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BOUNDARY CROSSING DISTRIBUTIONS OF RANDOM WALKS RELATED TO THE LAW OF THE ITERATED LOGARITHM

Ulrich Kerkhoff and Hans Rudolf Lerche

University of Freiburg

To David Siegmund on his 70th birthday.

Abstract: A result for the first passage densities of Brownian motion as $t \to \infty$ was given in Lerche (1986) for boundaries that grow faster than $\sqrt{t}$ as $t \to \infty$. From this result the Kolmogorov–Petrovski–Erdős test near infinity has been derived. Here we extend these results to first passage probabilities of random walks. The asymptotic formulas are the same as for Brownian motion and, especially, no overshoot term shows up.

Key words and phrases: Curved boundary crossing of random walks, overshoot calculations, stopping times and the law of the iterated logaritm.

1. Introduction

In several publications Robbins and Siegmund (1970, 1973) derived boundary crossing probabilities related to the law of the iterated logarithm. One of their methods was to use mixture stopping rules; these stop when a certain mixture of likelihood ratios crosses a certain level. Lai and Siegmund (1977, 1979) successfully applied this method to calculate the operating characteristics of a repeated significance test with a bounded horizon in a large deviation sense. For random walks overshoot terms show up, which the authors could handle with nonlinear renewal theory. A wide range of similar and related results can by found in the monograph of Siegmund (1985).

Motivated by these results the second author wrote his monograph Lerche (1986) using a different approach to curved boundary crossing. This approach is closely related to the Kolmogorov–Petrovski–Erdős test (see Itô and McKean (1974, p.33)) and to Strassen's result on first exit times near zero of Brownian motion (see Strassen (1967)). Theorem 3.5 there can be roughly stated as follows: Let ψ denote a smooth boundary (with $\psi(0) = 0$) that is monotone and belongs to the upper class near zero (in the sense of Itô–McKean). Let f denote the first hitting time density at ψ. Then

$$f(t) = \frac{\Lambda(t)}{t^{3/2}} \varphi\left(\frac{\psi(t)}{\sqrt{t}}\right)(1 + o(1)) \qquad \text{as } t \to 0. \tag{1.1}$$

Here $\Lambda(t) = \psi(t) - t\psi'(t)$ and $\varphi(x) = (1/\sqrt{2\pi})e^{-x^2/2}$. The expression on the right-hand side is the hitting density at time t at the tangent of ψ at t according to the Lévy–Bachelier formula (see Itô and McKean (1974, p.25)). The tangent intercept is $\Lambda(t)$. Strassen derived from this result the difficult half of the proof of the Kolmogorov–Petrovski–Erdős test at zero.

Starting from (1.1), Lerche (1986) described a theory which leads to boundary crossing probabilites quite similar to those derived by the method of mixture stopping rules. Some results are given there (in Chapter I, Section 5 of Lerche (1986)) which imply the Kolmogorov–Petrovski–Erdős test near infinity. In contrast to Strassen's approach, where one uses last entrance times (by time inversion), they are formulated for first passage times.

The purpose of this note is to show that similar results hold for random walks. One interesting point is that as the number of observations tends to infinity the overshoot term vanishes to the first order. The main part concentrates on the Gaussian random walk, but also some more general results are stated. Additionally, in those cases a central limit theorem effect turns up. This explains why the asymptotic expressions are the same as for the Gaussian random walk. For these more general situations one assumes that the moment generating function exists in a vicinity of zero; this leads to rather strong local and global central limit theorems. For their application see the Remark at the end of Section 4.1.

Finally we want to indicate the relations to other publications. At first we point at the book of Aldous (1989) on Poisson clumping, which discussed the tangent approximation on pg. 99–100. Further, there are more recent papers on boundary crossing that are more or less closely related, like that of Kesten and Maller (1998) on "Random Walks Crossing High Level Curved Boundaries" and that of Chan and Lai (2003) on "Saddlepoint Approximations and Nonlinear Boundary Crossing Probabilities of Markov Random Walks". Perhaps closest in the sense of integral tests is the paper of Hambly, Kersting, and Kyprianou (2003) on "Law of the iterated logarithm for oscillating random walks conditioned to stay non-negative". We also mention Doney and Maller (2000). Recent work on the law of iterated logarithm of random walks care more about the range in higher dimensions (see Bass and Kumagai (2002) and Hamana (2006)) or discuss more involved topics like local times of self intersections of planar random walks (see Bass, Chen, and Rosen (2005)).

2. Results

Let X_i, $i \geq 1$ denote a sequence of independent random variables that are identically distributed according to $N(0,1)$. Let $S_n = \sum_{i=1}^{n} X_i$, $S_0 = 0$. Let ψ be a nondecreasing continuously differentiable function $\psi : [1,\infty) \to \mathbb{R}$. Let

$T_\psi = \inf\{n \geq 1 \mid S_n \geq \psi(n)\}$ denote the first passage time of the random walk S_n of the boundary ψ, with $T_\psi = \infty$ if the infimum is taken over the empty set.

For boundaries ψ that grow faster to infinity than $\sqrt{t}$ if $t \to \infty$, we study the asymptotic behavior of $P(T_\psi = n)$ as $n \to \infty$. A typical example is $\psi(n) = \sqrt{2n \log\log n}$. Let $\Lambda(n) = \psi(n) - n\psi'(n)$ denote the intercept of the tangent at the curve ψ in point n. The following result is the discrete version of Theorem 5.1 in Lerche (1986) on Brownian motion.

Theorem 1. *Let ψ satisfy the conditions:*

(i) $\psi(t)/\sqrt{t} \to \infty$ *as* $t \to \infty$*;*

(ii) *there exists a constant* α *with* $1/2 < \alpha < 1$, *such that* $\psi(t)/t^\alpha$ *is finally decreasing;*

(iii) *for all* $\varepsilon > 0$ *there exists a* $\delta > 0$ *and a* $t_1 \geq 1$ *such that* $\left|s/t - 1\right| < \delta$ *implies* $\left|\Lambda(s)/\Lambda(t) - 1\right| < \varepsilon$ *if* $s, t \geq t_1$.

Then, as $n \to \infty$

$$P(T_\psi = n) = P(T_\psi \geq n)\frac{\Lambda(n)}{n^{3/2}}\varphi\left(\frac{\psi(n)}{\sqrt{n}}\right)(1 + o(1)). \tag{2.1}$$

Remark 1. We note that for Gaussian random walks the formula on the right-hand side is the same as that for Brownian motion, but evaluated at discrete time points instead of continuous ones. This means that an overshoot term does not show up. In Theorem 1 no restrictions are made concerning ultimate crossing: $P(T_\psi < \infty) < 1$ is as possible as $P(T_\psi < \infty) = 1$, which means that $P(T_\psi \geq n) \to 0$ as $n \to \infty$.

It is possible to generalize the statement for the case $P(T_\psi < \infty) < 1$ to random walks whose increments have finite moment generating functions. This is formulated in Theorem 2. For the case $P(T_\psi < \infty) = 1$ one needs an extra condition to have an inequality for the hazard functions. In the situation of Theorem 1, where one has normally distributed increments, a hazard inequality always holds:

$$h_{\psi_1}(n) \leq h_{\psi_2}(n) \tag{2.2}$$

with $h_i(n) = P(T_{\psi_i} = n)/P(T_{\psi_i} \geq n)$. Here $\psi_i : \{1, \ldots, n\} \to \mathbb{R}_+$ for $i = 1, 2$, with $\psi_1(j) \leq \psi_2(j)$ for $j = 1, \ldots, n-1$ and $\psi_1(n) = \psi_2(n)$.

In more general situations one needs an extra condition for (2.2) to hold. It is the total positivity of order 2 of the Lebesgue density of the increments of the random walk. We discuss this in more detail in Section 3.

Let X_i, $i \leq 1$ be independent identically distributed random variables whose distribution has a density g_1 with respect to the Lebesgue measure. Let $S_n = \sum_{i=1}^n X_i$. A condition is crucial and is always assumed in the following:

$$(*) \qquad E\exp(\theta X_1) < \infty \text{ for all } \theta \text{ with } |\theta| < \theta_0 \text{ with } \theta_0 > 0.$$

Theorem 2. *Let* $E(X_1) = 0$ *and* $\operatorname{Var}(X_1) = 1$. *Let the density* g_1 *satisfy condition* $(*)$ *and let*

$$(**) \qquad \text{there exists a } \theta_1 \text{ with } 0 < \theta_1 < \theta_0 \text{ such that } \sup_{|\theta| \le \theta_1} \sup_{x \in \mathbb{R}} \exp(\theta x) g_1(x) < \infty.$$

Suppose the boundary ψ *satisfies the conditions:*

(i) $\psi(t)/\sqrt{t}$ *is monotone increasing for sufficiently large* t*;*

(ii) *there exists a constant* α *with* $1/2 < \alpha < 2/3$, *such that* $\psi(t)/t^{\alpha}$ *is finally decreasing;*

(iii) *for all* $\varepsilon > 0$ *there exists a* $\delta > 0$ *and a* $t_1 \ge 1$ *such that* $\left|s/t - 1\right| < \delta$ *implies* $\left|\Lambda(s)/\Lambda(t) - 1\right| < \varepsilon$ *if* $s, t \ge t_1$.

If $P(T_\psi = \infty) > 0$ *holds then, as* $n \to \infty$,

$$P(T_\psi = n) = P(T_\psi = \infty) \frac{\Lambda(n)}{n^{3/2}} \varphi\left(\frac{\psi(n)}{\sqrt{n}}\right)(1 + o(1)). \tag{2.3}$$

Remark 2. The right-hand sides of (2.1) and (2.3) are the same. This is a kind of central limit effect in the situation of Theorem 2.

Now we formulate a result which gives up the restriction $P(T_\psi = \infty) > 0$. Families of distributions that satisfy the conditions of the next result are, for instance, the double exponential.

Theorem 3. *Let the conditions* $(*)$, $(**)$ *and* (I)–(III) *of Theorem* 1 *hold, and let* g_1 *be totally positive of order 2. Then, as* $n \to \infty$,

$$P(T_\psi = n) = P(T_\psi \ge n) \frac{\Lambda(n)}{n^{3/2}} \varphi\left(\frac{\psi(n)}{\sqrt{n}}\right)(1 + o(1)). \tag{2.4}$$

As a consequence of Theorems 1 and 3 one can state an asymptotic result for survival probabilities.

Corollary 1. *Let* $P(T_\psi < \infty) = 1$. *Then*

$$P(T_\psi \ge n) = \exp\left[\Big(-\sum_{m=1}^{n-1} \frac{\Lambda(m)}{m^{3/2}} \varphi\Big(\frac{\psi(m)}{\sqrt{m}}\Big)\Big)(1 + o(1))\right]. \tag{2.5}$$

This can be seen as follows. For

$$h_\psi(m) = \frac{P(T_\psi = m)}{P(T_\psi \ge m)},$$

one has

$$P(T_\psi \ge n) = \prod_{m=1}^{n-1} (1 - h_\psi(m)) = \exp\Big(-\sum_{m=1}^{n-1} \log(1 - h_\psi(m))\Big).$$

But $h_\psi(m) \to 0$ by (2.4), thus $-\log(1 - h_\psi(m)) = h_\psi(m)(1 + o(1))$. By (2.4) statement (2.5) follows.

Remark 3. For $\psi(m) = \sqrt{2m \log\log(m)}$ we obtain

$$P(T_\psi \geq n) = \exp\bigg(-\frac{1}{2\sqrt{\pi}}(\log\log n)^{3/2}(1 + o(1))\bigg).$$

We state a version of the Kolmogorov–Petrovski–Erdős test. For a version with last entrance times see Lerche (1986, p.87).

Corollary 2. *Let* $P(T_\psi > n) > 0$ *for all* $n > 0$ *and let the conditions of Theorem 3 hold. Then* $P(T_\psi < \infty) < 1$ *if and only if* $\sum_{n=1}^{\infty}(\psi(n)/n^{3/2})\varphi\big(\psi(n)/\sqrt{n}\big) < \infty$.

Proof. By (II) and the monotonicity of ψ, $(1-\alpha)\psi(n) \leq \Lambda(n) \leq \psi(n)$. Thus $\sum_{n\geq 1}(\psi(n)/n^{3/2})\varphi\big(\psi(n)/\sqrt{n}\big) < \infty$ is equivalent to $\sum_{n\geq 1}(\Lambda(n)/n^{3/2})\varphi\big(\psi(n)/\sqrt{n}\big) < \infty$. Let $P(T_\psi < \infty) = 1$. Then (2.5) implies

$$\sum_{n\geq 1}\frac{\Lambda(n)}{n^{3/2}}\varphi\bigg(\frac{\psi(n)}{\sqrt{n}}\bigg) = \infty.$$

Conversely if $P(T_\psi < \infty) < 1$, then $P(T_\psi = \infty) > 0$. Then by (2.3), as $n' \to \infty$,

$$\begin{aligned} P(n' < T_\psi < \infty) &= \sum_{n\geq n'} P(T_\psi = n) \\ &= P(T_\psi = \infty)(1 + o(1)) \sum_{n\geq n'} \frac{\Lambda(n)}{n^{3/2}}\varphi\bigg(\frac{\psi(n)}{\sqrt{n}}\bigg). \end{aligned}$$

This implies $\sum_{n\geq 1}(\Lambda(n)/n^{3/2})\varphi\big(\psi(n)/\sqrt{n}\big) < \infty$.

We will give a complete proof of Theorem 1. It follows the scheme of the proof of (1.1), which is a statement for $t \to 0$. Strassen's construction of the time sections also works here for $n \to \infty$. It is combined with results of Woodroofe and with Donsker's invariance principle.

The hazard inequality is discussed in Section 3. The proofs of Theorems 2 and 3 are rather lengthy and can be found in Kerkhoff (1990). See also the Remark at the end of Section 4.1.

3. The Hazard Inequality

We prove that the hazard inequality (2.2) holds when the density g_1 is totally positive of order 2 (its definition follows). This latter assumption enables one to prove monotonicity statements from which the hazard inequality follows. Let $\psi_i : \{1, \dots, n\} \to \mathbb{R}_+$ for $i = 1, 2$, with $\psi_1(j) \leq \psi_2(j)$. Then

$$P(S_n > z \mid T_{\psi_1} > n) \leq P(S_n > z \mid T_{\psi_2} > n). \tag{3.1}$$

We show that (3.1) implies the hazard inequality (2.1). First

$$P(T_{\psi_1} = n \mid T_{\psi_1} \geq n) \leq P(T_{\psi_2} = n \mid T_{\psi_2} \geq n).$$

Since $\psi_1(n) = \psi_2(n)$ we have

$$\begin{aligned} P(T_{\psi_1} = n \mid T_{\psi_1} \geq n) &= \int P(X_n + x \geq \psi_1(n)) P(S_{n-1} \in dx \mid T_{\psi_1} > n-1) \\ &= \int P(X_n + x \geq \psi_2(n)) P(S_{n-1} \in dx \mid T_{\psi_1} > n-1). \end{aligned}$$

The integrand is nondecreasing in x, thus by (3.1) we get

$$\begin{aligned} P(T_{\psi_1} = n \mid T_{\psi_1} \geq n) &\leq \int P(X_n + x \geq \psi_2(n)) P(S_{n-1} \in dx \mid T_{\psi_2} > n-1) \\ &= P(T_{\psi_2} = n \mid T_{\psi_2} \geq n). \end{aligned}$$

For (3.1) we appeal the notion of total positivity of order 2; for the concept see Karlin (1968).

Definition 1. A measurable function $h : \mathbb{R} \to \mathbb{R}_+$ that is different from zero for at least two points has property TP2 if

$$\begin{vmatrix} h(w_2 - w_1) & h(z_2 - w_1) \\ h(w_2 - z_1) & h(z_2 - z_1) \end{vmatrix} \geq 0$$

for all (z_1, z_2), $(w_1, w_2) \in \mathbb{R}^2$ with $w_1 \leq z_1$ and $w_2 \leq z_2$.

We turn to (3.1).

Lemma 1. *Let $\psi_i : \{1, \ldots, n\} \to \mathbb{R}_+$ with $\psi_1 \leq \psi_2$. Let $X_1, \ldots, X_n$ be independent and identically distributed with density g_1 with respect to Lebesgue-measure. Let g_1 have property TP2. Then*

$$P(S_n > z \mid T_{\psi_1} > n) \leq P(S_n > z \mid T_{\psi_2} > n). \tag{3.2}$$

Proof. For simplicity we write g instead of g_1. We show, that we can apply a FKG-type inequality (see e.g., Karlin and Rinott (1980)): Let h_i, $i = 1, 2$, denote densities on $\mathbb{R}^n$ with respect to the n-dimensional Lebesgue-measure λ_n. Then $\int f h_1 d\lambda_n \leq \int f h_2 d\lambda_n$ for any nondecreasing function f, if

$$h_1(x) h_2(y) \leq h_1(x \vee y) h_2(x \wedge y) \tag{3.3}$$

holds for all $x, y \in \mathbb{R}^n$. To apply the inequality we put

$$\tilde{g}(s_1, \ldots, s_n) = \prod_{i=1}^{n} g(s_i - s_{i-1}), \tag{3.4}$$

with $s_0 = 0$. This is the density of the joint distribution of $(S_1, \dots, S_n)$ with respect to λ_n. We also put, for $i = 1, 2$,

$$h_i := \frac{\tilde{g} I_{\{T_{\psi_i} > n\}}}{P(T_{\psi_i} > n)}$$

and let $f(s_1, \dots, s_n) = I_{(z,\infty)}(s_n)$. Then (3.2) follows.
It is left to show (3.3) for the choice of h_i. It is clear that

$$\prod_{i=1}^{n} I_{(-\infty,\psi_1(i))}(x_i) I_{(-\infty,\psi_2(i))}(y_i) \le \prod_{i=1}^{n} I_{(-\infty,\psi_1(i))}(x_i \wedge y_i) I_{(-\infty,\psi_2(i))}(x_i \vee y_i). \tag{3.5}$$

By the TP2 property of g one obtains for $i = 2, \dots, n$,

$$\begin{aligned} & g(x_i \vee y_i - x_{i-1} \vee y_{i-1}) g(x_i \wedge y_i - x_{i-1} \wedge y_{i-1}) \\ & \quad \ge g(x_i \vee y_i - x_{i-1} \wedge y_{i-1}) g(x_i \wedge y_i - x_{i-1} \vee y_{i-1}) \\ & \quad = g(y_i - y_{i-1}) g(x_i - x_{i-1}). \end{aligned} \tag{3.6}$$

The last equality follows by distinction of cases. Using (3.4) and (3.6) yields

$$\tilde{g}(x)\tilde{g}(y) \le \tilde{g}(x \vee y)\tilde{g}(x \wedge y).$$

This inequality combined with that of (3.5) yields (3.3) for our choice of h_i.

Lemma 2. *Let $\psi_i : \mathbb{N} \to \mathbb{R}_+$ for $i = 1, 2$ with $\psi_1(j) \le \psi_2(j)$ for $j = 1, \dots, n-1$, and $\psi_1(n) = \psi_2(n)$. If g_1 has the property TP2, then*

$$P(T_{\psi_1} = n \mid T_{\psi_1} \ge n) \le P(T_{\psi_2} = n \mid T_{\psi_2} \ge n).$$

We remark that normal densities have the property TP2. This follows, for instance, by a result of Schönberg (1951). We state it without proof and point for further information to Karlin's monograph (1968).

Theorem 4. *A function h has property TP2 if and only if it can be written as $h = \exp(-T)$, where T is a convex function.*

4. Proofs

4.1. The upper estimate

We prove Theorem 1. At first we show the upper estimate

$$P(T_\psi = n) \le P(T_\psi \ge n) \frac{\Lambda(n)}{n^{3/2}} \varphi\left(\frac{\psi(n)}{\sqrt{n}}\right)(1 + o(1)) \quad \text{as } n \to \infty. \tag{4.1}$$

Let $\varepsilon > 0$ with $\alpha + \varepsilon < 1$. Let $s(n) := n\left(1 - n^{\varepsilon}/(\psi(n))^{2\varepsilon}\right)$ and $k(n) := [s(n)]$ be the largest natural number less than $s(n)$. We apply Lemma 2 to ψ and ψ_k, where $\psi_k : \mathbb{N} \to \mathbb{R}_+$ with $\psi_k(i) \equiv \psi(k)$ for $i \leq k$ and $\psi_k(i) = \psi(i)$ if $i > k$.

We obtain

$$\begin{aligned} &P(T_\psi = n \mid T_\psi \geq n) \\ &\quad \leq P(T_{\psi_k} = n \mid T_{\psi_k} \geq n) \\ &\quad \leq P(T_{\psi_k} \geq n)^{-1} P(S_i < \psi_k(i) \text{ for } i = 1, \ldots, n-1, S_n \geq \psi_k(n)). \end{aligned} \tag{4.2}$$

We show that $P(T_{\psi_k} \geq n) \to 1$. Since ψ is nondecreasing, we get further

$$\begin{aligned} P(T_{\psi_k} \geq n) &\geq P(S_i < \psi(k) \text{ for } i = 1, \ldots, n-1) \\ &= P\Big(\frac{1}{\sqrt{n-1}} \max_{1 \leq i \leq n-1} S_i < \frac{\psi(k)\sqrt{k}}{\sqrt{k(n-1)}}\Big). \end{aligned} \tag{4.3}$$

By Donsker's invariance principle the right-hand side converges to the corresponding distribution of the maximum of Brownian motion, given by $2(\Phi(x) - 1)^+$, where $y^+ = \max(y, 0)$. But x here turns out to be infinity, since by the definition of $k(n)$, we have $k(n)/n \to 1$ and by (I) it holds $\psi(n)/\sqrt{n} \to \infty$. Thus the right-hand side of (4.3) tends to one. Then the inequality (4.2) can be written further as

$$P(T_\psi = n) \leq P(T_\psi \geq n) P(S_i \leq \psi(i) \text{ for } i = k, \ldots, n-1, S_n \geq \psi(n))(1 + o(1)).$$

We show that

$$P(S_i < \psi(i), i = k, \ldots, n-1, S_n \geq \psi(n)) \leq \frac{\Lambda(n)}{n^{3/2}} \varphi\Big(\frac{\psi(n)}{\sqrt{n}}\Big)(1 + o(1)).$$

For simplicity write

$$g_n(x) := \frac{1}{\sqrt{n}} \varphi\Big(\frac{x}{\sqrt{n}}\Big).$$

Let $\Lambda_1 = \sup\{\Lambda(u) \mid u \in [k, n]\}$. Let h denote the straight line determined by $h(0) = \Lambda_1$ and $h(n) = \psi(n)$. Then

$$\begin{aligned} &P(S_i < \psi(i) \text{ for } i = k, \ldots, n-1, S_n \geq \psi(n)) \\ &\quad \leq P(S_i < h(i) \text{ for } i = k, \ldots, n-1, S_n \geq \psi(n)) \\ &\quad = g_n(\psi(n)) \int_0^\infty P(S_i < h(i) \text{ for } i = k, \ldots, n-1 \mid S_n = h(n) + r) \frac{g_n(h(n) + r)}{g_n(h(n))} dr \\ &\quad = g_n(\psi(n)) \int_0^\infty I_1(r) I_2(r) dr. \end{aligned} \tag{4.4}$$

Now

$$\begin{aligned} I_1 &= P(S_i < h(i) \text{ for } i = k, \dots, n-1 \mid S_n = h(n) + r) \\ &= P(S_n - S_i > h(n) - h(i) + r \text{ for } i = k, \dots, n-1 \mid S_n = h(n) + r) \\ &= P(S_j > h(n) - h(n-j) + r \text{ for } j = 1, \dots, n-k \mid S_n = h(n) + r) \\ &= P\Big(S_i > \Big(\frac{\psi(n)}{n} - \frac{\Lambda_1}{n}\Big)i + r \text{ for } i = 1, \dots, n-k \mid S_n = \psi(n) + r\Big). \end{aligned}$$

We use the fact that this conditional distribution does not depend on the drift of the underlying distribution, by sufficiency of S_n. Let

$$g_n^\theta(x) := \frac{1}{\sqrt{n}}\varphi\Big(\frac{x - n\theta}{\sqrt{n}}\Big). \tag{4.5}$$

We choose as drift $\theta_n = \psi(n)/n$, and make a change of measure from that without drift to that with drift θ_n, and indicate this by a subscript of the measure. Then

$$\begin{aligned} I_1 &= P\Big(S_i > \Big(\frac{\psi(n)}{n} - \frac{\Lambda_1}{n}\Big)i + r \text{ for } i = 1, \dots, n-k \mid S_n = \psi(n) + r\Big) \\ &= P_{\theta_n}\Big(S_i > \Big(\frac{\psi(n)}{n} - \frac{\Lambda_1}{n}\Big)i + r \text{ for } i = 1, \dots, n-k \mid S_n = \psi(n) + r\Big) \\ &= \int dP_{\theta_n}^{X_1,\dots,X_{n-k}}(s_1, \dots, s_{n-k}) I_{\{s_i > (\frac{\psi(n)}{n} - \frac{\Lambda_1}{n})i + r, i=1,\dots,n-k\}} \\ &\quad \cdot \frac{g_k^{\theta_n}(\psi(n) + r - s_{n-k})}{g_n^{\theta_n}(\psi(n) + r)}. \end{aligned}$$

We express I_2 also with the drift θ_n. We have from (4.5)

$$\frac{g_n^{\theta_1}(x)}{g_n^{\theta_2}(x)} = \frac{\varphi\left((x - n\theta_1)/\sqrt{n}\right)}{\varphi\left((x - n\theta_2)/\sqrt{n}\right)} = \exp\Big((\theta_1 - \theta_2)x - \frac{n}{2}(\theta_1^2 - \theta_2^2)\Big).$$

Thus

$$I_2 = \frac{g_n(\psi(n) + r)}{g_n(\psi(n))} = \exp(-\theta_n r)\frac{g_n^{\theta_n}(\psi(n) + r)}{g_n^{\theta_n}(\psi(n))}.$$

Combining these expressions we get for the integral on the right-hand side of (4.4),

$$\begin{aligned} \int_0^\infty I_1(r) I_2(r) dr &\le (1 + o(1)) \int_0^r dr \int dP_{\theta_n}^{X_1,\dots,X_{n-k}} I_{\{s_i > (\psi(n)/n - \Lambda_1/n)i + r, i=1,\dots,n-k\}} \\ &\quad \cdot \exp\Big(-\frac{(r - s_{n-k} + [\psi(n)/n](n-k))^2}{2k}\Big) \\ &\le (1 + o(1)) \int_0^r dr P_{\Lambda_1/n}(S_i > r, i = 1, \dots, n-k) \\ &= (1 + o(1)) E_{\Lambda_1/n}(M_{n-k}^+). \end{aligned}$$

Here the expectation is taken with respect to the drift Λ_1/n and $M_m^+ := (\min\limits_{1\le i\le m} S_i)^+$ denotes the positive part of the minimum. By Lemma 3 we have, with $S_{n-k}^- := -\min(S_{n-k}, 0)$, that

$$E_{\Lambda_1/n}(M_{n-k}^+) = \frac{\Lambda_1}{n} + \frac{1}{n-k} E_{\Lambda_1/n}(S_{n-k}^-).$$

Here we can show that

$$\frac{1}{n-k} E_{\Lambda_1/n}(S_{n-k}^-) = o\left(\frac{\Lambda_1}{n}\right) \tag{4.6}$$

holds, so

$$\int I_1(r) I_2(r) dr \le \frac{\Lambda_1}{n}(1 + o(1)).$$

By (I) and (III), Λ_1 can be substituted by $\Lambda(n)$ asymptotically, from which (4.1) follows.

By using Fubini's theorem one obtains (4.6) as follows:

$$\begin{aligned}\frac{1}{n-k} E_{(\Lambda_1/n)}(S_{n-k}^-) &= \frac{1}{n-k} \int_0^\infty dx\, P_{(\Lambda_1/n)}(-S_{n-k} \ge x)\\ &= \frac{1}{n-k} \int_0^\infty dx \int_{-\infty}^\infty du\, I_{\{x \le u\sqrt{n-k}-(n-k)(\Lambda_1/n)\}} \varphi(u)\\ &= \frac{1}{n-k} \int_{\sqrt{n-k}(\Lambda_1/n)}^\infty du\, \varphi(u) \Big(u\sqrt{n-k} - (n-k)\frac{\Lambda_1}{n}\Big)\\ &= \frac{\Lambda_1}{n} \Big[\frac{\varphi\Big(\sqrt{n-k}(\Lambda_1/n)\Big)}{\sqrt{n-k}(\Lambda_1/n)} - \Big(1 - \Phi\Big(\sqrt{n-k}\,\frac{\Lambda_1}{n}\Big)\Big)\Big]\\ &= o\left(\frac{\Lambda_1}{n}\right) \text{ for } n \to \infty.\end{aligned}$$

This follows since $\Lambda_1 \ge \Lambda(n)$ and $\Lambda(n) \ge \psi(n)(1-\alpha)$ by (II) and, since $(\psi(n)/n)\sqrt{n-k} \to \infty$ by (I), the definition of $k(n)$.

Remark 4. We want to indicate how one derives the upper bound of Theorem 2 and how the central limit theorem comes in there. By $(*)$ one can define a related exponential family of measures with drift. This allows one to apply sufficiency arguments as in the proof of Theorem 1, (4.5) and the following.

Let s, k, r, l be as in the proof of Theorem 1. Then

$$P(T_\psi = n) \le \int_{-\infty}^{\psi(l)} P(T_\psi > l, S_l \in dx) P^{l,x}(S_i < \psi(i), i = k, \ldots, n-1, S_n \ge \psi(n)).$$

One can show similarly to the proof of Theorem 1, that

$$\begin{aligned} P^{l,x}\big(S_i < \psi(i), i = k, \dots, n-1, S_n \geq \psi(n)\big) &\leq g_{n-l}(\psi(n) - x) \int_0^\infty dr I_1 \cdot I_2 \\ &\leq g_{n-l}(\psi(n) - x)\Big(\frac{\Lambda(n)}{n} + R\Big), \end{aligned}$$

with g_k the k-fold density of g_1 and R is the remainder.

At this point one applies the central limit theorem twofold, to estimate R with a global version and $g_{n-\ell}(\psi(n) - x)$ with a local one.

4.2. The lower estimate

We want to show now, as $n \to \infty$,

$$P(T_\psi = n) \geq P(T_\psi \geq n) \frac{\Lambda(n)}{n^{3/2}} \varphi\Big(\frac{\psi(n)}{\sqrt{n}}\Big)(1 + o(1)). \tag{4.7}$$

We partition the event $\{T_\psi = n\}$ into three time sections. The middle section is to have low probability. The first and third sections contribute to the two terms on the right-hand side of formula (4.7).

We now define the two splitting points k and l of the sections, essentially following Strassen (1967). Let $\varepsilon > 0$ be chosen such that $\alpha + \varepsilon < 1$. Let $s = n\left(1 - n^\varepsilon/\psi(n)^{2\varepsilon}\right)$ and $k = [s]$, the largest integer smaller than s. Let β and γ be chosen such that $\alpha + \varepsilon < 2\beta - 1 < \gamma < \beta < 1$. Let r denote the solution of the implicit equation $\psi(k)^2/k\,(r/k)^\gamma = 1$. Let $l = [r]$ and let $k_l = \psi(k)\,(l/k)^\beta$. Since $k \sim n$ and, by (I) $l/k \to 0$ as $n \to \infty$. By the definition of β and γ and (I),

$$\frac{\psi(k)^2}{k}\Big(\frac{l}{k}\Big)^\beta \to 0, \tag{4.8}$$

$$\frac{\psi(k)^2}{k}\Big(\frac{l}{k}\Big)^{2\beta-1} \to \infty. \tag{4.9}$$

Let $P^{m,z}\{T_\psi = n\}$ denote the first passage probability of the curve ψ by the random walk $\{S_i; i \in \mathbb{N}\}$ that starts at time m in point $z < \psi(m)$. Then one has

$$\begin{aligned} P(T_\psi = n) &\geq \int_{-k_l}^{k_l} P(T_\psi > l, S_l \in dx) P^{l,x}(T_\psi = n) \\ &\geq P(T_\psi > l, |S_l| \leq k_l) \inf_{|x| \leq k_l} P^{l,x}(T_\psi = n). \end{aligned} \tag{4.10}$$

Now if

$$P(T_\psi > l, |S_l| \leq k_l) \sim P(T_\psi > l), \tag{4.11}$$

one has

$$P(T_\psi > l, |S_l| \le k_l) = (1+o(1))P(T_\psi > l) \ge (1+o(1))P(T_\psi \ge n).$$

To show (4.11), we prove

$$\lim_{n\to\infty} P(S_l \le k_l \mid T_\psi > l) = 1. \tag{4.12}$$

To derive this, we apply Lemma 1 to ψ and ψ_l, defined as $\psi_l : \mathbb{N} \to \mathbb{R}^+$ with $\psi_l(i) = \psi(l)$ for $i = 1, \dots, l$, and $\psi_l(i) = \psi(i)$ for $i > l$. By the monotonicity of ψ, $\psi \le \psi_l$ on $\{1, \dots, l\}$ we get from Lemma 1,

$$P(S_l \le k_l \mid T_\psi > l) \ge P(S_l \le k_l \mid T_{\psi_l} > l) \ge \frac{P(\max_{1\le i\le l} S_i \le k_l)}{P\left(\max_{1\le i\le l} S_i < \psi(l)\right)}. \tag{4.13}$$

Now, if $n \to \infty$, then $l \to \infty$ and $\psi(l)/\sqrt{l} \to \infty$ by assumption I). Further

$$\frac{k_l}{\sqrt{l}} = \frac{\psi(k)}{\sqrt{l}}\Big(\frac{l}{k}\Big)^{\beta} = \frac{\psi(k)}{\sqrt{k}}\Big(\frac{l}{k}\Big)^{\beta-1/2} \to \infty$$

by (4.9).

From Donsker's invariance principle it follows, that the right-hand side of (4.13) tends to 1 as $n \to \infty$, which implies (4.12).

We show now

$$\lim_{n\to\infty} P(S_l < -k_l \mid T_\psi > l) = 0. \tag{4.14}$$

Let $\psi_1 : \mathbb{N} \to \mathbb{R}_+$ with $\psi_1(i) = \psi(1)$ for $i = 1, \dots, l$ and $\psi_1(i) = \psi(i)$ for $i > l$. Then by the monotonicity of ψ, $\psi \ge \psi_1$ holds. By Lemma 1 one obtains

$$P(S_l \le -k_l \mid T_\psi > l) \le P(S_l \le -k_l \mid T_{\psi_1} > l) \le P(S_l \le -k_l \mid T_{\tilde\varepsilon} > l).$$

Here $\tilde\varepsilon$ denotes the constant function with value ε at $\mathbb{N}$, where $0 < \varepsilon \le f(1)$. Since the inequality holds for every such ε, one obtains by symmetry for the random walk

$$\begin{aligned}
&P(S_l \le -k_l \mid T_\psi > l)\\
&\quad\le P(S_l \le -k_l \mid S_i \le 0, \text{ for } i = 1, \dots, l)\\
&\quad= \frac{P(S_l \ge k_l, S_i \ge 0 \text{ for } i = 1, \dots, l)}{P(S_i \le 0, \text{ for } i = 1, \dots, l)}\\
&\quad\le \frac{1}{k_l}\frac{E(S_l; S_i \ge 0, \text{ for } i = 1, \dots, l)}{P(S_i \le 0; \text{ for } i = 1, \dots, l)}\\
&\quad= \left(\frac{k_l}{\sqrt{l}}\sqrt{l}\, P(S_i \le 0, \text{ for } i = 1, \dots, l)\right)^{-1} \cdot E(S_l; S_i \ge 0, \text{ for } i = 1, \dots, l).
\end{aligned}$$

But $k_l/\sqrt{l} \to \infty$ and there exists a positive constant K with $\sqrt{l}\, P(S_i \leq 0,\ \text{for } i = 1,\dots,l) \to K$. See Feller (1971, p.415). By Lemma 4 the expectation term remains bounded. Thus one obtains (4.14). But (4.12) together with (4.14) yields (4.11).

To estimate $\inf_{|x|\leq k_l} P^{l,x}(T_\psi > n)$ at (4.10), we linearize the boundary ψ as follows: Let $\lambda_1 = \inf\{\Lambda(u) \mid u \in [k,n]\}$ and $\nu \in (0,1]$, such that h_1 is a straight line with $h_1(0) = \nu\lambda_1$ and $h_1(n) = f(n)$. Let h_2 denote the straight line with $h_2(k) = \psi(k)$ and $h_2(l) = \psi(k)\,(l/k)^\alpha$. Then

$$\begin{aligned} &P^{l,x}(T_\psi = n) \\ &\quad \geq \int_{-\infty}^{h_1(k)} P^{l,x}(T_\psi > k, S_k \in dy) P^{k,y}(T_{h_1} = n) \\ &\quad \geq \int_{-\infty}^{h_1(k)} P^{l,x}(S_k \in dy) \\ &\qquad \cdot \Big[1 - P^{l,x}(S_i \geq h_2(i) \text{ for some } i = l+1,\dots,k-1 \mid S_k = y)\Big] P^{k,y}(T_{h_1} = n). \end{aligned} \tag{4.15}$$

It now holds that

$$P^{l,x}(S_i \geq h_2(i) \text{ for some } i = l+1,\dots,k-1 \mid S_k = y) = o(1) \tag{4.16}$$

uniformly in $|x| \leq k_l$ and $y \leq h_1(k)$ for $n \to \infty$. To see this, let W denote Brownian motion. Then

$$\begin{aligned} &P^{l,x}(S_i \geq h_2(i) \text{ for some } i = l+1,\dots,k-1 \mid S_k = y) \\ &\qquad \leq P^{l,x}(W_t \geq h_2(t) \text{ for some } t \in (l,k) \mid W_k = y) \\ &\qquad = \exp\Big(\frac{-2(h_2(l)-x)(h_2(k)-y)}{k-l}\Big) \\ &\qquad \leq \exp\Big(\frac{-2(h_2(l)-k_l)(h_2(k)-h_1(k))}{k-l}\Big). \end{aligned} \tag{4.17}$$

Since l, k, k_l as well as h_1 and h_2 are slightly modified versions of r, s, and k in the proof of Theorem 3.5 in Strassen (1967), one obtains with similar estimates and an appropriate choice of ν (tending to 1),

$$\frac{(h_2(l)-k_l)(h_2(k)-h_1(k))}{k-l} \to \infty.$$

This together with (4.17) implies (4.16).

It is therefore left to estimate the part regarding h_1 in (4.15). We show

$$P^{l,x}(S_i < h_1(i) \text{ for } i = k,\dots,n-1, S_n \geq h_1(n)) \geq \tilde{\theta}_n g_{n-l}(\psi(n)-x)(1+o(1)) \tag{4.18}$$

with $\tilde{\theta}_n = \nu(\lambda_1/n) - \psi(n)/n + (\psi(n) - x)/(n - l)$ uniformly in $|x| \le k_l$. By the definition of k_l and (II),

$$\tilde{\theta}_n \ge \nu\frac{\lambda_1}{n} - \frac{k_l}{n-l} = \nu\frac{\lambda_1}{n}(1 + o(1))$$

uniformly for $|x| \le k_l$. Since $l = o(k)$ and $(\psi(k)^2/k)\,(l/k)^\beta \to 0$ as $n \to \infty$, it follows that

$$g_{n-l}(\psi(n) - x) \ge g_{n-l}(\psi(n) + k_l) \ge \frac{1}{\sqrt{n}}\varphi\left(\frac{\psi(n)}{\sqrt{n}}\right)(1 + o(1)) \qquad (4.19)$$

uniformly for $|x| \le k_l$. Since $\nu \to 1$ as $n \to \infty$, by (I) and (III) λ_1 can be substituted by $\Lambda(n)$. This and (4.18) together with (4.19) yield, for (4.15),

$$P^{l,x}(T_\psi = n) \ge \frac{\Lambda(n)}{n^{3/2}}\varphi\left(\frac{\psi(n)}{\sqrt{n}}\right)(1 + o(1))$$

uniformly for $|x| \le k_l$. This together with (4.10) and (4.11) yields (4.7).

It remains to show (4.18). The argument for that is similar to that for the upper estimate of (4.4). We have

$$P^{l,x}(S_i < h_1(i) \text{ for } i = k, \dots, n-1, S_n > h_1(n)) = g_{n-l}(\psi(n) - x)\int_0^\infty I_1 I_2 \, dr \qquad (4.20)$$

with

$$\begin{aligned} I_1 &= P(S_i > h_1(n) - h_1(n-i) + r \text{ for } i = 1, \dots, n-k \mid x + S_{n-l} = h_1(n) + r) \\ &= P\left(S_i > \left(\frac{\psi(n)}{n} - \nu\frac{\lambda_1}{n}\right)i + r, i = 1, \dots, n-k \mid S_{n-l} = \psi(n) + r - x\right), \\ I_2 &= \frac{g_{n-l}(\psi(n) + r - x)}{g_{n-l}(\psi(n) - x)} = \exp(-\theta_n r)\frac{g_{n-l}^{\theta_n}(\psi(n) + r - x)}{g_{n-l}^{\theta_n}(\psi(n) - x)}. \end{aligned}$$

Now choose the drift $\theta_n = (\psi(n) - x)/(n - l)$. Evaluating I_1, noting $n - l \sim k - l$ and rewriting I_2 with drift θ_n, yields

$$\begin{aligned} I_1 I_2 = (1 + o(1))\exp(-\theta_n r) \int dP_{\theta_n}^{X_1,\dots,X_{n-k}} I_{\{s_i > (\psi(n)/n - \nu(\lambda_1/n))i + r, i = 1, \dots, n-k\}} \\ \cdot\sqrt{2\pi}\,\varphi\left(\frac{\psi(n) + r - x - s_{n-k} - (k-l)\theta_n}{(k-l)^{1/2}}\right). \end{aligned}$$

With $\gamma_n = \dfrac{\sqrt{n-k}}{\sqrt{k-l}}$,

$$\tilde{S}_{n-k} = \frac{S_{n-k} - (n-k)\tilde{\theta}_n}{\sqrt{n-k}},$$

$$r_n = \sqrt{n-k},$$

$$\tilde{\theta}_n = \nu\frac{\lambda_1}{n} - \frac{\psi(n)}{n} + \frac{\psi(n) - x}{n-l},$$

$$\begin{aligned}\int_0^\infty dr\, I_1 I_2 &\geq (1+o(1)) \int_0^{r_n} dr \exp\{-\theta_n r\} \\ &\quad \cdot \int dP_{\tilde{\theta}_n}^{X_1,\ldots,X_{n-k}} I_{\{s_i > r, i=1,\ldots,n-k\}} (2\pi)^{1/2} \varphi\left(\gamma_n\left(-\tilde{s}_{n-k} + \frac{r}{(n-k)^{1/2}}\right)\right) \\ &\geq (1+o(1)) \int_0^{r_n} dr \exp\{-\theta_n r\} \\ &\quad \cdot \int dP_{\tilde{\theta}_n}^{X_1,\ldots,X_{n-k}} I_{\{s_i > r, i=1,\ldots n-k\}} (2\pi)^{1/2} \varphi(\gamma_n(|\tilde{s}_{n-k}| + 1])).\end{aligned}$$

By the definition of l and k and assumption (I), $\gamma_n \to 0$ as $n \to \infty$. Let $(\varepsilon_n; n \geq 1)$ be a sequence with $\varepsilon_n > 0$, $\varepsilon_n \to 0$, and $\varepsilon_n/\gamma_n \to \infty$ as $n \to \infty$. Then

$$\int_0^\infty dr I_1 I_2 \geq (1+o(1)) \left[\int_0^{r_n} dr \exp(-\theta_n r) P_{\tilde{\theta}_n}(S_i > r \text{ for } i = 1, \ldots, n-k) + W_n\right] \tag{4.21}$$

with

$$|W_n| \leq \int_0^{r_n} dr \int dP_{\tilde{\theta}_n}^{X_1,\ldots,X_{n-k}} (1 - (2\pi)^{1/2} \varphi(\gamma_n(|\tilde{s}_{n-k}| + 1))) I_{\{s_i > r, i=1,\ldots,n-k\}}.$$

Now we study W_n. One has

$$\begin{aligned}|W_n| &\leq K_1 \sup_{|u| \leq \varepsilon_n} |(2\pi)^{1/2}\varphi(u) - 1| \int_0^\infty dr\, P_{\tilde{\theta}_n} I_{\{S_i > r, i=1,\ldots,n-k\}} \\ &\quad + K_2 \int_0^\infty dr \int dP_{\tilde{\theta}_n}^{X_1,\ldots,X_{n-k}} I_{\{\gamma_n(|\tilde{s}_{n-k}|+1) > \varepsilon_n\}} I_{\{s_i > r, i=1,\ldots,n-k\}} \\ &\leq K(R_1 + R_2).\end{aligned}$$

Since $\varepsilon_n \to 0$, by Lemma 3 and (4.6) with Λ_1/n replaced by $\tilde{\theta}_n$, we obtain $R_1 = o(\tilde{\theta}_n)$. Note that $\tilde{\theta}_n\sqrt{n-k} \to \infty$ since $\tilde{\theta}_n \geq \nu(\lambda_1/n) - x/(n-l) \geq \nu(\lambda_1/n) - k_l/(n-l)$. One further has $\nu \to 1$, $(\Lambda_1/n)\sqrt{n-k} \to \infty$, and $k_l/\sqrt{n} =$

$\psi(k)/\sqrt{n}\,(l/k)^{\beta} \to 0$ by (4.8). The estimate of R_2 is done as follows:

$$\begin{aligned} R_2 &= E_{\tilde{\theta}_n}(M^+_{n-k} I_{\{\gamma_n(|\tilde{S}_{n-k}|+1) > \varepsilon_n\}}) \\ &= E_{\tilde{\theta}_n}\big(M^+_{n-k} I_{\{S_{n-k} > (\varepsilon_n/\gamma_n - 1)\sqrt{n-k} + (n-k)\tilde{\theta}_n\}}\big) \\ &\quad + E_{\tilde{\theta}_n}\big(M^+_{n-k} I_{\{0 < S_{n-k} < -(\varepsilon_n/\gamma_n - 1)\sqrt{n-k} + (n-k)\tilde{\theta}_n\}}\big) \\ &= R_{2,1} + R_{2,2}. \end{aligned}$$

Applying Lemma 5 and Fubini's theorem to $R_{2,1}$ yields

$$\begin{aligned} R_{2,1} = {}& \frac{1}{n-k}\left(\tilde{\theta}_n(n-k) + \left(\frac{\varepsilon_n}{\gamma_n} - 1\right)\sqrt{n-k}\right) P_{\tilde{\theta}_n}\left\{\tilde{S}_{n-k} > \frac{\varepsilon_n}{\gamma_n} - 1\right\} \\ & + (n-k)^{-1/2} \int_{\varepsilon_n/\gamma_n - 1}^{\infty} du P_{\tilde{\theta}_n}\{\tilde{S}_{n-k} > u\} = o(\tilde{\theta}_n), \end{aligned}$$

since $\varepsilon_n/\gamma_n \to \infty$ and $\tilde{\theta}_n\sqrt{n-k} \to \infty$ as $n \to \infty$. By Lemma 5 and Fubini's theorem, since $\varepsilon_n/\gamma_n \to \infty$,

$$R_{2,2} = (n-k)^{-1/2} \int_{\left(-\tilde{\theta}_n\sqrt{n-k},\, -(\varepsilon_n/\gamma_n - 1)\right)} du P_{\tilde{\theta}_n}\Big(-\Big(\frac{\varepsilon_n}{\gamma_n} - 1\Big) > \tilde{S}_{n-k} > u\Big) = o(\tilde{\theta}_n).$$

Thus we have shown that $|W_n| = o(\tilde{\theta}_n)$, and, from (4.21),

$$\int_0^\infty dr\, I_1 I_2 \geq (1 + o(1)) \int_0^{r_n} \exp(-\theta_n r) P_{\tilde{\theta}_n}(S_i > r \text{ for all } i \in \mathbb{N}) dr + o(\tilde{\theta}_n).$$

Now we want to increase the right-hand side by O_n to have an integral from 0 to infinity. By Theorem 2.7 of Woodroofe (1982) we see that the resulting error is $o(\tilde{\theta}_n)$.

Let $\tau_+ = \inf\{n \geq 1 \mid S_n > 0\}$. By the definition of r_n and since $\theta_n\sqrt{n-k} \to \infty$,

$$\begin{aligned} O_n &= \int_{r_n}^\infty dr \exp\{-\theta_n r\} P_{\tilde{\theta}_n}\{S_i > r \text{ for all } i \in \mathbb{N}\} \\ &= \tilde{\theta}_n \int_{r_n}^\infty dr \exp\{-\theta_n r\} (E_{\tilde{\theta}_n} S_{\tau_+})^{-1} P_{\tilde{\theta}_n}\{S_{\tau_+} > r\} \\ &\leq \tilde{\theta}_n (r_n\theta_n)^{-1} = o(\tilde{\theta}_n). \end{aligned}$$

Thus we have

$$\int_0^\infty dr I_1 I_2 \geq (1 + o(1)) \int_0^\infty dr \exp(-\theta_n r) P_{\tilde{\theta}_n}(S_i > r \text{ for all } i \in \mathbb{N}) + o(\tilde{\theta}_n). \quad (4.22)$$

We evaluate the right-hand side of (4.22) using Woodroofe's formula and the estimate $\exp(x) \geq 1 + x$.

$$\begin{aligned}
&\int_0^\infty dr \exp(-\theta_n r) P_{\tilde{\theta}_n}(S_i > r \text{ for all } i \in \mathbb{N}) \\
&\quad \geq \int_0^\infty dr\, P_{\tilde{\theta}_n}(S_i > r \text{ for all } i \in \mathbb{N}) - \theta_n \int_0^\infty dr\, r P_{\tilde{\theta}_n}(S_i > r \text{ for all } i \in \mathbb{N}) \\
&\quad = \tilde{\theta}_n \int_0^\infty dr P_{\tilde{\theta}_n}(S_{\tau_+} > r)(E_{\tilde{\theta}_n} S_{\tau_+})^{-1} - \tilde{\theta}_n \theta_n \int_0^\infty dr r P_{\tilde{\theta}_n}(S_{\tau_+} > r)(E_{\tilde{\theta}_n}(S_{\tau_+}))^{-1} \\
&\quad = \tilde{\theta}_n \Big(1 - \theta_n \frac{1}{2} \frac{E_{\tilde{\theta}_n}(S^2_{\tau_+})}{E_{\tilde{\theta}_n}(S_{\tau_+})}\Big). \qquad (4.23)
\end{aligned}$$

Now it holds

$$E_\theta(S^i_{\tau_+}) < \infty \text{ with } i = 1, 2, \text{ for all } 0 \leq \theta < \theta_0 \text{ and } \lim_{\theta \to 0} E_\theta(S^i_{\tau_+}) = E_0(S^i_{\tau_+})$$

by dominated convergence. Since $E_0(S_{\tau_r}) > 0$ the term $E_{\tilde{\theta}_n}(S^2_{\tau_+})/E_{\tilde{\theta}_n}(S_{\tau_+})$ remains bounded as $n \to \infty$. Since $\tilde{\theta}_n$ and θ_n tend to zero as $n \to \infty$, it follows from the right-hand side of (4.22) and (4.23) that $\int I_1 I_2 dr \geq \tilde{\theta}_n(1 + o(1))$. By (4.20) we get (4.18) and the proof is completed.

5. Several Lemmata

Lemma 3. *Let X_i, $1 \leq i \leq m$ be independent identically distributed random variables with $E|X_1| < \infty$. Then*

$$E(M^+_m) = E(X_1) + \frac{1}{m} E(S^-_m).$$

Proof. One has $E(M_m) = E(M^+_m) - E(M^-_m)$ and

$$E(M_m) = E(X_1) + E\Big(\min_{1 \leq j \leq m}(S_j - X_1)\Big) = E(X_1) - E(M^-_{m-1}).$$

From

$$-M^-_m = \min_{0 \leq i \leq m} S_i \text{ and } E\big(\min_{0 \leq i \leq m} S_i\big) = -\sum_{k=1}^m \frac{1}{k} E(S^-_k),$$

the statement of the lemma follows. For the last equation see Siegmund (1985, p.187).

Lemma 4. *Let X_i, $i \geq 1$ be independent identically distributed random variables with $E(X_1) = 0$ and $\mathrm{Var}\,(X_1) = 1$. Let $S_0 = 0$ and $S_n = \sum_{i=1}^n X_i$. Let $N = \inf\{n \geq 1 \mid S_n < 0\}$. Then*

$$\lim_{n \to \infty} E\big(S_n I_{\{S_i \geq 0 \ \textit{for}\ i=1,\ldots,n\}}\big) = -E(S_N).$$

Proof. One has

$$-S_n I_{\{S_i\geq 0, i=0,\ldots,n\}} = -\sum_{j=0}^{n-1}\big(S_j I_{\{S_i\geq 0, i=0,\ldots,j\}} - S_{j+1} I_{\{S_i\geq 0, i=0,\ldots,j+1\}}\big).$$

Now take expectations on both sides. Since $\{S_i, 1 \leq i \leq j\}$ and X_{j+1} are independent and since $EX_{j+1} = 0$ we can write S_{j+1} instead of S_j in the first term on the right-hand side. We get

$$-E\big(S_n I_{\{S_i\geq 0, i=1,\ldots,n\}}\big) = \sum_{j=0}^{n} E\big(S_i I_{\{S_i\geq 0, i=1,\ldots,j-1, S_{j+1}<0\}}\big) = E(S_N I_{\{N\leq n\}}).$$

For $n \to \infty$ the statement of the lemma follows.

Lemma 5. *Let X_i, $i = 1, \ldots, m$ be identically distributed with $E(X_1) < \infty$. Let $M_m = \min_{1\leq i\leq m} S_i$. Then, for any real $z > 0$,*

$$E\big(M_m^+ I_{\{S_m>z\}}\big) = E\big(X_1 I_{\{S_m>z\}}\big) = E\Big(\frac{S_m}{m} I_{\{S_m>z\}}\Big).$$

The equality holds if $\{0 < S_m < z\}$ replaces $\{S_n > z\}$.

Proof. On the one hand one has

$$E\big(M_m^+ I_{\{S_m>z\}}\big) = E\big(M_m I_{\{S_m>z\}}\big) + E\big(M_{m-1}^- I_{\{S_m>z\}}\big),$$

on the other hand,

$$\begin{aligned} E\big(M_m I_{\{S_m>z\}}\big) &= E\big(X_1 I_{\{S_m>z\}}\big) + E\big(\min_{1\leq j\leq m}(S_j - X_1) I_{\{S_m>z\}}\big) \\ &= E\big(X_1 I_{\{S_m>z\}}\big) - E\big(M_{m-1}^- I_{\{S_m>z\}}\big). \end{aligned}$$

This implies the first equation, the second follows by conditioning.

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Mathematisches Institut Albert-Ludwigs, Universitaet Freiburg Eckerstrasse 1, 79104 Freiburg, Germany.

E-mail: Ulrich.Kerkhoff@kpmg.de

Mathematisches Institut Albert-Ludwigs-Universität Freiburg, Eckerstrasse 1, 79104 Freiburg, Germany.

E-mail: lerche@stochastik.uni-freiburg.de

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