Statistica Sinica 3(1993), 197-207

ESTIMABILITY AND EFFICIENCY IN NEARLY ORTHOGONAL $2^{m_1} \times 3^{m_2}$ DELETION DESIGNS

Joan M. Mahoney and John E. Angus

University of California and Claremont Graduate School

Abstract: This article considers single replicate factorial experimental designs in incomplete blocks. A single replicate $2^{m_1} \times 3^{m_2}$ deletion design in 3 incomplete blocks is obtained from a single replicate $3^m (m = m_1 + m_2)$ preliminary design by deleting all runs (or treatment combinations) with the first m_1 factors at level two. A systematic method for determining the unbiasedly estimable (u.e.) and not-unbiasedly estimable (n.u.e.) factorial effects is provided. Specifically, it is shown that, for $m_2 > 0$, all factorial effects of the form $F(\alpha_1 \dots \alpha_{m_1}, \alpha_{m_1+1} \dots \alpha_m)$, where $\alpha_i = 0, 1$ for $i = 1, \dots, m_1$, $\alpha_i = 0, 1, 2$, for $i = m_1 + 1, \dots, m$, with $(\alpha_1 \dots \alpha_m) \neq (0 \dots 0)$, and $(\alpha_{m_1+1} \dots \alpha_m) \neq \alpha(1 \dots 1)$ for $\alpha = 1, 2$, are u.e. and the remaining effects are n.u.e. The result that identifies u.e. factorial effects is derived as a special case of a theorem developed herein for general deletion designs obtained from a single replicate p^m preliminary design. It is noted that $(2^{m_1} - 1)$ factorial effects of 2^{m_1} factorial experiments, and $(3^{m_2} - 3)$ factorial effects of 3^{m_2} factorial experiments, which are embedded in $2^{m_1} \times 3^{m_2}$ experiments, are u.e. The $2 \times 3^{m-1}$ deletion designs were considered in the work by Voss (1986). By defining the single-degree-of-freedom components $F(\alpha_1 \dots \alpha_m)$ of the factorial effects of a $2^{m_1} \times 3^{m_2}$ factorial experiment in a form different from that of Voss (1986), our simple representation of u.e. and n.u.e. effects identifies more u.e. effects than is done in the representation by Voss (1986). The relative efficiency expressions, and their bounds, in the estimation of factorial effects of $2^{m_1} \times 3^{m_2}$ deletion designs are also given, along with methods for adjusting n.u.e. effects to be u.e. when certain higher order effects are assumed negligible.

Key words and phrases: Confounding, factorial experiment, single replicate, unbiasedly estimable.

1. Introduction

There is a vast literature on the construction of single replicate asymmetric factorial designs in incomplete blocks. The reader is referred to Voss (1986) and Raktoe, Hedayat and Federer (1981) for the list of references. The concept of deletion designs was introduced in Kishen and Srivastava (1959). The deletion technique in deletion designs was then used by many authors, among them Addelman (1962, 1972) and Voss (1986). This article considers $2^{m_1} \times 3^{m_2}$ deletion designs in three incomplete blocks and presents a systematic method for classifying all the factorial effects as either u.e. or n.u.e. Recently Chauhan (1988) has considered $(q-a)^{m_1} \times q^{m_2}$ deletion designs in $m = m_1 + m_2$ blocks, and obtained results for determining unbiased estimability of some (but not all) factorial effects (see discussion in Section 5). However, by considering the particularly useful special case $q - a = 2$ and $q = 3$, we are able to classify all the factorial effects as either u.e. or n.u.e., a much stronger result than that of Chauhan (1988). While the smaller values of m_1 and m_2 are the most practically important cases, we do not consider the case $m_2 = 0$, since, there, the blocks are of unequal sizes and main effects are confounded. This work is based on Mahoney (1988) and Ghosh and Mahoney (1988), where several generalizations are discussed.

The model assumed is the linear fixed effects model. A factorial effect is estimable if, and only if, it can unbiasedly estimated with a linear combination of the observations. An unadjusted estimator of a factorial effect is simply the factorial effect with the treatment effects replaced by the observed response at the corresponding treatment combination. The unadjusted estimators can be unbiased or biased. When they are biased, then under the assumption that certain higher order factorial effects are negligible, it is possible to adjust them to be unbiased in minimum variance fashion. The unbiased estimators of factorial effects obtained in this fashion are called adjusted estimators.

The relative efficiency in the estimation of a factorial effect is the ratio of the variance of its unadjusted estimator to its adjusted estimator. Under the assumption that certain higher order effects are negligible, the relative efficiency considered in this paper is identical to the standard efficiency factor. (See John (1987), Equation (2.1), page 24.) If the unadjusted estimator is unbiased there is no need for adjustment, and hence the relative efficiency is unity. Otherwise, the relative efficiency is less than unity, and the closer the relative efficiency to unity, the lesser the effect of adjustment on the variance of the estimator.

For the general definition of estimable parametric functions, the reader is referred to Scheffe (1959), page 13, and Lehmann (1983), page 75. In this paper the parametric functions are factorial effects and contrasts of block effects. Definitions of factorial effects and deletion designs are given in Section 2. An orthogonal block design is a block design that has the property that the least squares estimators of all factorial effects are not only orthogonal to each other but also orthogonal to the least squares estimators of a complete orthogonal set of block effect contrasts. (See Raktoe, Hedayat and Federer (1981), Definition 8.1, page 102.) For a single replicate factorial design in incomplete blocks, an orthogonal design does not exist. It is, however, observed in Section 4, under the assumption that two of the highest order factorial effects are zero, that the deletion designs are nearly orthogonal. Section 3 presents the systematic method (Theorem 2) of determining which factorial effects are unbiasedly estimable (u.e.) by their unadjusted estimators. Determination of the u.e. effects is accomplished as a special case of a general theorem proved in Section 3 (Theorem 1) that identifies estimable contrasts related to factorial effects for general deletion designs derived from a preliminary p^m design. Section 4 discusses the relative efficiency with an illustrative example.

2. Definitions and Notation

Consider a single replicate $2^{m_1} \times 3^{m_2}$ factorial experiment in incomplete blocks. There are m factors ($m = m_1 + m_2$) in the experiment. Runs and their effects are denoted by the same notation $(x_1 \ldots x_{m_1}, x_{m_1+1} \cdots x_m)$, where $x_i = 0, 1$ for $1 \le i \le m_1$, and $x_i = 0, 1, 2$ for $m_1 + 1 \le i \le m$. The observation on the run $(x_1 \ldots x_m)$ is denoted by $y(x_1 \ldots x_m)$. With this notation, the model can be written as

$$
\begin{aligned}
Ey(x_1 \ldots x_m) &= (x_1 \ldots x_m) + \beta_j \\
\operatorname{Var} y(x_1 \ldots x_m) &= \sigma^2 > 0 \\
\operatorname{Cov}\Big(y(x_1 \ldots x_m), y(x_1' \ldots x_m')\Big) &= 0,
\end{aligned} \tag{1}
$$

where β_j is the effect of the jth block containing the run $(x_1 \ldots x_m)$, and $(x_1 \ldots x_m) \neq (x_1' \ldots x_m')$. The model assumed is equivalent to the linear fixed effects model:

$$Y_{(n\times 1)} = \tau_{(n\times 1)} + N_{(n\times k)} B_{(k\times 1)} + \varepsilon_{(n\times 1)}.$$

Here, $\tau = (\tau_x)$ is the vector of run or treatment effects, where $x = (x_1 \ldots x_m)$ is a treatment combination and is ordered in lexicographical order, with x_i being the level of factor i. The matrix N is the incidence matrix. That is, $N = (\delta_{x,h})$ where $\delta_{x,h} = 1$ if treatment combination x appears in block h, and is 0 otherwise.

Factorial effects are denoted by $F(\alpha_1 \ldots \alpha_{m_1}, \alpha_{m_1+1} \cdots \alpha_m), \alpha_i = 0, 1$ for $1 \le i \le m_1$, and $\alpha_i = 0, 1, 2$ for $m_1 + 1 \le i \le m$. (This notation is equivalent to $F_1^{\alpha_1} \cdots F_m^{\alpha_m}$, which is only used in examples.) A factorial effect is a contrast in τ, $c^t\tau$, where $c^t\underline{1} = 0$ and $\underline{1}(\underline{0})$ denotes a column vector of 1s (0s) whose dimension will be clear from context. The factorial effect $c^t\tau$ is estimable if, and only if, $c^tN = \underline{0}$. This result, due to Dean (1978), simply states that if $c^tN = \underline{0}$ (i.e., the block effects cancel one another), then c^tY is an unbiased estimator of $c^t\tau$ which, by the Gauss-Markov theorem, is the Best Linear Unbiased Estimator (BLUE). In general, the estimator c^tY of the factorial effect $c^t\tau$ will be called the *unadjusted estimator* of the effect, and will be denoted by $\widehat{c^t\tau}$. When $c^tN \neq \underline{0}$, then $E(\widehat{c^t\tau}) = c^t\tau + c^tNB$. For our problem, each row of N contains exactly one entry equal to 1 and the rest equal to 0. Hence, $c^tN\underline{1} = 0$, so $\psi^t = c^tN$ is a contrast. Now, if e_i are vectors and $e_1^t\tau = e_2^t\tau = \cdots = e_r^t\tau = 0$ are negligible factorial effects, $e_i^tc = 0$, $i = 1, \ldots, r$, $e_i^te_j = 0$, $i \neq j$, and $E(\widehat{e_i^t\tau}) = \psi^tB$,

$i = 1, \cdots, r$, then it is possible to adjust $c^t\hat{\tau}$ to be unbiased in minimum variance fashion, by subtracting a linear combination of the $e_i^t\hat{\tau}$, $i = 1, \ldots, r$. The resulting estimator, denoted by $(c^t\hat{\tau})_{\text{adj}}$, is given by

$$(c^t\hat{\tau})_{\text{adj}} = c^t\hat{\tau} - \sum_{i=1}^{r} w_i\left(e_i^t\hat{\tau}\right), \text{ where } w_i = \frac{\left(e_i^t e_i\right)^{-1}}{\sum_{i=1}^{r}\left(e_i^t e_i\right)^{-1}}, \quad i = 1, \ldots, r. \tag{2}$$

The notation $\{\alpha_1 x_1 + \cdots + \alpha_{m_1} x_{m_1} = u_1\}$ represents the sum of all points $(x_1 \ldots x_{m_1})$ which are solutions of $\alpha_1 x_1 + \cdots + \alpha_{m_1} x_{m_1} = u_1$ over the Galois Field GF(2), $u_1 = 0, 1$. Note that it is the sum of all points $(x_1 \ldots x_{m_1})$, $x_i \in \{0, 1\}$, if $(\alpha_1 \ldots \alpha_{m_1}) = (0 \ldots 0)$ and $u_1 = 0$ and is taken to be 0 if $(\alpha_1 \ldots \alpha_{m_1}) = (0 \ldots 0)$ and $u_1 = 1$. Similarly, the notation $\{\alpha_{m_1+1} x_{m_1+1} + \cdots + \alpha_m x_m = u_2\}$ represents the sum of all points $(x_{m_1+1} \ldots x_m)$ which are solutions of $\alpha_{m_1+1} x_{m_1+1} + \cdots + \alpha_m x_m = u_2$ over the Galois Field GF(3), $u_2 = 0, 1, 2$, with the same conventions involving the case $(\alpha_{m_1+1} \ldots \alpha_m) = (0 \ldots 0)$.

The "$\times$" product of $\{\alpha_1 x_1 + \cdots + \alpha_{m_1} x_{m_1} = u_1\}$ and $\{\alpha_{m_1+1} x_{m_1+1} + \cdots + \alpha_m x_m = u_2\}$ is denoted by:

$$\left\{\alpha_1 x_1 + \cdots + \alpha_{m_1} x_{m_1} = u_1\right\} \times \left\{\alpha_{m_1+1} x_{m_1+1} + \cdots + \alpha_m x_m = u_2\right\}.$$

It represents the sum of all run effects $(x_1 \ldots x_{m_1}, x_{m_1+1} \ldots x_m)$ where $(x_1 \ldots x_{m_1})$ is a solution of $\alpha_1 x_1 + \cdots + \alpha_{m_1} = u_1$ over GF(2) and $(x_{m_1+1} \ldots x_m)$ is a solution of $\alpha_{m_1+1} x_{m_1+1} + \cdots + \alpha_m x_m = u_2$ over GF(3). To handle the cases $(\alpha_1 \ldots \alpha_{m_1}) = (0 \ldots 0)$ and/or $(\alpha_{m_1+1} \ldots \alpha_m) = (0 \ldots 0)$, make the convention that $A \times B = 0$ if either $A = 0$ or $B = 0$.

The factorial effects of a $2^{m_1} \times 3^{m_2}$ factorial experiment are defined as contrasts of run effects as follows. Let $\underline{\alpha} = (\alpha_1 \ldots \alpha_{m_1}, \alpha_{m_1+1} \ldots \alpha_m)$, and define row vectors $u(\underline{\alpha})$ and $v(\underline{\alpha})$ as

$$u(\underline{\alpha}) = \begin{cases} (1,1,1,1,1), \text{ if } \alpha_1 = \alpha_2 = \cdots = \alpha_{m_1} = 0 \\ (-1,1,1,1,1), \text{ otherwise;} \end{cases}$$

$$v(\underline{\alpha}) = \begin{cases} (1,1,1,1,1), \text{ if } \alpha_{m_1+1} = \cdots = \alpha_m = 0 \\ (1,1,-1,0,1), \text{ if the first non zero value in } (\alpha_{m_1+1} \ldots \alpha_m) \text{ is } 1 \\ (1,1,1,-2,1), \text{ if the first non zero value in } (\alpha_{m_1+1} \ldots \alpha_m) \text{ is } 2. \end{cases}$$

Finally, define the coefficients $(c_0, c_1, d_0, d_1, d_2) \equiv u(\underline{\alpha})v(\underline{\alpha})$ where the product of u and v is taken coordinate by coordinate. Here, the dependence of c_i and d_j on $\underline{\alpha}$ is suppressed, but will be clear from context. A factorial effect is now defined as

$$F(\underline{\alpha}) = \left[\sum_{i=0}^{1} c_i\Big\{\alpha_1 x_1 + \cdots + \alpha_{m_1} x_{m_1} = i\Big\}\right] \times\left[\sum_{i=0}^{2} d_i\Big\{\alpha_{m_1+1} x_{m_1+1} + \cdots + \alpha_m x_m = i\Big\}\right]. \qquad (3)$$

Example 1. Consider a $2^2 \times 3^2$ factorial experiment. We have $m_1 = 2, m_2 = 2$, and $m = 4$. The notation $\{x_1+x_2 = 0\}$ represents (00)+(11); the notation $\{x_3+2x_4 = 1\}$ represents (10)+(02)+(21). Finally, the notation $\{x_1+x_2 = 0\}\times\{x_3+2x_4 = 1\}$ represents (0010) + (0002) + (0021) + (1110) + (1102) + (1121). The factorial effect $F(0120) = F_2F_3^2 = [-\{x_2 = 0\} + \{x_2 = 1\}] \times [\{x_3 = 0\} - 2\{x_3 = 1\} + \{x_3 = 2\}] = [-(00) - (10) + (01) + (11)] \times [(00) + (01) + (02) - 2(10) - 2(11) - 2(12) + (20) + (21) + (22)] = -(0000) - \cdots + 2(0010) + \cdots - (0020) - \cdots + (1122)$.

A $2^{m_1} \times 3^{m_2}$ deletion design D in three incomplete blocks is described as follows. Consider a 3^m $(m = m_1 + m_2)$ factorial experiment in 3 blocks by confounding the two degrees of freedom in $F(11\ldots1)$ and $F(22\ldots2)$. The block u consists of runs which are solutions over GF(3) of the equation $x_1+\cdots+x_m = u$ for $u = 0, 1, 2$. From every block, the runs with level 2 for the first m_1 factors are deleted. The resulting design is D with $2^{m_1} \times 3^{m_2-1}$ runs in every block. It is assumed that $m_2 \geq 2$.

Example 2. The runs in the three blocks of a $2^2 \times 3^2$ deletion design D are given by:
Block 0: 0000, 0012, 0021, 1020, 1002, 1011, 0120, 0102, 0111, 1110, 1101, 1122
Block 1: 0010, 0001, 0022, 1000, 1012, 1021, 0100, 0112, 0121, 1120, 1102, 1111
Block 2: 0020, 0002, 0011, 1010, 1001, 1022, 0110, 0101, 0122, 1100, 1112, 1121.

The unadjusted estimators of factorial effects $F(\alpha_1 \ldots \alpha_{m_1}, \alpha_{m_1+1} \ldots \alpha_m)$ are obtained by replacing the run effect $(x_1 \ldots x_m)$ with the observation $y(x_1 \ldots x_m)$ in (3), and are denoted by $\hat{F}(\alpha_1 \ldots \alpha_{m_1}, \alpha_{m_1+1} \ldots \alpha_m)$.

Let B_u $(u = 0, 1, 2)$ be the sum of all run effects in the uth block. Let $X = -B_1 + B_2$ and $Y = 2B_0 - B_1 - B_2$. Clearly, X and Y are confounded with blocks in D.

Let $B_u(\alpha_1 x_1 + \cdots + \alpha_{m_1} x_{m_1} = i)$, for $i = 0, 1$, and $u = 0, 1, 2$, denote the sum of all run effects satisfying the equation $\alpha_1 x_1 + \cdots + \alpha_{m_1} x_{m_1} = i$ over GF(2) in the uth block. Consistent with previous conventions, $B_u(\alpha_1 x_1 + \cdots + \alpha_{m_1} x_{m_1} = i) = B_u$ if $(\alpha_1 \ldots \alpha_{m_1}) = (0 \ldots 0)$ and $i = 0$, and is 0 otherwise, so that in general, $B_u = \sum_{i=0}^{1} B_u(\alpha_1 x_1 + \cdots + \alpha_{m_1} x_{m_1} = i)$.

Now, define the following linear combinations of factorial effects:

$$\begin{aligned}&F(\alpha_1 \ldots \alpha_{m_1})X \\ = \; & -\Big[B_1(\alpha_1 x_1 + \cdots + \alpha_{m_1} x_{m_1} = 1) - B_1(\alpha_1 x_1 + \cdots + \alpha_{m_1} x_{m_1} = 0)\Big] \\ & +\Big[B_2(\alpha_1 x_1 + \cdots + \alpha_{m_1} x_{m_1} = 1) - B_2(\alpha_1 x_1 + \cdots + \alpha_{m_1} x_{m_1} = 0)\Big], \quad \text{(4a)}\end{aligned}$$

$$\begin{aligned}&F(\alpha_1 \ldots \alpha_{m_1})Y \\ = \; & 2\Big[B_0(\alpha_1 x_1 + \cdots + \alpha_{m_1} x_{m_1} = 1) - B_0(\alpha_1 x_1 + \cdots + \alpha_{m_1} x_{m_1} = 0)\Big] \\ & -\Big[B_1(\alpha_1 x_1 + \cdots + \alpha_{m_1} x_{m_1} = 1) - B_1(\alpha_1 x_1 + \cdots + \alpha_{m_1} x_{m_1} = 0)\Big] \\ & -\Big[B_2(\alpha_1 x_1 + \cdots + \alpha_{m_1} x_{m_1} = 1) - B_2(\alpha_1 x_1 + \cdots + \alpha_{m_1} x_{m_1} = 0)\Big]. \quad \text{(4b)}\end{aligned}$$

3. Properties

Consider, for the moment, a preliminary p^m (p a prime or prime power) design in p blocks with the hth block containing the runs satisfying $\sum_{i=1}^m x_i = h$ over GF(p). Now, delete the treatment combinations having any of the q_i levels of the ith factor, $i = 1, \ldots, s$, $1 \le q_i \le p-2$, yielding a $(p-q_1) \times \cdots \times (p-q_s) \times p^{m-s}$ deletion design. Call this design D^G, to distinguish it from the previously defined deletion design D. The following general theorem identifies u.e. contrasts of the run effects in D^G.

Theorem 1. *Assume the design is D^G. Let $\alpha_{s+1}, \ldots, \alpha_m$ be elements of $\{0, 1, \ldots, p-1\}$ such that for some $i < j$, $\alpha_i \ne \alpha_j$. Then $(a \otimes c)^t \tau$ is* u.e. *by $(a \otimes c)^t Y$, where a is any $(p-q_1) \ldots (p-q_s)$ by one, nonzero vector, $c = (c_{x_{s+1} \ldots x_m})$ is any p^{m-s} by one contrast vector such that the component $c_{x_{s+1} \ldots x_m}$ depends only on the value of $\sum_{i=s+1}^m \alpha_i x_i$, and $\otimes$ is the Kronecker product.*

Proof. If $\alpha_i = 0$ for some $i > s$, the result follows from Theorem 2 of Chauhan (1988). Assume, then, that $\alpha_i \ne 0$ for all $i > s$. For each subtreatment combination $x_1 \ldots x_s (0 \le x_i \le p - q_i)$, each block $B_h = \{x_1 \ldots x_m : \sum_{i=1}^m x_i = h$ over GF(p)$\}$ of D^G, and each j in $\{0, 1, \ldots, p-1\}$, the block contains exactly p^{m-s-2} treatment combinations which simultaneously have $x_1 \ldots x_s$ as a subtreatment combination and satisfy $\sum_{i=s+1}^m \alpha_i x_i = j$ over GF(p). To see this, without loss of generality, suppose that $\alpha_{m-1} \ne \alpha_m$. Then for each possible subtreatment combination $x_1 \ldots x_{m-2}$ in D^G, and for fixed j and h, it follows from the properties of GF(p) that there exists a unique 2-tuple $x_{m-1} x_m$ such that $x_1 \ldots x_{m-2} x_{m-1} x_m$ satisfies both $\sum_{i=s+1}^m \alpha_i x_i = j$ and $\sum_{i=1}^m x_i = h$. The block effects in $E((a \otimes c)^t Y)$ therefore cancel, and the result follows.

We now consider the $2^{m_1} \times 3^{m_2}$ deletion design D previously defined. We say that a factorial effect in D is unbiasedly estimable (u.e.) if its unadjusted estimator is unbiased, and that it is not unbiasedly estimable (n.u.e) otherwise. (Note that we are not at this point assuming that any factorial effects are negligible.) The following theorem allows one to classify all the factorial effects of D as u.e. or n.u.e. It is assumed that $m_2 \geq 1$.

Theorem 2. *If* $(\alpha_1 \ldots \alpha_m) \neq (0 \ldots 0)$, *the factorial effects* $F(\alpha_1 \ldots \alpha_{m_1}, \alpha_{m_1+1} \ldots \alpha_m)$ *are:*
(i) u.e. *under* D *for* $(\alpha_{m_1+1} \ldots \alpha_m) \neq \alpha(1 \ldots 1),\ \alpha = 1, 2;$
(ii) n.u.e *under* D *for* $(\alpha_{m_1+1} \ldots \alpha_m) = \alpha(1 \ldots 1),\ \alpha = 1, 2.$

Proof. Part (i) follows from Theorem 1 and the definition of the factorial effects in D given by (3).

For the proof of part (ii), if $(\alpha_1 \ldots \alpha_{m_1}) = (0 \ldots 0)$, then the result follows from the fact that the grand total of run effects in the embedded 2^{m_1} design is n.u.e. Next, if $(\alpha_1 \ldots \alpha_{m_1}) \neq (0 \ldots 0)$, consider a fixed block, block u say. Then the runs in that block satisfy $x_1 + \cdots + x_m = u$ over GF(3), and those runs can be divided into six mutually exclusive sets of runs that satisfy

(*) $\alpha_1 x_1 + \cdots + \alpha_{m_1} x_{m_1} = i$ over GF(2),

(**) $\alpha_{m_1+1} x_{m_1+1} + \cdots + \alpha_m x_m = j$ over GF(3),

for $i = 0, 1$, and $j = 0, 1, 2$. The runs in each of these six sets therefore satisfy three linear equations. Suppose that $(\alpha_{m_1+1} \ldots \alpha_m) = \alpha(1 \ldots 1)$ for either $\alpha = 1$ or 2. Then (**) and the linear equation defining block u can be combined to form a new linear equation, placing another constraint on $x_1 \ldots x_{m_1}$, in addition to (*). In this case, the six sets of runs have different numbers of elements, and the net block effect will not cancel from $E(\hat{F}(\alpha_1 \ldots \alpha_{m_1}, \alpha_{m_1+1} \ldots \alpha_m))$. This completes the proof of (ii).

Example 3. In example 2, the following factorial effects are n.u.e. in addition to the general mean $\mu = \underline{1}^t \tau$: F_3F_4, $F_3^2F_4^2$, $F_1F_3F_4$, $F_2F_3F_4$, $F_1F_2F_3F_4$, $F_1F_3^2F_4^2$, $F_2F_3^2F_4^2$, $F_1F_2F_3^2F_4^2$. The other factorial effects are u.e.

Corollary 1. *Under* $D, F(\alpha_1 \ldots \alpha_{m_1})X$ *and* $F(\alpha_1 \ldots \alpha_{m_1})Y$ *with* $(\alpha_1 \ldots \alpha_{m_1}) \neq (0 \ldots 0)$, *defined in* (4), *are* u.e.

Proof. By Theorem 2, all effects $F(\alpha_1 \ldots \alpha_{m_1}, 0 \ldots 0)$ are u.e., so each run subcombination $(x_1 \ldots x_{m_1})$ occurs equally often in each block. It now follows from (4) that the block effects cancel in the unadjusted estimators of the effects defined by (4).

Observe that μ (the general mean), X and Y are confounded with blocks in D. The $(2^{m_1}(3^{m_2} - 2) - 1)$ factorial effects $F(\alpha_1 \ldots \alpha_{m_1}, \alpha_{m_1+1} \ldots \alpha_m)$ with $(\alpha_{m_1+1} \ldots \alpha_m) \neq \alpha(1 \ldots 1)$, $\alpha = 1, 2$, and $(\alpha_1 \ldots \alpha_m) \neq (0 \ldots 0)$, are u.e. under

D. The $(2^{m_1}-1)2$ linear functions of factorial effects $F(\alpha_1 \ldots \alpha_{m_1})X$ and $F(\alpha_1 \ldots \alpha_{m_1})Y$ with $(\alpha_1 \ldots \alpha_{m_1}) \neq (0 \ldots 0)$, are u.e. under D. Thus, we have $[3+(2^{m_1}(3^{m_2}-2)-1)+(2^{m_1}-1)2] = 2^{m_1} \times 3^{m_2}$ linear functions of factorial effects which are also orthogonal linear functions of run effects (orthogonality follows from the definitions (3) and (4)).

4. Relative Efficiency

When certain higher order factorial effects are assumed negligible, then n.u.e. factorial effects become estimable through adjustment as discussed earlier. In this section the relative efficiencies of adjusted estimators of n.u.e. factorial effects are calculated. It can be easily verified that for $\alpha = 1$ or 2,

$$\operatorname{Var}\Big(\hat{F}(\alpha_1 \ldots \alpha_{m_1}, \alpha \ldots \alpha)\Big) = \sigma^2 2^{m_1+1} 3^{m_2+\alpha-2}. \tag{5}$$

Let $S = wt(\alpha_1 \ldots \alpha_{m_1})$ be the number of non zero elements in $(\alpha_1 \ldots \alpha_{m_1})$, $B^t = (\beta_0, \beta_1, \beta_2)$ the block effects, and $[\,\bullet\,]$ the usual greatest integer function. It follows after straightforward, but tedious, computations that for $\alpha = 1$ or 2,

$$E\Big(\hat{F}(\alpha_1 \ldots \alpha_{m_1}, \alpha \ldots \alpha)\Big) = F(\alpha_1 \ldots \alpha_{m_1}, \alpha \ldots \alpha) + 3^{m_2-1+[(S+\alpha-1)/2]} \beta_{S,\alpha}(B). \tag{6}$$

In (6), the absolute value of the term $\beta_{S,\alpha}(B)$ is

$$\Big|\beta_{S,\alpha}(B)\Big| = \Big|\Big\{Q_{S,\alpha}(1,-2,1) + (1-Q_{S,\alpha})(-1,0,1)\Big\} P_{m_1 \operatorname{mod}(3)} B\Big| \tag{7}$$

where $Q_{S,\alpha} = (S+\alpha-1)\operatorname{mod}(2)$, and P_0, P_1 and P_2 are the permutation matrices

$$P_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

It follows from (6) and (7) that if $F(1 \ldots 1, 1 \ldots 1)$ and $F(1 \ldots 1, 2 \ldots 2)$ are zero, the bias in the unadjusted estimator of $F(\alpha_1 \ldots \alpha_{m_1}, \alpha \ldots \alpha)$, $(\alpha_1 \ldots \alpha_{m_1}) \neq (1 \ldots 1)$, can be removed by subtracting from it a constant multiple of the unadjusted estimator of either (but not both) $F(1 \ldots 1, 1 \ldots 1)$ or $F(1 \ldots 1, 2 \ldots 2)$. Whether $\hat{F}(1 \ldots 1, 1 \ldots 1)$ or $\hat{F}(1 \ldots 1, 2 \ldots 2)$ is used depends on whether or not $(S - m_1) = 0 \operatorname{mod}(2)$. The adjusted estimator will thus be of the form

$$\hat{F}(\alpha_1 \ldots \alpha_{m_1}, \alpha \ldots \alpha)_{\text{adj}} = \hat{F}(\alpha_1 \ldots \alpha_{m_1}, \alpha \ldots \alpha) + w\hat{F}(1 \ldots 1, \gamma \ldots \gamma) \tag{8}$$

where $\gamma = 1$ or 2. Thus, using (5), the relative efficiency in the estimation of $F(\alpha_1 \ldots \alpha_{m_1}, \alpha \ldots \alpha)$, $(\alpha_1 \ldots \alpha_{m_1}) \neq (1 \ldots 1)$, $\alpha = 1, 2$, is

$$\text{RE} = \frac{\operatorname{Var} \hat{F}(\alpha_1 \ldots \alpha_{m_1}, \alpha \ldots \alpha)}{\operatorname{Var} \hat{F}(\alpha_1 \ldots \alpha_{m_1}, \alpha \ldots \alpha)_{\text{adj}}} = \frac{1}{1 + w^2 3^{(\gamma-\alpha)}}. \tag{9}$$

It is seen from (6) and (7) by considering the two cases $(m_1 - S) = 0, 1 \bmod(2)$ that

$$w = 3^{(S-m_1+\alpha-\gamma)/2}. \tag{10}$$

Using (10) in (9), it follows that in every case,

$$3/4 \leq \text{RE} = \frac{3^{m_1}}{\left(3^{m_1} + 3^S\right)} \leq \frac{3^{m_1}}{\left(3^{m_1} + 1\right)}. \tag{11}$$

It is noted that under the assumption that $F(1\ldots1, \alpha\ldots\alpha), \alpha = 1$ and 2, are negligible, the RE in (11) is the standard relative efficiency or the efficiency factor for the factorial effect $F(\alpha_1 \ldots \alpha_{m_1}, \alpha \ldots \alpha)$, $(\alpha_1 \ldots \alpha_{m_1}) \neq (1 \ldots 1)$, $\alpha = 1, 2$. (See John (1987), Equation (2.1) on page 24.)

We thus observe that under the assumption that $F(1\ldots1, \alpha\ldots\alpha)$, $\alpha = 1, 2$, are negligible, all the factorial effects (except the general mean) are estimable in these deletion designs. Furthermore, the unbiased estimators which are unadjusted are mutually orthogonal and also orthogonal to the unbiased estimators which are adjusted. Pairs of unbiased estimators which have been adjusted are orthogonal when they are adjusted with orthogonal bias corrections. Hence, the deletion design is a nearly orthogonal design under the aforementioned negligibility assumptions.

Example 4. In Example 2, $m_1 = 2$. Assume that $F_1F_2F_3F_4$ and $F_1F_2F_3^2F_4^2$ are zero. For the factorial effects F_3F_4 and $F_3^2F_4^2$, we have $S = 0$. For the factorial effects $F_iF_3F_4$ and $F_iF_3^2F_4^2$, $i = 1, 2$, we have $S = 1$. The REs for estimating F_3F_4 and $F_3^2F_4^2$ attain the maximum value .90. The REs for estimating $F_iF_3F_4$ and $F_iF_3^2F_4^2, i = 1, 2$, attain the value .75. There are $(2^23^2 - 1 - 2) = 33$ factorial effects, excluding the general mean and the two negligible factorial effects. Thus, out of these 33 factorial effects, all but 4 factorial effects attain the maximum values of RE, 1 (for u.e. factorial effects) or .90 (for n.u.e. factorial effects). These 4 factorial effects are all three factor interactions. We note that in this example, there are at most 3 pairs of correlated effect estimators (possibly correlated due to adjustment). All others are uncorrelated, and uncorrelated with the adjusted estimators. This illustrates that the design is nearly orthogonal.

5. Conclusions

A general theorem that identifies certain estimable functions in a general deletion design has been developed, and systematic methods for determining the u.e. and n.u.e unbiasedly estimable factorial effects in $2^{m_1} \times 3^{m_2}$ deletion designs in three incomplete blocks of equal sizes have been presented. In addition, a method for correcting the bias in estimators of n.u.e. factorial effects and the resulting relative efficiency in doing so was discussed.

These deletion designs, with two and three level factors and relatively small values of m_1 and m_2 are of the most practical significance, since larger values for the numbers of levels result in much larger total run numbers. Indeed, if much larger run numbers are actually feasible, it would be advisable to use a more standard, well-studied and easily analyzed design (e.g. a full, symmetrical factorial design). In more generality, Chauhan (1988) has considered the $(q-a)^{m_1} \times q^{m_2}$ deletion design obtained from a q^m $(m = m_1 + m_2)$ design in m blocks with initial block containing the runs $(x_1 \ldots x_m)$ satisfying $x_1 + \cdots + x_m = 0 \bmod(m)$, and showed that any main effect or interaction not involving all of the factors $F_{m_1+1}, \ldots, F_m$ is estimable. Theorem 1 of this paper generalizes that result. Also, by considering the special case of $q - a = 2$ and $q = 3$, we were able to obtain a stronger result (Theorem 2) than that of Chauhan. Specifically, Theorem 2 identifies the same effects as being estimable as does the result of Chauhan (1988), but in addition identifies some single-degree-of-freedom factorial effects involving all of the factors $F_{m_1+1}, \ldots, F_m$ which are also estimable. For example, in Example 3, of the effects determined to be estimable by an application of Theorem 2, Chauhan's result establishes estimability of all but $F_3F_4^2, F_3^2F_4, F_1F_3F_4^2, F_1F_3^2F_4, F_2F_3F_4^2, F_2F_3^2F_4,$ $F_1F_2F_3F_4^2$ and $F_1F_2F_3^2F_4$. Therefore, while Chauhan (1988) covers more general designs, the results contained herein for the special case of $q - a = 2$ and $q = 3$ are stronger.

Acknowledgement

The work of the first author was supported in part by the National Science Foundation under grant CCR-8521398 and was part of her doctoral dissertation at the University of California at Irvine (copyright June 1988). The author hereby acknowledges the support, contributions, and encouragement of Professor Subir Ghosh, University of California at Riverside, in his capacity as co-chairman of the dissertation committee. We wish to thank an anonymous referee who pointed out the reference by Chauhan, whose comments on earlier versions of this paper led to many improvements in the presentation, and for showing us how our earlier results could be generalized into the result of Theorem 1.

References

Addelman, S. (1962). Orthogonal main-effect plans for asymmetrical factorial experiments. *Technometrics* **4**, 21–46.

Addelman, S. (1972). Recent developments in the design of factorial experiments. *J. Amer. Statist. Assoc.* **67**, 103–111.

Chauhan, C. K. (1988). *Construction of Efficient Single Replicate Designs.* Presented at the Joint Statistical Meetings, New Orleans.

Dean, A. M. (1978). The analysis of interactions in single replicate generalized cyclic designs. *J. Roy. Statist. Soc. Ser.B* **40**, 79–84.

Ghosh, S. and Mahoney, J. (1988). Efficient nearly orthogonal deletion designs. *Proc. R. C. Bose Memorial Conference*, to appear.

John, J. A. (1987). *Cyclic Designs*. Chapman and Hall, New York.

Kishen, K. and Srivastava, J. N. (1959). Mathematical theory of confounding in asymmetrical and symmetrical factorial designs. *J. Indian Soc. Agricultural Statist.* **11**, 73–110.

Lehmann, E. L. (1983). *Theory of Point Estimation*. John Wiley, New York.

Mahoney, J. M. (1988). Efficient nearly orthogonal deletion designs. Ph.D. dissertation, University of California at Irvine, copyright June 1988.

Raktoe, B. L., Hedayat, A. and Federer, W. T. (1981). *Factorial Designs*. John Wiley, New York.

Scheffe, H. (1959). *The Analysis of Variance*. John Wiley, New York.

Voss, D. T. (1986). First-order deletion designs and the construction of efficient nearly orthogonal factorial designs in small blocks. *J. Amer. Statist. Assoc.* **81**, 813–818.

University of California, Irvine, CA 92717, U.S.A.
Department of Mathematics, The Claremont Graduate School, Claremont, CA 91711, U.S.A.

(Received April 1990; accepted May 1992)