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MINIMUM ABERRATION DESIGNS FOR MIXED FACTORIALS IN TERMS OF COMPLEMENTARY SETS

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Abstract: Minimum aberration designs are obtained for two types of mixed-level fractional factorial: (i) $(s^r) \times s^n$ factorial, and (ii) $(s^{r_1}) \times (s^{r_2}) \times s^n$ factorial, where s is any prime or prime power, and r, r_1, r_2 and n are positive integers. Projective geometric tools are employed to find the wordlength pattern of a given design in terms of that of its complementary set. Many useful designs are found and tabulated.

Key words and phrases: Fractional factorial designs, main effect fractions, projective geometry.

1. Introduction

Let $s(\geq 2)$ be a prime or prime power. In this article, we consider the following two types of mixed factorial settings: (i) $(s^r) \times s^n$ factorial, involving one factor at s^r levels $(r \geq 2)$ and n factors each at s levels, (ii) $(s^{r_1}) \times (s^{r_2}) \times s^n$ factorial, involving one factor at s^{r_1} levels, one factor at s^{r_2} levels $(r_1, r_2 \geq 2)$ and n factors each at s levels. The cases $s = 2, 3$ will be of particular interest and typically n will be large. This is in keeping with most practical situations which involve a large number of factors each with a small number of levels and one or two factors with more levels.

For symmetric factorials, there has been much recent interest in the characterization of minimum aberration designs in terms of complementary sets $-$ see Tang and Wu (1996) and Suen, Chen and Wu (1997). We investigate the corresponding developments for mixed factorials. The only reference on minimum aberration mixed factorial designs is Wu and Zhang (1993) who studied $4^m 2^n$ designs with $m = 1$ and 2. Here we develop a general theoretical approach to the problem. A novel feature of our approach is that it simplifies the derivation of minimum aberration designs for large $n-$ a situation which corresponds to the nearly saturated case and hence is of practical interest. Finite projective geometry provides an elegant and unified tool in our theoretical formulation.

2. Designs for $(s^r) \times s^n$ Factorial

2.1. Preliminaries

Consider the setup of an $(s^r)\times s^n$ factorial with one factor, say Z_0, involving s^r levels and n factors, say $Z_1, \ldots, Z_n$, each involving s levels. Its regular main effect fraction of s^t runs can be geometrically described as follows. Denote the finite projective geometry $PG(t-1, s)$ by P, which consists of the nonzero points $x = (x_1, \ldots, x_t)'$ with x_i from the Galois field $GF(s)$ over s, and x and y are identical if $x_i = \lambda y_i$ for some $\lambda \in GF(s)$ and all i. Recall that an $(r-1)-$flat of P is a subspace (of P) with cardinality $(s^r-1)/(s-1) = g$. For any nonempty subset Q of P, let $V(Q)$ denote a matrix with columns given by the points in Q. (Note that each column is a $t\times 1$ vector.) Then a regular fraction as mentioned above is specified by a pair of subsets (C_0, C) of P such that (a) C_0 and C are disjoint, (b) C_0 is an $(r-1)-$flat of P, (c) C has cardinality n, and (d) the matrix $V(C_0 \cup C)$ has full row rank t. The resulting fractional factorial design is constructed as follows. Consider the s^t vectors in the row space of $V(C_0 \cup C)$. Any such vector will be of the form $(\rho_1, \ldots, \rho_g, \rho_{g+1}, \ldots, \rho_{g+n})$, where $\rho_i \in GF(s)$ for each i and $(\rho_1, \ldots, \rho_g)$ is the contribution arising from C_0. Since C_0 is an $(r-1)-$flat, there are exactly s^r possibilities for $(\rho_1, \ldots, \rho_g)$. Identifying each of these possibilities with a level of Z_0 and interpreting $\rho_{g+1}, \ldots, \rho_{g+n}$ as the levels of $Z_1, \ldots, Z_n$ respectively, each of the s^t vectors in the row space of $V(C_0 \cup C)$ represents a treatment combination of an $(s^r) \times s^n$ factorial. The collection of s^t treatment combinations so obtained gives a regular main effect fraction, to be denoted by $d = d(C_0, C)$, of an $(s^r) \times s^n$ factorial.

The above construction is in the spirit of Wu, Zhang and Wang (1992) who took $C = P - C_0$ in order to construct saturated asymmetrical orthogonal arrays of strength two. In general, however, C can be a proper subset of $P - C_0$ and we intend to address the problem of choosing the pair (C_0, C) so that the resulting fraction has minimum aberration. Considering the cardinalities of C_0, C and P, the construction described in the previous paragraph is possible if and only if $s^r + n(s-1) \le s^t$, a condition which is supposed to hold throughout this section.

Example 1. Consider a regular main effect fraction of a 9×3^3 factorial in 27 runs. Then $s = 3, r = 2, n = 3$ and $t = 3$. Taking, for example, $C_0 = \{(1,0,0)', (0,1,0)', (1,1,0)', (1,2,0)'\}$ and $C = \{(1,1,2)', (1,2,1)', (1,2,2)'\}$, we have

$$V(C_0 \cup C) = \begin{bmatrix} 1\,0\,1\,1\,1\,1\,1 \\ 0\,1\,1\,2\,1\,2\,2 \\ 0\,0\,0\,0\,2\,1\,2 \end{bmatrix}.$$

Since C_0 is a $1-$flat, it is easy to see that, if $(\rho_1, \ldots, \rho_7)$ is any vector in the row space of $V(C_0 \cup C)$, then there are exactly nine $(= 3^2)$ possibilities for the subvector $(\rho_1, \ldots, \rho_4)$, the contribution from C_0. Hence the 27 vectors in the row space of $V(C_0 \cup C)$ give a regular main effect fraction of a 9×3^3 factorial in 27 runs.

The criterion of minimum aberration is based on the defining relation of a regular fraction. To explain the latter we need the notion of a pencil. With reference to an $(s^r) \times s^n$ factorial a typical pencil, carrying $s-1$ degrees of freedom (d.f.), is a nonnull vector of the form $\xi = (\xi_1, \ldots, \xi_g, \xi_{g+1}, \ldots, \xi_{g+n})'$, where $\xi_i \in GF(s)$ and among $\xi_1, \ldots, \xi_g$ *at most one is nonzero*. As with symmetric prime powered factorials, any two pencils with proportional coordinates are considered identical. Such a pencil ξ belongs to the main effect of the s^r- level factor Z_0 if $\xi_{g+1} = \cdots \xi_{g+n} = 0$. Thus there are $g = (s^r-1)/(s-1)$ distinct pencils belonging to the main effect of Z_0. Since each of these carries $s-1$ d.f., this accounts for the $s^r - 1$ d.f. belonging to the main effect of Z_0. Similarly, a pencil ξ with $\xi_{g+i} \neq 0$ for some $i(1 \leq i \leq n)$ and $\xi_j = 0$ for every $j \neq g+i$ represents the main effect of the $s-$level factor Z_i. Any pencil with exactly $i(\geq 2)$ nonzero elements is an $i-$factor interaction pencil. An interaction pencil ξ can involve only the $s-$level factors (i.e. have $\xi_1 = \cdots = \xi_g = 0$) or the s^r-level factor together with some s-level factors (i.e., have one of $\xi_1, \ldots, \xi_g$ nonzero). As in Wu and Zhang (1993), pencils of these two types are called type 0 and type 1 respectively.

A pencil ξ appears in the defining relation of the fraction $d = d(C_0, C)$ if $V(C_0 \cup C)\xi = 0$. Since $C_0 \cup C$ consists of distinct points of P, the columns of $V(C_0 \cup C)$ are nonnull and no two of them are proportional to one another. As such, each pencil appearing in the defining relation of d corresponds to an interaction involving at least three factors i.e., d has resolution at least three and so is called a regular "main effect" fraction. For $i = 3, 4, \ldots,$ let $A_{i0}(d)$ and $A_{i1}(d)$ denote the numbers of distinct $i-$factor interaction pencils, of types 0 and 1 respectively, appearing in the defining relation of d. The sequence $\{A_{30}(d), A_{31}(d), A_{40}(d), A_{41}(d), \ldots\}$ is called the wordlength pattern of d.

Example 1. (continued) Here the distinct pencils appearing in the defining relation, i.e., the solution vectors to $V(C_0 \cup C)\xi = 0$ are

$$(1,0,0,0,1,1,0)',\ (0,1,0,0,1,0,2)',\ (0,0,0,1,0,1,1)',\ (0,0,1,0,2,1,2)'.$$

Suppose we denote the three $3-$level factors by A,B, C and the four components of the $9-$level factor by (a, b, ab, ab^2). Then the defining relation can be represented by $aAB = bAC^2 = (ab)A^2BC^2 = (ab^2)BC = I$, where I is the identity. All these are of type 1 and we have $A_{30}(d) = 0, A_{31}(d) = 3, A_{40}(d) = 0, A_{41}(d) = 1$. Thus the wordlength pattern is $\{0, 3, 0, 1\}$.

Let $f = (s^t - s^r)/(s-1) - n$ be the cardinality of the complementary set, F, of $C_0 \cup C$ in P. If $f = 0$ or 1 then all designs are isomorphic (cf. Chen, Sun and

Wu (1993)) and have the same wordlength pattern. Hence, only the case $f \geq 2$ will be considered in this section. Also, we are concerned only with $n \geq 3$, since elementary considerations apply for $n \leq 2$; for example, if $n = 1$ then d reduces to the complete factorial.

We now present some notation and two useful lemmas. Let Q be any nonempty subset of P and C_0 be any $(r-1)-$flat of P such that C_0 and Q are disjoint. For $i \geq 1$, define

$$G_i(Q) = (s-1)^{-1}\#\{\beta : \beta \in \Omega_{iq}, V(Q)\beta = 0\}, \tag{1}$$

$$H_i(C_0, Q) = (s-1)^{-1}\#\{\beta : \beta \in \Omega_{iq}, V(Q)\beta \text{ is nonnull but proportional to some point in } C_0\}, \tag{2}$$

where $\#$ is the cardinality of a set, q is the cardinality of Q, and Ω_{iq} is the set of $q \times 1$ vectors over $GF(s)$ involving exactly i nonzero elements. Clearly, $G_1 = H_1 = 0$ and

$$G_i(Q) = H_i(C_0, Q) = 0 \quad \text{for} \quad i > q. \tag{3}$$

Since two pencils with proportional coordinates are identical, with reference to any design $d = d(C_0, C)$, from (1) and (2) it is not hard to see that for $i \geq 3$,

$$A_{i0}(d) = G_i(C), \quad A_{i1}(d) = H_{i-1}(C_0, C). \tag{4}$$

The following lemmas hold in the above set-up.

Lemma 1. (i) $G_3(C_0 \cup Q) = constant + G_3(Q) + H_2(C_0, Q)$,
(ii) $G_4(C_0 \cup Q) = constant + G_4(Q) + H_3(C_0, Q) + \frac{1}{2}(s^r - s)H_2(C_0, Q)$.

Lemma 2. *Let $\overline{Q} = P - Q$ be nonempty. Then*
(i) $G_3(Q) = constant - G_3(\overline{Q})$, (ii) $G_4(Q) = constant + (3s-5)G_3(\overline{Q}) + G_4(\overline{Q})$.

The constants in Lemmas 1 and 2 may depend on s, r, q and t, but not on the particular choice of C_0 and Q. Using (1) – (3), these lemmas follow from Mukerjee and Wu (1999) and Suen, Chen and Wu (1997) respectively. In fact, following them, we could give expressions for $G_i(C_0 \cup Q)$ and $G_i(Q)$ for $i \geq 5$. However, such details are rarely needed in the present approach.

2.2. Minimum aberration designs of type 0

As in Wu and Zhang (1993) we argue that interaction pencils of type 0 are more important than those of type 1. Because the s^r-level factor has g components $\xi_1, \ldots, \xi_g$ (and g is at least 3), it is unlikely in typical situations that all these components are significant. *A priori* knowledge may allow the experimenter to choose the least significant component to be included in an interaction pencil of type 1, which explains why type 1 pencils are less serious. We first consider minimum aberration designs of type 0. With designs d_1 and d_2, let u be the smallest integer i such that $(A_{i0}(d_1), A_{i1}(d_1)) \neq (A_{i0}(d_2), A_{i1}(d_2))$. If $A_{u0}(d_1) < A_{u0}(d_2)$ or $A_{u0}(d_1) = A_{u0}(d_2)$ but $A_{u1}(d_1) < A_{u1}(d_2)$, then d_1 is said to have less aberration of type 0 than d_2. A design d has *minimum aberration* of type 0 if no other design has less aberration of type 0 than d.

Lemma 3. *For any design $d = d(C_0, C)$, let $F = P - (C_0 \cup C)$. Then*
(i) $A_{30}(d) = G_3(C) = \text{constant} - G_3(C_0 \cup F)$,
(ii) $A_{31}(d) = H_2(C_0, C) = \text{constant} + G_3(C_0 \cup F) - G_3(F)$,
(iii) $A_{40}(d) = G_4(C) = \text{constant} + (3s - 5)G_3(C_0 \cup F) + G_4(C_0 \cup F)$,
(iv) $A_{41}(d) = H_3(C_0, C) = \text{constant} - \frac{1}{2}(s^r + 5s - 10)\{G_3(C_0 \cup F) - G_3(F)\} - G_4(C_0 \cup F) + G_4(F)$.

Proof. Parts (i) and (iii) are immediate from (4) and Lemma 2. Part (ii) is immediate from (4), Lemma 1(i) and Lemma 2(i) . Also by (4) and Lemma 1,

$$A_{41}(d) = H_3(C_0, C) = \text{ constant } + G_4(C_0 \cup C) - G_4(C) - \frac{1}{2}(s^r - s)\{G_3(C_0 \cup C) - G_3(C)\},$$

whence using Lemma 2, part (iv) follows.
We now define the following classes of designs:

$$\begin{aligned} D_1 &= \{d = d(C_0, C) : d \text{ maximizes } G_3(C_0 \cup F)\}, \\ D_2 &= \{d : d \in D_1,\ d \text{ maximizes } G_3(F) \text{ over } D_1\}, \\ D_3 &= \{d : d \in D_2,\ d \text{ minimizes } G_4(C_0 \cup F) \text{ over } D_2\}, \\ D_4 &= \{d : d \in D_3,\ d \text{ minimizes } G_4(F) \text{ over } D_3\}. \end{aligned}$$

Recalling the definition of a minimum aberration design of type 0, Lemma 3 yields the following result, which serves as a tool for the identification of such designs.

Theorem 1. *For any i $(1 \leq i \leq 4)$, suppose d belongs to D_i and, up to isomorphism, is the unique member of D_i. Then d has minimum aberration of type 0.*

Corollary 1. *Let $f = 2$. Then a design $d(C_0, C)$ has minimum aberration of type 0 provided $F = P - (C_0 \cup C)$ is of the form*

$$F = \{\alpha_1, \alpha_1 + \rho\alpha_0\}, \tag{5}$$

for some $\alpha_1 \notin C_0, \alpha_0 \in C_0$ and $\rho(\neq 0) \in GF(s)$.

Proof. Since C_0 is a flat and C_0 and F are disjoint, from (2) one can check that $H_2(C_0, F)$ equals unity if F is as in (5) and zero otherwise. The result now follows from Theorem 1 (with $i = 1$) noting that (a) for $f = 2$, by (3) and Lemma 1 (i), $G_3(C_0 \cup F) =$ constant $+ H_2(C_0, F)$, and (b) all designs with F as in (5) are isomorphic.

Remark 1. For $f = 2$ and $t - r \geq 2$, not all designs have F as in (5). Another choice of F is $\{\alpha_1, \alpha_2\}$, where $\alpha_1 \notin C_0, \alpha_2 \notin C_0$ and $V(C_0 \cup \{\alpha_1, \alpha_2\})$ has rank $r + 2$. Hence, even for $f = 2$, one can discriminate among designs with respect to minimum aberration of type 0. This may be contrasted with the case of symmetric factorials where, for $f = 2$, all designs are equivalent under the minimum aberration criterion - see Suen, Chen and Wu (1997).

Corollary 2. *Let $f = (s^u - s^r)/(s - 1)$, where $u > r$. Then a design $d(C_0, C)$ has minimum aberration of type 0 provided $C_0 \cup F$ is a $(u - 1)-$flat of P, where $F = P - (C_0 \cup C)$.*

While Corollary 2 follows from Theorem 1 with $i = 1$, we get the following additional result for $s = 2$.

Theorem 2. *Let $s = 2$ and $f = 2^u - 2^r - w$, where $u > r$ and $1 \leq w \leq 3$. Let $\alpha_1, \ldots, \alpha_u$ be any u linearly independent points of P and C_0^* and $\hat{C}$ be the $(r-1)-$ and $(u-1)-$flats spanned by $\{\alpha_1, \ldots, \alpha_r\}$ and $\{\alpha_1, \ldots, \alpha_u\}$ respectively. Let $F^* = \hat{C} - (C_0^* \cup T^*)$, where*
(a) $T^* = \{\alpha_{r+1}\}$ *if* $w = 1$,
(b) $T^* = \{\alpha_{r+1}, \alpha_1 + \alpha_{r+1}\}$ *if* $w = 2$ *and* $u = r + 1$,
(c) $T^* = \{\alpha_{r+1}, \alpha_{r+2}\}$ *if* $w = 2$ *and* $u > r + 1$,
(d) $T^* = \{\alpha_{r+1}, \alpha_1 + \alpha_{r+1}, \alpha_2 + \alpha_{r+1}\}$ *if* $w = 3$ *and* $u = r + 1$,
(e) $T^* = \{\alpha_{r+1}, \alpha_{r+2}, \alpha_1 + \alpha_{r+1} + \alpha_{r+2}\}$ *if* $w = 3$ *and* $u = r + 2$,
(f) $T^* = \{\alpha_{r+1}, \alpha_{r+2}, \alpha_{r+3}\}$ *if* $w = 3$ *and* $u > r + 2$.

Then $d^* = d(C_0^*, C^*)$, where $C^* = P - (C_0^* \cup F^*)$, has minimum aberration of type 0.

Proof. Consider the case $w = 3$ and $u > r + 2$ which corresponds to (f) above. Then for any design $d = d(C_0, C)$, the cardinality of $C_0 \cup F$ is $2^u - 4$ where, as usual $F = P - (C_0 \cup C)$. Hence, using the results in Section 3 of Cheng and Mukerjee (1998) for symmetric two-level factorials, d belongs to D_1, i.e., maximizes $G_3(C_0 \cup F)$ if and only if $C_0 = C_0^*$ and $F = \hat{C} - (C_0^* \cup T)$, where C_0^* and $\hat{C}$ are defined as in the statement of the theorem and T consists of any three non-collinear points of $\hat{C} - C_0^*$. Since $G_3(T) = G_4(T) = 0$, by Lemmas 1 and 2 with $\hat{C}$ playing the role of P in Lemma 2, for any $d \in D_1$,

$$G_3(F) = \text{constant} - G_3(C_0^* \cup T) = \text{constant} - H_2(C_0^*, T),$$

$$
\begin{aligned}
G_4(C_0^* \cup F) &= \text{constant} + G_3(T) + G_4(T) = \text{constant}, \\
G_4(F) &= \text{constant} + G_3(C_0^* \cup T) + G_4(C_0^* \cup T) \qquad (6) \\
&= \text{constant} + 2^{r-1} H_2(C_0^*, T) + H_3(C_0^*, T).
\end{aligned}
$$

Now $H_2(C_0^*, T) \geq 0, H_3(C_0^*, T) \geq 0$, with equalities attained in both if and only if the three non-collinear points of T do not span any point of C_0^*. In view of (6), D_4 consists of designs d which belong to D_1 and for which the three non-collinear points of T do not span any point of C_0^*. But the members of D_4 are isomorphic and the design d^*, with T^* as given in (f) above, is one of them. Hence by Theorem 1, with $i = 4$, d^* has minimum aberration of type 0.

Similarly, the proofs for cases $(a) - (e)$ can be completed using Theorem 1 with $i = 1, 1, 2, 1$ and 2 respectively.

We remark that Theorem 2 does not hold for general s, since then the results in Section 3 of Cheng and Mukerjee (1998) cannot be used.

In the practically important nearly saturated case, n is large and f is small. Hence it is much easier to handle the set F than the set C and, in this sense, Theorem 1 can be of particular help. Some useful special cases are discussed in the next subsection where, in view of Corollary 1, we consider only $f \geq 3$.

2.3. Special class

Considering 4×2^n factorials, we have $s = r = 2$, and for $f = 3, 4, 9, 10, 11, 12$, minimum aberration designs of type 0 are given by Corollary 2 or Theorem 2. For $3 \leq f \leq 12$, Table 1 shows minimum aberration designs of type 0 and indicates how they are obtained. In Table 1 and elsewhere, a typical point of $PG(t-1, 2)$ is denoted by $i_1 \ldots i_h$ which represents a $t \times 1$ vector with 1 in the $i_1 th, \ldots, i_h th$ positions and 0 elsewhere.

Similarly, for 8×2^n factorials, if $5 \leq f \leq 8$ then minimum aberration designs of type 0 are given by Corollary 2 or Theorem 2. On the other hand, for $f = 3$ or 4, such designs are given by $C_0 = \{1, 2, 12, 3, 13, 23, 123\}$ and $F = \{4, 14, 24\}$ or $F = \{4, 14, 24, 34\}$ respectively; this follows from Theorem 1 with $i = 1$ or 3 respectively.

Turning to 9×3^n factorials, Corollary 2 yields the minimum aberration design of type 0 for $f = 9$ while Table 2 lists such designs for $3 \leq f \leq 8$. The designs in Table 2 are obtained using Theorem 1, with $i = 1$, and the findings in Suen, Chen and Wu (1997) for symmetric three-level factorials help in identifying the set D_1 in each case. In Table 2, a typical point of $PG(t-1, 3)$ is denoted by $i_1^{j_1} \ldots i_h^{j_h}$ which represents a $t \times 1$ vector with $j_1, \ldots, j_h$ in the $i_1 th, \ldots, i_h th$ positions respectively and 0 elsewhere. From Table 2 with $f = 6$, we find that the design considered in Example 1 has minimum aberration of type 0. It follows from writing C_0 and C in Example 1 as $\{1, 2, 12, 12^2\}$ and $\{123^2, 12^23, 12^23^2\}$ and taking $F = PG(2, 3) - (C \cup C_0)$.

Table 1. Minimum aberration designs of type 0 for 4×2^n factorials†.

f	F	Source
3	{3, 13, 23}	Theorem 2(a)
4	{3, 13, 23, 123}	Corollary 2
5	{3, 13, 23, 123, 4}	Theorem 1 ($i = 1$)
6	{3, 13, 4, 14, 34, 134}	Theorem 1 ($i = 2$)
7	{3, 13, 4, 14, 24, 34, 134}	Theorem 1 ($i = 2$)
8	{3, 13, 23, 4, 14, 24, 34, 134}	Theorem 1 ($i = 2$)
9	{3, 13, 23, 4, 14, 24, 34, 134, 234}	Theorem 2 (e)
10	{3, 13, 23, 123, 4, 14, 24, 34, 134, 234}	Theorem 2 (c)
11	{3, 13, 23, 123, 4, 14, 24, 124, 34, 134, 234}	Theorem 2 (a)
12	{3, 13, 23, 123, 4, 14, 24, 124, 34, 134, 234, 1234}	Corollary 2

† Each design $d(C_0, C)$ is a fraction of a 4×2^n factorial in 2^t runs, where $C_0 = \{1, 2, 12\}$ corresponds to the 4−level factor, $C = PG(t-1, 2) - (C_0 \cup F)$ corresponds to the n 2−level factors, F is the complementary set given in the table and $f = \#F$.

Example 2. Consider a 4×2^{25} factorial in 32 runs. Then $s = r = 2, n = 25, t = 5, f = 32 - 4 - 25 = 3$, and Theorem 2 is applicable with $f = 3 = 2^3 - 2^2 - 1, u = 3, w = 1$. Using Theorem 2(a), we get $C_0^* = \{1, 2, 12\}, \hat{C} = \{1, 2, 12, 3, 13, 23, 123\}, T^* = \{3\}, F^* = \{13, 23, 123\}$. The design $d(C_0^*, C^*)$, with $C^* = PG(4, 2) - (C_0^* \cup F^*)$, has minimum aberration of type 0 and is equivalent to the one given in Table 1 with $f = 3$.

Table 2. Minimum aberration designs of type 0 for 9×3^n factorials†.

f	F
3	$\{3, 12^23, 12^23^2\}$
4	$\{3, 12^23, 12^23^2, 23^2\}$
5	$\{3, 13^2, 23, 12^23, 12^23^2\}$
6	$\{3, 13, 23, 123, 13^2, 23^2\}$
7	$\{3, 13, 23, 123, 13^2, 23^2, 123^2\}$
8	$\{3, 13, 23, 123, 13^2, 23^2, 123^2, 12^23^2\}$

†Each design $d(C_0, C)$ is a fraction of a 9×3^n factorial in 3^t runs, where $C_0 = \{1, 2, 12, 12^2\}$ corresponds to the 9−level factor, $C = PG(t-1, 3) - (C_0 \cup F)$ corresponds to the n 3−level factors, F is the complementary set given in the table and $f = \#F$.

2.4. Designs with minimum overall aberration

We now briefly discuss the situation where interactions of types 0 and 1 are considered equally important. Then it is appropriate to consider minimum overall aberration designs which are defined as follows. With designs d_1 and d_2, let u be the smallest integer i such that $A_{i0}(d_1) + A_{i1}(d_1) \neq A_{i0}(d_2) + A_{i1}(d_2)$. If $A_{u0}(d_1) + A_{u1}(d_1) < A_{u0}(d_2) + A_{u1}(d_2)$, then d_1 is said to have less overall aberration than d_2. A design d has minimum overall aberration if no other design has less overall aberration than d. From Lemmas 1-3, we get the following result.

Lemma 4. *For any design* $d = d(C_0, C)$, *let* $F = P - (C_0 \cup C)$. *Then*
(i) $A_{30}(d) + A_{31}(d) = constant - G_3(F)$
(ii) $A_{40}(d) + A_{41}(d) = constant + (3s-5)G_3(F) + G_4(F) - \frac{1}{2}(s^r - s)H_2(C_0, F)$.

One can obtain an analogue of Theorem 1 from Lemma 4 and use it to derive further results. For example, with $f = 2$, it can be seen that a design $d(C_0, C)$ has minimum overall aberration if and only if $F = P - (C_0 \cup C)$ is as given by (5). To save space, we omit such details here and present only Table 3 which, for 4×2^n factorials, shows designs with minimum overall aberration for $3 \leq f \leq 12$ and, for each such design, indicates the parts of Lemma 4 that are needed for the derivation. Comparing Table 3 with Table 1, it is clear that in most cases the criteria of minimum overall aberration and minimum aberration of type 0 yield different results.

Table 3. Designs with minimum overall aberration for 4×2^n factorials. (Description of these designs is the same as in the footnote of Table 1.)

f	F	Needed Part(s) of Lemma 4
3	$\{3, 4, 34\}$	(i)
4	$\{3, 4, 34, 13\}$	(i),(ii)
5	$\{3, 4, 34, 14, 134\}$	(i),(ii)
6	$\{3, 4, 34, 13, 14, 134\}$	(i),(ii)
7 ($t = 4$)	$\{3, 4, 34, 13, 14, 134, 24\}$	(i)
7 ($t \geq 5$)	$\{3, 4, 34, 5, 35, 45, 345\}$	(i)
8 ($t = 4$)	$\{3, 4, 34, 13, 14, 134, 23, 24\}$	(i)
8 ($t \geq 5$)	$\{3, 4, 34, 5, 35, 45, 345, 13\}$	(i),(ii)
9 ($t = 4$)	$\{3, 4, 34, 13, 14, 134, 23, 24, 234\}$	(i)
9 ($t \geq 5$)	$\{3, 4, 34, 5, 35, 45, 345, 14, 134\}$	(i),(ii)
10	$\{3, 4, 34, 5, 35, 45, 134, 135, 145, 1345\}$	(i),(ii)
11	$\{3, 4, 34, 5, 35, 45, 345, 134, 135, 145, 1345\}$	(i),(ii)
12	$\{3, 4, 34, 5, 35, 45, 345, 13, 14, 134, 15, 1345\}$	(i),(ii)

3. Designs for $(s^{r_1}) \times (s^{r_2}) \times s^n$ Factorial

3.1. Preliminaries

Let P be the set of points of $PG(t-1, s)$. As in Section 2, in the spirit of Wu, Zhang and Wang (1992), a regular main effect fraction of an $(s^{r_1}) \times (s^{r_2}) \times s^n$ factorial in s^t runs is specified by a triplet of subsets (C_1, C_2, C) of P such that (a) C_1, C_2 and C are mutually exclusive, (b) C_j is an $(r_j - 1)-$flat of $P (j = 1, 2)$, (c) C has cardinality n, and (d) $V(C_1 \cup C_2 \cup C)$ has full row rank. The resulting fractional factorial design, to be denoted by $d = d(C_1, C_2, C)$, consists of the s^t level combinations represented by the vectors in the row space of $V(C_1 \cup C_2 \cup C)$, with the contribution arising from C_j in any such vector identified with a level of the $s^{r_j}-$level factor $(j = 1, 2)$. Considering the cardinalities of C_1, C_2, C and P, this construction is possible if and only if $r_1+r_2 \leq t$ and $s^{r_1} + s^{r_2} + n(s-1) - 1 \leq s^t$, a condition which is imposed throughout this section.

As in Section 2, each pencil appearing in the defining equation of d corresponds to an interaction involving at least three factors. Any such pencil can involve either only the $s-$level factors, or one of the two $s^{r_j}-$level factors $(j = 1, 2)$ together with some $s-$level factors, or both the $s^{r_j}-$level factors $(j = 1, 2)$ together with some $s-$level factors. Pencils of these three types are called type 0, type 1 and type 2 respectively. For $i = 3, 4, \ldots$ and $u = 0, 1, 2$, let $A_{iu}(d)$ denote the number of distinct $i-$factor interaction pencils of type u appearing in the defining relation of d. The sequence $\{A_{iu}(d)\}$ is called the wordlength pattern of d.

Let $f = (s^t - s^{r_1} - s^{r_2} + 1)/(s-1) - n$ be the cardinality of the complementary set, F, of $C_1 \cup C_2 \cup C$ in P. If $f = 0$ then all designs are isomorphic. Hence we are concerned only with $f \geq 1$ and, to avoid trivialities, also assume that $n \geq 2$. With any mutually exclusive subsets C_1, C_2 and Q of P such that C_j is an $(r_j - 1)-$flat $(j = 1, 2)$ and Q has cardinality $q(\geq 1)$, define for $i \geq 1$,

$$\begin{aligned} L_i(C_1, C_2, Q) = (s-1)^{-1}\#\{\beta : \quad & \beta \in \Omega_{iq}, \text{ there exist nonzero } \rho_j \in GF(s), \\ & \alpha_j \in C_j (j = 1, 2), \text{ such that } V(Q)\beta = \rho_1\alpha_1 + \rho_2\alpha_2\}. \end{aligned} \tag{7}$$

Analogously to (4), with reference to any design $d = d(C_1, C_2, C)$, for $i \geq 3$,

$$A_{i0}(d) = G_i(C),\ A_{i1}(d) = H_{i-1}(C_1, C) + H_{i-1}(C_2, C),\ A_{i2}(d) = L_{i-2}(C_1, C_2, C). \tag{8}$$

In the above set-up, we have the following lemma whose proof is given in the appendix.

Lemma 5. (i) $H_2(C_1, C_2 \cup Q) = H_2(C_1, Q) + L_1(C_1, C_2, Q)$,
(ii) $H_3(C_1, C_2 \cup Q) = H_3(C_1, Q) + L_2(C_1, C_2, Q) + \frac{1}{2}(s^{r_2} - s)L_1(C_1, C_2, Q)$.

3.2. Minimum aberration designs of type 0

As argued in Section 2.2, in most practical situations interaction pencils of type 0 are most serious and those of type 2 are least serious. Hence we again consider minimum aberration designs of type 0 which are defined as follows. With designs d_1 and d_2, let u be the smallest integer i such that $(A_{i0}(d_1), A_{i1}(d_1), A_{i2}(d_1)) \neq (A_{i0}(d_2), A_{i1}(d_2), A_{i2}(d_2))$. If either (i) $A_{u0}(d_1) < A_{u0}(d_2)$, or (ii) $A_{u0}(d_1) = A_{u0}(d_2)$ but $A_{u1}(d_1) < A_{u1}(d_2)$, or (iii) $A_{u0}(d_1) = A_{u0}(d_2), A_{u1}(d_1) = A_{u1}(d_2)$ but $A_{u2}(d_1) < A_{u2}(d_2)$, then d_1 is said to have less aberration of type 0 than d_2. A design d has *minimum aberration* of type 0 if no other design has less aberration of type 0 than d. The proof of the following lemma is given in the appendix.

Lemma 6. *For any design $d = d(C_1, C_2, C)$, let $F = P-(C_1 \cup C_2 \cup C)$. Then*

(i) $A_{30}(d) = \textit{constant} - G_3(C_1 \cup C_2 \cup F)$,

(ii) $A_{31}(d) = \textit{constant} + G_3(C_1 \cup C_2 \cup F) + \Psi_1(C_1, C_2, F)$,

(iii) $A_{32}(d) = L_1(C_1, C_2, C) = \textit{constant} - L_1(C_1, C_2, F)$,

(iv) $A_{40}(d) = \textit{constant} + (3s-5)G_3(C_1 \cup C_2 \cup F) + G_4(C_1 \cup C_2 \cup F)$,

(v) $A_{41}(d) = \textit{constant} - (3s-5)\{G_3(C_1 \cup C_2 \cup F) + \Psi_1(C_1, C_2, F)\} - \frac{1}{2}(s^{r_1} + s^{r_2} - 2s)L_1(C_1, C_2, F) - 2G_4(C_1 \cup C_2 \cup F) + \Psi_2(C_1, C_2, F)$,

(vi) $A_{42}(d) = L_2(C_1, C_2, C) = \textit{constant} + (s^{r_1} + s^{r_2} + s - 5)L_1(C_1, C_2, F) + L_2(C_1, C_2, F)$,

where

$$\begin{aligned} \Psi_1(C_1, C_2, F) &= L_1(C_1, C_2, F) - G_3(F), \\ \Psi_2(C_1, C_2, F) &= 2G_4(F) + H_3(C_1, F) + H_3(C_2, F), \end{aligned} \tag{9}$$

and the constants may depend on s, r_1, r_2, n and t but not on the particular choice of C_1, C_2 and C.

We now define the following classes of designs:

$$\begin{aligned} \Delta_1 &= \{d = d(C_1, C_2, C) : d \text{ maximizes } G_3(C_1 \cup C_2 \cup F)\}, \\ \Delta_2 &= \{d : d \in \Delta_1,\ d \text{ minimizes } \Psi_1(C_1, C_2, F) \text{ over } \Delta_1\}, \\ \Delta_3 &= \{d : d \in \Delta_2,\ d \text{ maximizes } L_1(C_1, C_2, F) \text{ over } \Delta_2\}, \\ \Delta_4 &= \{d : d \in \Delta_3,\ d \text{ minimizes } G_4(C_1 \cup C_2 \cup F) \text{ over } \Delta_3\}, \\ \Delta_5 &= \{d : d \in \Delta_4,\ d \text{ minimizes } \Psi_2(C_1, C_2, F) \text{ over } \Delta_4\} \\ \Delta_6 &= \{d : d \in \Delta_5,\ d \text{ minimizes } L_2(C_1, C_2, F) \text{ over } \Delta_5\}. \end{aligned}$$

Recalling the definition of a minimum aberration design of type 0, Lemma 6 yields the following result which serves as a tool for the identification of such designs.

Theorem 3. *For any i $(1 \leq i \leq 6)$, suppose d belongs to Δ_i and, up to isomorphism, is the unique member of Δ_i. Then d has minimum aberration of type* 0.

Theorem 3, with $i = 1$, yields the following corollaries.

Corollary 3. *Let $f = 1$. Then a design $d(C_1, C_2, C)$ has minimum aberration of type* 0 *provided $F = P - (C_1 \cup C_2 \cup C)$ is of the form $F = \{\alpha_1 + \rho\alpha_2\}$ for some $\alpha_j \in C_j (j = 1, 2)$ and $\rho(\neq 0) \in GF(s)$.*

Corollary 4. *Let $f = (s^u - s^{r_1} - s^{r_2} + 1)/(s - 1)$, where $u \geq r_1 + r_2$. Then a design $d(C_1, C_2, C)$ has minimum aberration of type* 0 *provided $C_1 \cup C_2 \cup F$ is a $(u - 1)-$flat of P, where $F = P - (C_1 \cup C_2 \cup C)$.*

In the spirit of Remark 1, we note that if $t > r_1 + r_2$ then for $f = 1$, not all designs have F as in Corollary 3 so that even for $f = 1$ discrimination among designs is possible with respect to minimum aberration of type 0. In particular, Theorem 3 can be employed to get a counterpart of Theorem 2 in the present set-up. The result so obtained will, however, be somewhat messy and is omitted here.

Theorem 3 can considerably simplify the derivation of minimum aberration designs of type 0 when f is small which corresponds to the nearly saturated case. As a specific application, we consider $4^2 \times 2^n$ factorials. Then $s = r_1 = r_2 = 2$ and Corollaries 3 and 4 settle the cases $f = 1$ and $f = 9$ respectively. For $2 \leq f \leq 8$, Table 4 shows minimum aberration $4^2 \times 2^n$ designs of type 0 and indicates how these are obtained via Theorem 3. The findings in Tang and Wu (1996) for symmetric two-level factorials often help in the identification of Δ_1.

We conclude with a discussion of the significance of results obtained, especially those in the tables. The results given in Tables 2 and 3 are new. There is some overlap between Tables 1 and 4 and those given in Wu and Zhang(1993). Wu and Zhang gave minimum aberration $4^1 2^n$ designs with 16 runs for $3 \leq n \leq 11$ and with 32 runs for $4 \leq n \leq 9$. Our Table 1 covers their results for 16 runs but complement their results for 32 runs. The latter can be seen as follows. For $4^1 2^n$ designs with 32 runs, $f = 32 - 4 - n = 28 - n$. In Table 1 $3 \leq f \leq 12$, which is equivalent to $16 \leq n \leq 25$ and not addressed by Wu and Zhang. Similarly, our Table 4 covers their results for $4^2 2^n$ designs with 16 runs but complements their results for 32 runs, i.e., their approach covers $2 \leq n \leq 10$, while ours covers $17 \leq n \leq 23$. Another difference is that their approach fixes the value of n and the run size 2^t while ours does not fix these two values as long as the value of f is fixed.

We finally remark that, with mixed factorials, various other modifications of the criterion of minimum aberration can be relevant in practice. For example, in an $(s^{r_1}) \times (s^{r_2}) \times s^n$ factorial with $r_1 > r_2$, it may so happen that among pencils of type 1, those involving the $s^{r_2}-$ level factor are more serious than those involving the $s^{r_1}-$level factor. Then one may consider a modified version of minimum aberration of type 0 which calls for successive minimization of

$$A_{30}(d), A_{31}^{(2)}(d), A_{31}^{(1)}(d), A_{32}(d), A_{40}(d), A_{41}^{(2)}(d), A_{41}^{(1)}(d), A_{42}(d), \ldots,$$

where, for $j = 1, 2$, and $i = 3, 4, \ldots, A_{i1}^{(j)}(d)$ is the number of distinct $i-$factor interaction pencils of type 1, involving the $s^{r_j}-$level factor, that appear in the defining relation of d. Since $A_{i1}^{(j)}(d) = H_{i-1}(C_j, C)$, the techniques developed here can be readily adapted in such situations to obtain results in terms of complementary subsets.

Table 4. Minimum aberration designs of type 0 for $4^2 \times 2^n$ factorials†.

f	F	Use Theorem 3 with $i =$
2	$\{13, 23\}$	1
3	$\{13, 23, 123\}$	1
4	$\{13, 23, 14, 24\}$	4
5	$\{13, 23, 14, 24, 1234\}$	2
6	$\{13, 23, 123, 14, 24, 1234\}$	2
7	$\{13, 23, 123, 14, 24, 134, 1234\}$	2
8	$\{13, 23, 123, 14, 24, 124, 134, 1234\}$	1

†Each design $d(C_1, C_2, C)$ is a fraction of a $4^2 \times 2^n$ factorial in 2^t runs, where $C_1 = \{1, 2, 12\}$ and $C_2 = \{3, 4, 34\}$ correspond to the two 4−level factors, $C = PG(t-1, 2) - (C_0 \cup C_1 \cup F)$ corresponds to the n 2−level factors, F is the complementary set given in the table and $f = \#F$.

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Appendix. Proofs of Two Lemmas

Proof of Lemma 5. We only prove part (ii) of the lemma. The proof of part (i) is similar and simpler. For $j = 1, 2$, let $\sum_j$ denote the double sum over ρ_j and α_j, where $\rho_j(\neq 0) \in GF(s)$ and $\alpha_j \in C_j$. Also, let $k = (s^{r_2} - 1)/(s - 1)$. Then by (2),

$$
\begin{aligned}
&(s-1)H_3(C_1, C_2\cup Q)\\
&=\sum_{i=0}^{3}\sum_{1}\#\{\beta=(\beta_2',\beta_1')' : \beta_1\in\Omega_{iq},\beta_2\in\Omega_{(3-i)k}, V(C_2)\beta_2+V(Q)\beta_1=\rho_1\alpha_1\}\\
&=\sum_{i=0}^{3}\sum_{1}[\#\{\beta_2 : \beta_2\in\Omega_{(3-i)k}, V(C_2)\beta_2=0\}][\#\{\beta_1 : \beta_1\in\Omega_{iq}, V(Q)\beta_1=\rho_1\alpha_1\}]\\
&\quad+\sum_{i=0}^{3}\sum_{1}\sum_{2}[\#\{\beta_2 : \beta_2\in\Omega_{(3-i)k}, V(C_2)\beta_2=-\rho_2\alpha_2\}]\\
&\quad\times[\#\{\beta_1 : \beta_1\in\Omega_{iq}, V(Q)\beta_1=\rho_1\alpha_1+\rho_2\alpha_2\}]. \qquad (A.1)
\end{aligned}
$$

By (2), since Q and C_1 are disjoint, the first term in the right hand side of (A.1) equals $(s-1)H_3(C_1, Q)$. Similarly, noting in particular that

$$\#\{\beta_2 : \beta_2\in\Omega_{2k}, V(C_2)\beta_2=-\rho_2\alpha_2\}=\frac{1}{2}(s^{r_2}-s),$$

for any $\rho_2(\neq 0)\in GF(s)$ and $\alpha_2\in C_2$, and recalling (7), one can check that the second term in the right hand side of (A.1) equals $(s-1)\{\frac{1}{2}(s^{r_2}-s)L_1(C_1, C_2, Q)+L_2(C_1, C_2, Q)\}$. Hence from (A.1), part (ii) of the lemma follows.

Proof of Lemma 6. We begin with some useful identities. Note that by Lemma 5 and repeated application of Lemma 1,

$$G_3(C_1\cup C_2\cup Q)=\text{constant}+G_3(Q)+H_2(C_1, Q)+H_2(C_2, Q)+L_1(C_1, C_2, Q), \qquad (A.2)$$

$$
\begin{aligned}
G_4(C_1\cup C_2\cup Q)=\text{constant}&+G_4(Q)+H_3(C_1, Q)+H_3(C_2, Q)+L_2(C_1, C_2Q)\\
&+\frac{1}{2}(s^{r_1}-s)H_2(C_1, Q)+\frac{1}{2}(s^{r_2}-s)H_2(C_2, Q)\\
&+\frac{1}{2}(s^{r_1}+s^{r_2}-2s)L_1(C_1, C_2, Q), \qquad (A.3)
\end{aligned}
$$

where Q equals C or F. By (9) and (A.2),

$$H_2(C_1, F)+H_2(C_2, F)=\text{constant}+G_3(C_1\cup C_2\cup F)+\Psi_1(C_1, C_2, F)-2L_1(C_1, C_2, F). \qquad (A.4)$$

By Lemmas 1, 2 and 5,

$$H_2(C_1, C)=\text{constant }+H_2(C_1, C_2\cup F)=\text{constant }+H_2(C_1, F)+L_1(C_1, C_2, F), \qquad (A.5)$$

$$
\begin{aligned}
H_3(C_1, C)&=\text{constant}-H_3(C_1, C_2\cup F)-(s^{r_1}+2s-5)H_2(C_1, C_2\cup F)\\
&=\text{constant}-H_3(C_1, F)-L_2(C_1, C_2, F)-\frac{1}{2}(s^{r_2}-s)L_1(C_1, C_2, F)\\
&\quad-(s^{r_1}+2s-5)\{H_2(C_1, F)+L_1(C_1, C_2, F)\}. \qquad (A.6)
\end{aligned}
$$

Interchanging the roles of C_1 and C_2 in (A.5), we also have

$$H_2(C_2, C) = \text{constant} + H_2(C_2, F) + L_1(C_1, C_2, F). \tag{A.7}$$

Parts (i) and (iv) of the lemma are immediate from (8) and Lemma 2. Part (ii) follows from (8), (A.4), (A.5) and (A.7). Next, by (8), (A.2) and Lemma 2, $A_{30}(d) + A_{31}(d) + A_{32}(d) = \text{constant} - G_3(F)$, and part (iii) follows using (9), and parts (i) and (ii). By (8), (A.6) and a dual of (A.6) with roles of C_1 and C_2 interchanged,

$$\begin{aligned} A_{41}(d) = \text{constant } & -H_3(C_1, F) - H_3(C_2, F) - 2L_2(C_1, C_2, F) \\ & -(s^{r_1} + 2s - 5)H_2(C_1, F) - (s^{r_2} + 2s - 5)H_2(C_2, F) \\ & -\{\frac{3}{2}(s^{r_1} + s^{r_2}) + 3s - 10\}L_1(C_1, C_2, F), \end{aligned}$$

whence, using (9), (A.2), (A.3) and (A.4), part (v) can be proved after some algebra. Finally, in order to prove (vi), observe that by (8) and (A.3),

$$\begin{aligned} A_{42}(d) = \text{constant } & + G_4(C_1 \cup C_2 \cup C) - A_{40}(d) - A_{41}(d) - \frac{1}{2}(s^{r_1} + s^{s_2} - 2s)A_{32}(d) \\ & -\frac{1}{2}(s^{r_1} - s)H_2(C_1, C) - \frac{1}{2}(s^{r_2} - s)H_2(C_2, C). \end{aligned} \tag{A.8}$$

Now by Lemma 2, $G_4(C_1 \cup C_2 \cup C) = \text{constant} + (3s - 5)G_3(F) + G_4(F)$. Hence (vi) follows from (A.8) after some simplification using (9), (A.3), (A.5), (A.7) and parts (iii), (iv) and (v).

References

Chen, J., Sun, D. X. and Wu, C. F. J. (1993). A catalogue of two-level and three-level fractional factorial designs with small runs. *Internat. Statist. Rev.* **61**, 131-145.

Cheng, C. S. and Mukerjee, R. (1998). Regular fractional factorial designs with minimum aberration and maximum estimation capacity. *Ann. Statist.* **26**, 2289-2300.

Mukerjee, R. and Wu, C. F. J. (1999). Blocking in regular fractional factorials: a projective geometric approach. *Ann. Statist.* **27**, 1256-1271.

Suen, C. Y., Chen, H. and Wu, C. F. J. (1997). Some identities on q^{n-m} designs with application to minimum aberration designs. *Ann. Statist.* **25**, 1176-1188.

Tang, B. and Wu, C. F. J. (1996). Characterization of minimum aberration 2^{n-k} designs in terms of their complementary designs. *Ann. Statist.* **24**, 2549-2559.

Wu, C. F. J. and Zhang, R. (1993). Minimum aberration designs with two-level and four-level factors. *Biometrika* **80**, 203-209.

Wu, C. F. J., Zhang, R. and Wang, R. (1992). Construction of asymmetrical orthogonal arrays of the type $OA(s^k, (s^{r_1})^{n_1} \dots (s^{r_t})^{n_t})$. *Statist. Sinica* **2**, 203-219.

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