t} be the $v \times q$ incidence matrix of treatments versus columns in the tth block, and define

$$N_{d1} = (N_{d11} : N_{d12} : \cdots : N_{d1b}), \quad N_{d2} = (N_{d21} : N_{d22} : \cdots : N_{d2b}).$$

Then, the coefficient matrix of the reduced normal equation for the treatment effects is

$$C_d = r^\delta - \frac{1}{q} N_{d1} N'_{d1} - \frac{1}{p} N_{d2} N'_{d2} + \frac{1}{pq} N_d N'_d.$$

The matrix C_d is also called the information matrix or the C-matrix of the design d. To simplify notation, the subscript d will be dropped whenever it is clear which design is being referred to.

Let $\Xi_{v,p,q,b}$ denote the collection of all treatment connected NRC's with v treatments, p rows, and q columns in each of the b blocks. Also let $\Omega_{v,b,k}$ be the collection of all treatment connected block designs with v treatments and b blocks of sizes k. For each $d \in \Xi_{v,p,q,b}$, the two associated block designs, d_R and d_c, where in d_R one views the bp rows of d as bp blocks of size q and in d_c one views the bq columns of d as bq blocks of size p, belong to $\Omega_{v,bp,q}$ and $\Omega_{v,bq,p}$, respectively. Note that N_1 and N_2 defined above are the corresponding incidence matrices of d_R and d_c. Finally we define C_d^R and C_d^c as the coefficient matrices of the reduced normal equations for estimating treatment effects in d_R and d_c.

3. Optimality of NRC's with Maximum Trace of the Information Matrix

In this section, we develop a general procedure for constructing universally optimal NRC's. This procedure is based on an initial row-column design which has maximum trace of the information matrix (C_d). Methods of constructing the initial design are also given.

Because an NRC consists of b row-column designs, if each of the row-column design has maximum trace of the information matrix, then the NRC has maximum trace of C_d. Let $n_{dik}(R)$ be the number of times treatment i occurs in the kth row and $n_{dih}(C)$ be the number of times treatment i occurs in the hth column. A sufficient condition (see, for example, Kiefer (1975)) for a row-column design to have maximum trace of the information matrix is that: $n_{dih}(C)$'s are as nearly equal as possible and $n_{dik}(R) = \frac{1}{p} r_{di}$, for all i, k. Based on this condition, we derive the following theorem.

Theorem 3.1. *Let $d^* = \{d_1, d_2, \ldots, d_b\}$ be a design in $\Xi_{v,p,q,b}$ which satisfies:*

(a) $|n_{itk}(R) - \frac{q}{v}| < 1$ *and* $|n_{ith}(C) - \frac{p}{v}| < 1, \ \forall\, i, k, h, t,$
where

$n_{itk}(R)$ = *the number of times treatment i appears in the kth row of the tth block.*

$n_{ith}(C)$ = *the number of times treatment i appears in the hth column of the tth block.*

(b) $r_{it} = p(I_1 + 1)$ *or* $pI_1, \ \forall\, i, t,$
where

$r_{it} =$ *the number of times treatment i appears in the tth block,*

$I_1 = \mathrm{int}\left(\frac{q}{v}\right).$

Then, (1) $\mathrm{tr}C_{d^*} = \max_{\{d: d \in \Xi\}} \mathrm{tr}C_d,$
(2) $C_{d^*} = C^c_{d^*}.$

Proof. The proof is straightforword.

Note that condition (a) in Theorem 3.1 is a sufficient condition for maximal trace, in light of the comment preceding the statement of the Theorem.

We say Φ is a nonincreasing criterion if $\Phi(C) \leq \Phi(D)$ when $C - D$ is a nonnegative definite matrix. If a design d satisfies the conditions in Theorem 3.1 and its corresponding column design, and d_c, is an optimal block design with respect to a nonincreasing criterion, then d itself is an optimal nested row-column design under the same criterion. This result can be found in Bagchi, et al. (1990) as well as in Chang (1989). Methods for constructing optimal nested row-column designs using this condition, can also be found in Bagchi, et al. (1990).

Optimal nested row-column designs obtained by using the above results exist only for certain parameter combinations. In fact, one can always construct a universally optimal NRC with $v!$ blocks if a row-column design with maximum trace is given.

Theorem 3.2. *Let $\hat{d} \in \Xi_{v,p,q,1}$ be a design such that $\mathrm{tr}C_{\hat{d}}$ is the maximum over $\Xi_{v,p,q,1}$. Suppose we permute the treatment labels and let $d^* = \{\hat{d}_1, \hat{d}_2, \ldots, \hat{d}_{v!}\}$, i.e. d^* is the NRC with block i as the row-column design $\hat{d}_i \in \Xi_{v,p,q,1}$ which is the design resulting from $\hat{d}$ by one of the $v!$ permutations of the v treatment labels. Then d^* is universally optimal in the class $\Xi_{v,p,q,v!}$.*

Proof. Let P denote a permutation matrix. Since

$$C_{d^*} = \sum_{i=1}^{v!} C_{\hat{d}_i} = \sum_{\text{all } P} PC_{\hat{d}_1}P',$$

it follows that C_{d^*} is completely symmetric.

If $d \in \Xi_{v,p,q,v!}$, then there exist $d_1, d_2, \ldots, d_{v!} \in \Xi_{v,p,q,1}$, such that $d = \{d_1, d_2, \ldots, d_{v!}\}$. Then

$$\text{tr}C_d = \text{tr}\sum_{i=1}^{v!} C_{d_i} = \sum_{i=1}^{v!} \text{tr}C_{d_i} \leq \sum_{i=1}^{v!} \text{tr}C_{\hat{d}_i} = \text{tr}C_{d^*}$$

which implies d^* maximizes $\text{tr}C_d$ over $\Xi_{v,p,q,v!}$. Thus, by the first proposition in Kiefer (1975), d^* is universally optimal in $\Xi_{v,p,q,v!}$.

It can be shown that if we repeat d^* n times, the resulting design is also universally optimal in the class of $\Xi_{v,p,q,nv!}$.

In Theorem 3.2, an initial row-column design with maximum trace of C_d is required for obtaining a universally optimal NRC. We next show how one can construct such a design. Let $\rho = q - \text{int}(q/v)v$. Designating the v treatments in $\hat{d}$ by the integers $1, 2, \ldots, v$, the procedures for constructing $\hat{d}$ for different cases are given below:

Case 1. If $p = q$

Step 1. For row 1, assign treatment i to the ith element. If $q > v$, repeat the assignment until all of the cells in row 1 are assigned.

Step 2. For row k, copy row $k-1$, and then move the last entry of the row to the head and shift the rest of the entries to the right by 1 position.

Step 3. If $k < p$, go to Step 2.

Otherwise, stop.

The design $\hat{d}$ will look like the following:

$$\begin{bmatrix} 1 & 2 & \cdots & \rho \\ \rho & 1 & \cdots & \rho - 1 \\ \vdots & \vdots & & \vdots \\ 2 & 3 & \cdots & 1 \end{bmatrix}.$$

It can be seen that treatment 1, treatment 2, ..., treatment ρ have $I_1 + 1$ replications and treatment $\rho + 1$, treatment $\rho + 2$, ..., treatment v have I_1 replications in each row. Also, each treatment occurs as equally as possible in each row and in each column.

Case 2. If $p \neq q$

(a) $p \leq \rho$

The same procedure used in Case 1 can also be used here. Design $\hat{d}$ will look like the following:

$$\begin{bmatrix} 1 & 2 & \cdots & & & \rho \\ \rho & 1 & \cdots & & & \rho - 1 \\ \vdots & \vdots & & & & \vdots \\ \rho - p + 2 & \rho - p + 3 & \cdots & 1 & \cdots & \rho - p + 1 \end{bmatrix}.$$

Obviously, treatment 1, treatment $2, \ldots,$ and treatment ρ have $I_1 + 1$ replications, and treatment $\rho + 1$, treatment $\rho + 2, \ldots,$ and treatment v have I_1 replications in each row. Thus, each treatment occurs as equally as possible in each row and at most once in each column.

(b) $q \leq v$

Randomly choosing p rows from a $q \times q$ Latin square gives the design $\hat{d}$. We can see that this design has maximum trace of the information matrix.

However, when $p \neq q$, $p > \rho$, and $q > v$, the procedures above do not apply and other procedures are needed.

By examining the proof of Theorem 3.2, the reason one needs $v!$ permutations (blocks) is to guarantee that one has a completely symmetric information matrix for the resulting design. However, without requiring all $v!$ permutations, one can still obtain a completely symmetric information matrix for the resulting design in many cases. If we choose a good initial row-column design, the number of permutations can be significantly reduced and yet the design whose blocks consist of these permutations of the initial row-column will be completely symmetric. It is very possible that there are many row-column designs which have maximum trace. Among these designs, if there exists a design d such that C_d has the following form

$$\begin{bmatrix} (\boldsymbol{A}_1)_{v_1 \times v_1} & & & -\lambda \\ & (\boldsymbol{A}_2)_{v_2 \times v_2} & & \\ & & \ddots & \\ & -\lambda & & (\boldsymbol{A}_n)_{v_n \times v_n} \end{bmatrix}, \tag{3.1}$$

where each $\boldsymbol{A}_i$ is a completely symmetric matrix with constant $-\lambda_i$ in all off-diagonal entries, then instead of $v!$ blocks, the minimal number of blocks $s_{\hat{d}}$ (say) yielding a completely symmetric design satisfies

$$s_{\hat{d}} \leq \frac{v!}{\prod_{i=1}^{n} v_i!}, \quad \text{where} \quad \sum_{i=1}^{n} v_i = v.$$

It is easy to find a design which has maximum trace, but it is a more complex combinatorial problem to find such a design with its information matrix of the same form as in (3.1). For certain row-column designs with information matrices of the form (3.1), an upper bound for s_d can be explicitly given. For example, some well-known row-column designs such as Latin square, and regular generalized Youden have a completely symmetric information matrix. If such a design is used as the initial row-column design, any number of blocks consisting of this design will be universally optimal. In the above notation, this implies $s_d = 1$.

In the case where a Latin Square or regular Generalized Youden Design (GYD) do not exist, we have found the upper bound for s_d for a particular class of Partially Balanced Block Designs (PBBD). The simplest class of PBBD's is the group divisible designs (GD PBBD). These are PBBD's where the treatments can be divided into several groups, each group containing the same number of treatments. Two treatments are called first associates if they belong to the same group and second associates otherwise (see Cheng (1978)). If GD($v = mn$) PBBD denotes a group divisible design in which the $v = mn$ treatments are divided into m groups with n treatments in each group, then we have the following two results.

Corollary 3.3. *Given v, p and q, if*
(a) $v|p$ $(v|q)$, *and*
(b) *a* GD($v = mn$) PBBD *exists with the p rows (q columns) considered as the blocks, then, there exists a $\hat{d} \in \Xi_{v,p,q,1}$, such that*

$$s_{\hat{d}} \leq \frac{v!}{m!(n!)^m}.$$

Proof. Straightforward.

Corollary 3.4. *If a design $\hat{d} \in \Xi_{v,p,q,1}$ has parameters $p = q = s$ (say) where s is not an integral multiple of v, and satisfies conditions* (a) *and* (b) *in Theorem* 3.1, *then*

$$s_{\hat{d}} \leq \frac{v!}{\rho!(v-\rho)!}, \quad \textit{where} \quad \rho = \left(\frac{s}{v} - \text{int}\left(\frac{s}{v}\right)\right)v.$$

Proof. Without loss of generality, we assign the treatment replications of $\hat{d}, r_i$, as

$$r_i = \begin{cases} s(I_1 + 1), & \text{for } 1 \leq i \leq \rho; \\ sI_1, & \text{for } \rho + 1 \leq i \leq v. \end{cases}$$

Then by the construction in case 1 discussed earlier, $C_{\hat{d}}$ has the form

$$\begin{bmatrix} \boldsymbol{A}_1 & c\boldsymbol{J} \\ c\boldsymbol{J}' & \boldsymbol{A}_2 \end{bmatrix}_{v\times v} \quad \text{where } \boldsymbol{A}_1 \text{ and } \boldsymbol{A}_2 \text{ are completely symmetric,}$$

and $\boldsymbol{A}_1$ is $\rho\times\rho$, $\boldsymbol{J}$ is $\rho\times(v-\rho)$, $\boldsymbol{A}_2$ is $(v-\rho)\times(v-\rho)$.

Hence, by (3.1), the result follows.

4. Optimality of NRC's without Maximum Trace of the Information Matrix

In this section, we show that some NRC's, which have completely symmetric information matrices but not maximum traces, are Φ_a-optimal for some a. A design is said to be Φ_a-optimal if it minimizes

$$\Phi_a(\lambda_d) = \begin{cases} \left(\dfrac{1}{v-1}\displaystyle\sum_{i=1}^{v-1}\lambda_{di}^{-a}\right)^{\frac{1}{a}}, & 0<a<\infty; \\ \displaystyle\prod_{i=1}^{v-1}(\lambda_{di})^{-\frac{1}{v-1}}, & a=0; \\ \displaystyle\max_{1\le i\le v-1}\lambda_{di}^{-1}, & a=\infty, \end{cases}$$

where $\lambda_{d1},\lambda_{d2},\ldots,$ and $\lambda_{d(v-1)}$ are the nonzero eigenvalues of C_d. Let $\tilde{d}=d^*\cup d^\#$ be an NRC where $d^*\in\Xi_{v,p,q,b-1}$ has a completely symmetric information matrix with maximum trace among all possible designs, and $d^\#\in\Xi_{v,p,q,1}$ is a nonregular GYD (see Kiefer (1975)). Then $\tilde{d}$ has a completely symmetric information matrix but does not have maximum trace. We shall consider the Φ_a-optimality of $\tilde{d}$. The method used is similar to the one used by Kiefer (1975) for proving the A-optimality of nonregular GYD's. Although we cannot guarantee A-optimality, we can always prove the E-optimality of $\tilde{d}$. When $p=q$, Φ_a-optimality can also be obtained if a is greater than a constant which depends on the values of v,p and b. Let $H=\{d\in\Xi_{v,p,q,b}: n_{ita}(C)\text{'s are as equal as possible}\}$. For each treatment i and $s=p$ or q, we define

$$\begin{aligned} h(r_i,s) &= \min_{\{d:\Sigma\Sigma n_{ita}=r_i\}}\sum_{t=1}^{b}\sum_{a=1}^{s} n_{ita}^2, \\ &= \left(r_i - bs\times\text{int}\left(\frac{r_i}{bs}\right)\right)\left(1+\text{int}\left(\frac{r_i}{bs}\right)\right)^2 \\ &\quad + \left(bs - r_i + bs\times\text{int}\left(\frac{r_i}{bs}\right)\right)\left(\text{int}\left(\frac{r_i}{bs}\right)\right)^2 \\ &= -bsI_s^2+(2r_i-bs)I_s+r_i, \quad \text{where } I_s=\text{int}\left(\frac{r_i}{bs}\right) \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} m(r_i) &= \max_d \sum_{t=1}^{b} r_{it}^2 \\ &= \operatorname{int}\left(\frac{r_i - bqI_q}{q}\right)\{q(I_q+1)\}^2 + \left(b - 1 - \operatorname{int}\left(\frac{r_i - bqI_q}{q}\right)\right)(qI_q)^2 \\ &\quad + \left\{qI_q + r_i - bqI_q - q \times \operatorname{int}\left(\frac{r_i - bqI_q}{q}\right)\right\}^2, \end{aligned} \tag{4.2}$$

where the maximum is over all $d \in H$ with treatment i replicated r_i times. Also, let

$$c(r_i) = r_i - \frac{1}{q}h(r_i, p) - \frac{1}{p}h(r_i, q) + \frac{1}{pq}m(r_i).$$

By the definitions of $h(r_i, q)$ and $m(r_i)$,

$$m(r_i) - qh(r_i, q) = 0, \tag{4.3}$$

if r_i is an integral multiple of q.

We start with the following lemma.

Lemma 4.1. $\max_d C_{dii}(r_i) \le c(r_i)$, *where* $C_{dii}(r_i)$ *is the* (i,i)-th *entry in* C_d *and the maximum is over all designs in* $\Xi_{v,p,q,b}$ *with the* ith *treatment replicated* r_i *times.*

Proof. Let d' be any design in $\Xi_{v,p,q,b}$ with the ith treatment replicated r_i times; then

$$\begin{aligned} C_{d'ii}(r_i) &= r_i - \frac{1}{q}\sum_{t=1}^{b}\sum_{k=1}^{p} n_{d'itk}^2(R) - \frac{1}{p}\sum_{t=1}^{b}\sum_{h=1}^{q} n_{d'ith}^2(C) + \frac{1}{pq}\sum_{t=1}^{b} r_{d'it}^2 \\ &\le r_i - \frac{1}{q}h(r_i, p) - \frac{1}{p}\sum_{t=1}^{b}\sum_{h=1}^{q} n_{d'ith}^2(C) + \frac{1}{pq}\sum_{t=1}^{b} r_{d'it}^2 \\ &= r_i - \frac{1}{q}h(r_i, p) - \frac{1}{p}\sum_{t=1}^{b}\sum_{h=1}^{q}\left(n_{d'ith}(C) - \frac{1}{q}r_{d'it}\right)^2. \end{aligned}$$

If d' is a design in which r_i is an integral multiple of q, then, by Equation (4.3), we have $C_{d'ii}(r_i) \le c(r_i)$. Otherwise, $r_{d'i} = cq + u$ for some integers c and u, $0 < u < q$. In this case, by a straightforward calculation,

$$h(r_i, q) - \frac{1}{q}m(r_i) = u - \frac{u^2}{q}.$$

Therefore it is sufficient to prove that

$$\sum_{t=1}^{b}\sum_{h=1}^{q}\left(n_{d'ith}(C)-\frac{1}{q}r_{d'it}\right)^2 \geq u-\frac{u^2}{q}.$$

If d' has b^* (≥ 1) blocks in which $r_{d'it}$ is not an integral multiple of q, then

$$\begin{aligned}
&\sum_{t=1}^{b}\sum_{h=1}^{q}\left(n_{d'ith}(C)-\frac{1}{q}r_{d'it}\right)^2 \\
\geq\ &\sum_{t=1}^{b^*}\sum_{h=1}^{q}\left(n_{d'ith}(C)-\frac{1}{q}r_{d'it}\right)^2 \\
=\ &\sum_{t=1}^{b^*}\sum_{h=1}^{q}\left(n_{d'ith}(C)-\frac{1}{q}(c_t q+f_t)\right)^2 \quad \text{for some } c_t \text{ and } f_t,\quad 1\leq f_t\leq q-1 \\
\geq\ &\sum_{t=1}^{b^*}\left(f_t-\frac{f_t^2}{q}\right).
\end{aligned}$$

Case 1. $1 \leq u \leq \frac{q}{2}$

(i) If there exists a t^* such that $f_{t^*} \in [u, q-u]$, then

$$\sum_{t=1}^{b^*}\left(f_t-\frac{f_t^2}{q}\right) \geq f_{t^*}-\frac{f_{t^*}^2}{q} \geq u-\frac{u^2}{q}.$$

(ii) If $f_t \in [1, u-1]$ or $f_t \in [q-u+1, q-1]$ for all t, let $S = \{t|f_t \in [1, u-1]\}$ and $S' = \{t|f_t \in [q-u+1, q-1]\}$. Then,

$$\begin{aligned}
\sum_{t=1}^{b^*}\left(f_t-\frac{f_t^2}{q}\right) &= \sum_{t\in S}\left(f_t-\frac{f_t^2}{q}\right)+\sum_{t\in S'}\left(f_t-\frac{f_t^2}{q}\right) \\
&= \sum_{t\in S} f_t\left(1-\frac{f_t}{q}\right)+\sum_{t\in S'} f'_t\left(1-\frac{f'_t}{q}\right) \quad \text{where } f'_t = q-f_t \\
&\geq \left(1-\frac{u}{q}\right)\left(\sum_{t\in S} f_t+\sum_{t\in S'} f'_t\right) \\
&\geq u-\frac{u^2}{q}.
\end{aligned}$$

The last inequality is obvious if $\sum_{t\in S} f_t \geq u$. If $\sum_{t\in S} f_t < u$, then, since

$$\sum_{t\in S} f_t+\sum_{t\in S'} f_t = k_1 q+u$$

for some integer k_1, which implies

$$k_2 q - \sum_{t \in S'} f_t' = k_1 q + u - \sum_{t \in S} f_t$$

for some integer k_2, it follows that

$$\sum_{t \in S'} f_t' = (k_2 - k_1)q - u + \sum_{t \in S} f_t \geq q - u \geq u \quad \left(\text{since } \sum_{t \in S'} f_t' \geq 0 \text{ and } 1 \leq u \leq \frac{q}{2}\right).$$

Case 2. $\frac{q}{2} < u \leq q - 1$

This follows from Case 1 by replacing u with $q - u$ and noting that

$$u - \frac{u^2}{q} = (q - u) - \frac{(q - u)^2}{q}.$$

To prove E-optimality of $\tilde{d}$, we define two functions, $g(r)$ and $\Delta(r)$. By equations (4.1) and (4.3),

$$h(r+1, p) - h(r, p) = 1 + 2\text{int}\left(\frac{r}{bp}\right) = 1 + 2I_p, \tag{4.4}$$

and

$$m(r+1) - m(r) = 2\left(qI_q + r - bqI_q - q \times \text{int}\left(\frac{r - bqI_q}{q}\right)\right) + 1, \quad \forall\, r. \tag{4.5}$$

Define

$$g(r) = pqc(r) = pqr - ph(r, p) - qh(r, q) + m(r), \tag{4.6}$$

$$\begin{aligned}\Delta(r) &= g(r+1) - g(r) \\ &= pq - p(1 + 2I_p) - q(1 + 2I_q) + 2\left(qI_q + r - bqI_q - q \times \text{int}\left(\frac{r - bqI_q}{q}\right)\right) + 1 \\ &= (p-1)(q-1) - 2pI_p + 2\left(r - bqI_q - q \times \text{int}\left(\frac{r - bqI_q}{q}\right)\right).\end{aligned}$$

Since a nonregular GYD exists in each block, $\bar{r} = \frac{bpq}{v}$ must be an integer. We define

$$\Re = \left\{(r_1, r_2, \ldots, r_v) : r_i \text{ is a nonnegative integer } \forall\, i \text{ and } \sum_{i=1}^{v} r_i = v\bar{r}\right\}.$$

Theorem 4.2. *$\tilde{d}$ is E-optimal.*

Proof. By equations (3.5) and (3.9) in Kiefer (1975), if C_{d^*} is completely symmetric and $\max_{\Re} \min_i g(r_i) = g(\bar{r})$, then d^* is E-optimal. Since $C_{\tilde{d}}$ is completely symmetric, it suffices to show

$$\max_{\Re} \min_i g(r_i) = g(\bar{r}).$$

If $g(r_i)$ is an increasing function for $r_i \leq \bar{r}$, then we have $g(r_i) \leq g(\bar{r})$, which implies

$$\min_i g(r_i) \leq g(\bar{r}) \quad \text{where} \quad (r_1, r_2, \ldots, r_v) \in \Re.$$

Therefore, to complete the proof, we need to show $g(r_i)$ is an increasing function of r_i, for every $r_i \leq \bar{r}$. Since $\Delta(r) = g(r+1) - g(r)$, it is sufficient to show that $\Delta(r) \geq 0$, for $r < \bar{r}$. Now

$$r < \bar{r} \Rightarrow \frac{r}{bp} < \frac{bpq}{vbp} = \frac{q}{v} \Rightarrow I_p = \text{int}\Big(\frac{r}{bp}\Big) \leq \text{int}\Big(\frac{q}{v}\Big) < \frac{q}{v}.$$

Thus,

$$\begin{aligned}
\Delta(r) &= (p-1)(q-1) - 2pI_p + 2\Big(r - bqI_q - q \times \text{int}\Big(\frac{r - bqI_q}{q}\Big)\Big) \\
&\geq (p-1)(q-1) - 2pI_p \\
&> (p-1)(q-1) - 2p\frac{q}{v} \quad \Big(\text{since } I_p < \frac{q}{v}\Big) \\
&> (p-1)(q-1) - \frac{1}{2}pq \quad (\text{since } v \geq 4 \text{ if a GYD exists}) \\
&> 0. \quad (\text{since } p, q > 4 \text{ if a GYD exists})
\end{aligned}$$

Next we show that $\tilde{d}$ has an stronger optimality property when the number of rows equals the number of columns, i.e., $p = q$. We let $N = \{n : 0 \leq n \leq bpq,\ n = tp,\ t \text{ integer},\ b > 1\}$, and $M = \{n \in N : n \leq \frac{bpq}{2}\}$.

Definition 4.1. (Kiefer (1975)). If $C, D \in N$, $C < D$, and no integer between C and D is in N, then $[C, D]$ is called an elementary interval. Also, the elementary interval $[C_0, D_0]$ containing $\bar{r} = \frac{bpq}{v}$ is called the basic interval.

When $p = q = s$ (say), it is straightforward to show that $g(r)$ in Equation (4.6) has all of the following properties which were given by Kiefer (1975).

(i) For each elementary interval $[C, D]$, $\Delta(r)$ is linear in r and increasing for $C \leq r < D$, i.e., g is a convex quadratic on each elementary interval.

(ii) g is increasing in each elementary interval $[C, D]$ with $D \leq D_0$.

(iii) g is symmetric about $\frac{bpq}{2}$.

(iv) If $C_1, C_2 \in N$ and $C_1 < C_2$, then $\Delta(C_1) \geq \Delta(C_2)$ and $\Delta(C_1 - 1) \geq \Delta(C_2 - 1)$.

(v) g is nondecreasing on M and is nonincreasing on $N\backslash M$.

By the same argument as in Equation (3.17) of Kiefer (1975), $\tilde{d}$ is Φ_a-optimal if

$$\frac{d^2(g(r))^{-a}}{dr^2} \geq 0, \quad C_0 < r < D_0.$$

Let $I_0 = \text{int}(\frac{C_0}{bs})$, which equals $\text{int}(\frac{r}{bs})$ when $C_0 \leq r < D_0$. Then

$$\begin{aligned} g(r) &= s^2 r - 2s(-bsI_0^2 + 2rI_0 - bsI_0 + r) + \text{int}\left(\frac{r - bsI_0}{s}\right)(s(I_0+1))^2 \\ &+ \left(b - 1 - \text{int}\left(\frac{r - bsI_0}{s}\right)(sI_0)^2\right) + \left(sI_0 + r - sbI_0 - s \times \text{int}\left(\frac{r - bsI_0}{s}\right)\right)^2 \\ &= r^2 + \alpha_0 r + \beta_0, \end{aligned}$$

where

$$\begin{aligned} \alpha_0 &= s^2 - 2s - 2sI_0 - 2sbI_0 - 2s \times \text{int}\left(\frac{r - bsI_0}{s}\right), \\ \beta_0 &= 2bs^2 I_0^2 + 2bs^2 I_0 + \text{int}\left(\frac{r - bsI_0}{s}\right)(s(I_0+1))^2 \\ &+ \left(b - 1 - \text{int}\left(\frac{r - bsI_0}{s}\right)(sI_0)^2\right) + \left(sI_0 - sbI_0 - s \times \text{int}\left(\frac{r - bsI_0}{s}\right)\right)^2. \end{aligned}$$

Notice that α_0, β_0 are constants when $r \in (C_0, D_0)$. Let $\Gamma(r) = (g(r))^{-a}$. Then

$$\begin{aligned} &\frac{d^2\Gamma(r)}{dr^2} \geq 0 \quad \forall\, r \in (C_0, D_0) \\ \Longleftrightarrow\ & a(a+1)(g(r))^{-a-2}(g'(r))^2 - g''(r)a(g(r))^{-a-1} \geq 0 \qquad \forall\, r \in (C_0, D_0) \\ \Longleftrightarrow\ & (a+1)(g'(r))^2 - 2g(r) \geq 0 \\ \Longleftrightarrow\ & (a+1)(2r + \alpha_0)^2 - 2(r^2 + \alpha_0 r + \beta_0) \geq 0 \\ \Longleftrightarrow\ & a \geq \frac{2(r^2 + \alpha_0 r + \beta_0)}{(2r + \alpha_0)^2} - 1. \qquad \forall\, r \in (C_0, D_0) \end{aligned}$$

Theorem 4.2. *$\tilde{d}$ is a Φ_a-optimal NRC if*

$$a \geq \max_{\{r : r \in (C_0, D_0)\}} \frac{2(r^2 + \alpha_0 r + \beta_0)}{(2r + \alpha_0)^2} - 1.$$

Example 4.1. Let $\tilde{d} = d^* \cup d^\#$, where $d^* \in \Xi_{4,6,6,6}$ is a universally optimal NRC (easy to construct) and $d^\#$ is a GYD. Then $\tilde{d} \in \Xi_{4,6,6,7}$ is a Φ_a-optimal design when a is (approximately) greater than 4.5. Hence, $\tilde{d}$ is E-optimal in $\Xi_{4,6,6,7}$. However, $\tilde{d}$ is not A-optimal. Let $\bar{d} = \{d^*, d'\}$, where d' is

1	4	2	4	3	2
2	1	4	3	3	4
2	3	1	3	4	2
4	3	3	1	2	4
4	2	4	2	1	3
3	2	3	4	2	1

The nonzero eigenvalues of $C_{\tilde{d}}$ are $\lambda_{\tilde{d}} = (58.67, 61.67, 61.67)$. And, the nonzero eigenvalues of $C_{\bar{d}}$ are $\lambda_{\bar{d}} = (60.33, 60.33, 60.33)$. Thus, $\bar{d}$ is superior to $\tilde{d}$ with respect to A- and D-optimality.

5. Discussion

Given v, p, and q, we have been able to find universally optimal designs among NRC's with $v!$ blocks. When v is large, such designs involve a relatively large number of blocks, and, in turn, have limited use in practice. However, requiring a large number of blocks is a common drawback in the area of constructing NRC's (John (1987, p.113)). For small numbers of blocks, repetition of a nonregular GYD, if possible, is highly efficient, if not optimal. In order to reduce the number of blocks in the optimal designs above, an appropriate way to arrange the experimental units is desired. We have presented one way to obtain some reduction, but often one must resort to trial and error to obtain an initial design whose information matrix has the form given in (3.1).

When the number of rows and columns within blocks does not allow for a Latin square or a regular GYD, optimal designs are somewhat asymmetric in blocks, although symmetric overall. Such designs, however, may not be very robust against losses of several blocks. As a result, designs which are more balanced in blocks, such as repeating a non-regular GYD, may be preferable. Such designs are not optimal but they are highly efficient.

Finally, if the numbers of rows and columns within blocks are equal and allow for a nonregular GYD, we have shown that a design which has completely symmetric information matrix but does not have maximum trace is Φ_a-optimal for some a.

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