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LOG-CONCAVITY AND INEQUALITIES FOR CHI-SQUARE, F AND BETA DISTRIBUTIONS WITH APPLICATIONS IN MULTIPLE COMPARISONS

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Abstract: In several recent papers log-concavity results and related inequalities for a variety of distributions were obtained. This work is supposed to derive a nearly complete list of corresponding properties concerning the cdf's and some related functions for Beta as well as for central and non-central Chi-square and F distributions, where hitherto only partial results were available. To this end we introduce a generalized reproductive property, thereby extending the relationships between total positivity of order 2, log-concavity and reproductivity developed in Das Gupta and Sarkar (1984). The key to our results are log-concavity properties of the non-central Chi-square distribution with zero degrees of freedom introduced by Siegel (1979). Finally one of the results for the central F distribution is used to solve a monotonicity problem for a stepwise multiple F-test procedure for all pairwise comparisons of k means.

Key words and phrases: Beta distribution, Chi-square distribution, convolution theorem, doubly non-central F distribution, eccentric part of Chi-square, F distribution, log-concave, log-convex, multiple comparisous, pairwise comparisons, Pólya frequency function, Prekopa's theorem, probability inequality, reproductive property, stepwise multiple F-test, total positivity of order 2.

1. Introduction

It is often straightforward to show that a cumulative distribution function (cdf) $F(x|\vartheta)$ (say) depending on a parameter $\vartheta \in \Theta \subseteq \mathbf{R}$ is increasing or decreasing in $\vartheta \in \Theta$. Once such a monotonicity is established, e.g. $F(x|\vartheta_1) \geq F(x|\vartheta_2)$ for all $\vartheta_1 < \vartheta_2$, the next question may be whether these inequalities can be improved. Similarly, one may ask whether the trivial inequality $F(x_1|\vartheta) \leq F(x_2|\vartheta)$, $x_1 < x_2$, can be sharpened. A method which often leads to improved inequalities is to show that a cdf is log-concave in x or ϑ. Suppose for a moment that a cdf $F(x|\vartheta)$, depending on a parameter $\vartheta \in \mathbf{N}_0 = \{0, 1, \ldots\}$, is log-concave and non-increasing in ϑ, then we obtain e.g.

$$\forall \vartheta \geq 1 : F(x|\vartheta) \geq d(\vartheta) F(x|\vartheta + 1) \tag{1.1}$$

with $d(\vartheta) = F(x|\vartheta - 1)/F(x|\vartheta) \geq 1$, hence in case of $d(\vartheta) > 1$ an improvement of the monotonicity in ϑ. Moreover, if in addition $F(x|0)$ is the cdf of the point-mass at 0, then the log-concavity of F implies another type of (somewhat weaker) inequality, namely

$$\forall \vartheta \geq 1 : F(x|\vartheta)^{1/\vartheta} \geq F(x|\vartheta + 1)^{1/(\vartheta+1)}, \tag{1.2}$$

which is an improvement of the monotonicity in ϑ as well (cf. Finner (1990)).

While log-concavity properties or inequalities of types (1.1) and (1.2) for cdf's (or related functions) are certainly of independent interest, they also have some applications, for instance, in reliability theory (cf. e.g. the monograph by Barlow and Proschan (1975)) and in multiple hypotheses testing problems (cf. e.g. Hayter (1986), Finner (1990, 1993)), where the monotonicity of certain critical values in various stepwise multiple test procedures is closely related to (1.1) and/or (1.2). An open problem of this type concerning the F distribution will be discussed and solved in Section 5.

A series of log-concavity results in x and ϑ for cdf's and related functions is given in Das Gupta and Sarkar (1984). Moreover, they studied the relationship between log-concavity, reproductivity (which is a convolution property), and total positivity of order 2 (TP_2). Based on a slight extension of their approach to discrete distributions Finner and Roters (1993a) studied the most common univariate distributions with regard to log-concavity. It became evident that the underlying theory was not general enough to treat certain distributions to a sufficient extent, namely the important F and Beta distributions. In view of a negative result for the cdf of the studentized range of n normal random variables (cf. Finner (1990)) concerning log-concavity in n, which seems to be an effect of positive dependence of studentized random variables, at first sight a positive result for the F distribution (which can also be considered as the distribution of a studentized random variable) cannot be expected. Fortunately, this apprehension does not prove to be true.

This paper provides a variety of log-concavity results not only for central but also for non-central Chi-square and F as well as for Beta distributions. These improve some well-known monotonicity results for the cdf and some related functions available in the literature (cf. e.g. Johnson and Kotz (1970), p. 135, or Ghosh (1973)).

In Section 2 we first introduce a generalized reproductive property and show that all relationships between reproductivity, total positivity of order 2 and log-concavity as considered in Finner and Roters (1993a) remain valid.

Section 3 deals with central and non-central Chi-square distributions. Among other things we study the eccentric part of a decomposition of the non-central Chi-square distribution. These considerations finally allow us to extend already known log-concavity-results for the cdf and some related functions in connection with the (non-)central Chi-square distribution to a nearly complete list concerning the ranges of the values of x, the degrees of freedom and the non-centrality parameter. These results are summarized in Theorems 3.4, 3.8, and Remark 3.6.

In Section 4 we utilize the generalized reproductive property together with the results for the Chi-square distribution to prove a considerable number of log-concavity properties for the "unnormed" doubly non-central (including the central and non-central) F distribution, which carry over to the Beta distribution by means of the well-known relationship between F and Beta. The main statements can be found in Theorems 4.1, 4.3, 4.5 and Corollaries 4.4, 4.6.

While the results for the Chi-square distribution in Section 3 are obtained by using the original reproductive property (without utilizing the so-called mixture property applied by Das Gupta and Sarkar (1984)) we make essential use of the generalized reproductive property and the corresponding assertions of Section 2 in order to treat the F and Beta distributions. One of the results in Section 4 concerning the Beta distribution was already obtained by Das Gupta and Sarkar (1984) with the help of a different method, i.e., the so-called restricted reproductive property, which can be dispensed with in our approach. A further but fragmentary result based on the original reproductive property and the interrelation between the Negative Binomial, the F and the Beta distributions can be found in Finner and Roters (1993a). So the generalized reproductive property can be viewed as a key property for obtaining most of the results in a unifying way.

Finally, in Section 5 we apply one of the results for the unnormed central F distribution to a monotonicity problem occurring in a widely used stepwise multiple F-test procedure for all pairwise comparisons of k means.

2. A Generalized Reproductive Property

In this section we present a definition of reproductivity which is slightly more general than that given in Das Gupta and Sarkar (1984) but has, as will be seen especially in Section 4, a considerably wider range of applications.

To set notation, let $(\mathbf{X}, \mathbf{A}, \mu)$ denote a measure space which in general is assumed (unless specified otherwise) to be equal to $(\mathbf{R}, \mathbf{B}, \lambda)$ or $(\mathbf{Z}, \mathcal{P}(\mathbf{Z}), \kappa)$, where λ denotes the Lebesgue measure on the Borel-σ-field $\mathbf{B}$ of the set of real numbers $\mathbf{R}$ and κ denotes the counting measure on the power set $\mathcal{P}(\mathbf{Z})$ of the set of integers $\mathbf{Z}$. Let $\Theta \subseteq \mathbf{R}$ denote a parameter space and let $f(x|\vartheta)$, $\vartheta \in \Theta$, be probability density functions (pdf's) with respect to μ. Furthermore, let $F(x|\vartheta)$ denote the corresponding cdf and set $\overline{F}(x|\vartheta) = 1 - F(x|\vartheta)$ and $J_c(x|\vartheta) = F(x + c|\vartheta) - F(x|\vartheta)$, $c > 0$.

A function $g : A \to [0, \infty),\ A \subseteq \mathbf{R}^m$, is said to be log-concave (log-convex) in $x \in A$ (short: g is lcc(x) (lcx(x))), if for all $x_1, x_2 \in A$ and all $\alpha \in [0, 1]$ such that $\alpha x_1 + (1 - \alpha)x_2 \in A$ we have

$$g(\alpha x_1 + (1 - \alpha)x_2) \ \geq \ (\leq)\ g(x_1)^\alpha g(x_2)^{1-\alpha}.$$

Setting $\log 0 = -\infty$, these inequalities mean that $\log g$ is concave (convex) in $x \in A$ (short: log g is cc(x) (cx(x))).

For a collection of some basic and fundamental results concerning log-concavity we refer to Eaton (1987), Chapter 4.

A well-known method to derive log-concavity results, which does not only apply for the Lebesgue measure, is based on the concept of total positivity of order 2 (cf. Karlin (1968)). A function $g : A \to [0, \infty),\ A \subseteq \mathbf{R}^2$, is said to be totally positive of order 2 (short: $g(x, y)$ is $\mathrm{TP}_2(x, y)$) if for all $(x_i, y_j) \in A,\ i, j = 1, 2$, with $x_1 < x_2,\ y_1 < y_2$, we have

$$g(x_1, y_2)g(x_2, y_1) \leq g(x_1, y_1)g(x_2, y_2).$$

Furthermore, $f : \mathbf{X} \to [0, \infty)$ is called a Pólya frequency function of order 2 (PF_2) if $K(x, y) = f(x - y),\ x, y \in \mathbf{X}$, is $\mathrm{TP}_2(x, y)$. We note that a location family generated by a pdf f has monotone likelihood ratio iff f is PF_2. For $\mathbf{X} = \mathbf{Z}$ a PF_2 function f is also said to be a Pólya frequency sequence of order 2.

The following result which can be found e.g. in Marshall and Olkin (1979), Chapter 18, may be considered as a basic tool to derive log-concavity results.

Proposition 2.1. (i) *A measurable function* $g : \mathbf{X} \to [0, \infty)$ *is* PF_2 *if and only if* g *is* lcc(x). (ii) *Let* $g, h : \mathbf{X} \to [0, \infty)$ *be measurable* PF_2 *functions. Then the convolution* $k(x) = \int_{\mathbf{X}} g(x - y)h(y)d\mu(y)$ *is* PF_2.

The following approach to reproductivity generalizes the one given in Das Gupta and Sarkar (1984) insofar as it is also applicable for certain families of distributions the members of which are not necessarily concentrated on the non-negative real line. Denote by $\mathbf{N}$ the set of positive integers, and let $\Theta \in \{(0, \infty), [0, \infty), \mathbf{N}, \mathbf{N}_0\}$ and $g : \mathbf{X} \times \Theta \to [0, \infty)$ be measurable in the first component.

Definition 2.2. The function $g(x|\vartheta)$ is said to have the reproductive property in $\vartheta \in \Theta$ (short: $g(x|\vartheta)$ has $\mathrm{RP}(\vartheta)$) if for every $\eta \in \Theta$ there exists a probability measure P_η on $(\mathbf{X}, \mathbf{A})$ with $P_\eta([0, \infty) \cap \mathbf{X}) = 1$ such that for all $\vartheta \in \Theta$ and μ-almost all $x \in \mathbf{X}$

$$\int_{\mathbf{X}} g(x - y|\vartheta)dP_\eta(y) = g(x|\vartheta + \eta). \tag{2.1}$$

Now let $f(x|\vartheta)$, $x \in \mathbf{X} \in \{\mathbf{R}, \mathbf{Z}\}$, be probability density functions with the same properties as $g(x|\vartheta)$ defined above. In this case the reproductive property for pdf's $f(x|\vartheta)$ is just a convolution property of the corresponding probability measures.

Remark 2.3. In terms of random variables, the reproductive property of the corresponding pdf's $f(x|\vartheta)$ can be characterized as follows: $f(x|\vartheta)$ has $\mathrm{RP}(\vartheta)$ iff there exists a stochastic process $(Y_\vartheta)_{\vartheta\in\Theta}$ with stationary, independent and non-negative increments such that Y_ϑ has the pdf $f(x|\vartheta)$, $\vartheta \in \Theta$. In the special case $\Theta = \mathbf{N}_0$ (or similarly for $\Theta = \mathbf{N}$ with obvious modifications) the random variable Y_ϑ can be expressed as $Y_\vartheta = Y_0 + \sum_{i=1}^{\vartheta} X_i$, $\vartheta \in \mathbf{N}$, where the X_i, $i \in \mathbf{N}$, are non-negative independent, identically distributed (i.i.d.) random variables independent of Y_0.

The following two propositions generalize the results obtained for $\mathbf{X} = \mathbf{R}$ in Das Gupta and Sarkar (1984) and Finner and Roters (1993a).

Proposition 2.4. (i) *If* $f(x|\vartheta)$ has $\mathrm{RP}(\vartheta)$, *then* $F(x|\vartheta)$, $\overline{F}(x|\vartheta)$, *and* $J_c(x|\vartheta)$ *have* $\mathrm{RP}(\vartheta)$. (ii) *If* $f(x|\vartheta)$ *is* $\mathrm{TP}_2(x,\vartheta)$, *so are* $F(x|\vartheta)$, $\overline{F}(x|\vartheta)$, *and* $J_c(x|\vartheta)$.

Proposition 2.5. *Suppose* $G(x|\vartheta) \in \{F(x|\vartheta), \overline{F}(x|\vartheta), J_c(x|\vartheta)\}$ *is Borel-measurable in* $\vartheta \in \Theta$ *and has* $\mathrm{RP}(\vartheta)$. (i) *If* $G(x|\vartheta)$ *is* $\mathrm{lcc}(x)$, *then* $G(x|\vartheta)$ *is* $\mathrm{TP}_2(x,\vartheta)$. (ii) *If* $G(x|\vartheta)$ *is* $\mathrm{TP}_2(x,\vartheta)$, *then* $G(x|\vartheta)$ *is* $\mathrm{lcc}(\vartheta)$.

Remark 2.6. In contrast to Theorem 3.2 (i) in Finner and Roters (1993a) Proposition 2.4 (i) states that not only $F(x|\vartheta)$ but also $\overline{F}(x|\vartheta)$ and $J_c(x|\vartheta)$ have $\mathrm{RP}(\vartheta)$ whenever $f(x|\vartheta)$ has this property. This is a direct consequence of the new definition of $\mathrm{RP}(\vartheta)$.

Part (i) of Proposition 2.4 follows by integrating (2.1) for $f(x|\vartheta)$ and applying Fubini's theorem, whereas part (ii) follows by integration of $f(x|\vartheta)$ and application of the basic composition formula (Karlin (1968), p. 17). Part (i) of Proposition 2.5 can be proved in the same way as in Theorem 3 of Das Gupta and Sarkar (1984), whereas the proof of (ii) is similar to the one of Theorem 1 (ii) in the same paper. However, it should be noted that the integrations occurring in the proofs have to be carried out over the set $\mathbf{X}$ to obtain the results. At this point it is essential that the probability measures P_η, $\eta \in \Theta$, are concentrated on $\mathbf{X} \cap [0,\infty)$. For rigorous proofs we refer to a technical report (cf. Finner and Roters (1996)).

3. Properties of the Non-Central Chi-Square Distribution

This section provides some contributions concerning distributional properties of the non-central Chi-square distribution, especially, log-concavity properties and TP_2-properties of the cdf and some related functions. A part of these results is well-known if the degrees of freedom n (say) are greater than or equal to 2. It will be shown that many of these properties carry over for $n < 2$. Moreover, in almost all cases the methods used here for proving such results apply for all real $n > 0$. The main reason for the problems occurring for $n < 2$ is based on the fact that the pdf of the Chi-square distribution is no longer log-concave as it is for $n \geq 2$. The reader interested in further analytical properties of these pdf's is referred to a recent doctoral thesis by Stader (1992).

We begin with some notation. The central Chi-square distribution with degrees of freedom $n > 0$ will be denoted by χ^2_n, its density is given by

$$h(x|n) = (\exp(-x/2)x^{n/2-1}/(2^{n/2}\Gamma(n/2)))I_{(0,\infty)}(x), \tag{3.1}$$

where I_A denotes the indicator function of a set A.

The non-central Chi-square distribution with degrees of freedom $n > 0$ and non-centrality parameter $\xi \geq 0$ with density

$$h(x|n,\xi) \ = \ \sum_{j=0}^{\infty} \frac{\exp(-\xi/2)}{j!} (\frac{\xi}{2})^j h(x|n+2j) \tag{3.2}$$

will be denoted by $\chi^2_{n,\xi}$. For $\xi = 0$ we obtain $h(x|n,0) = h(x|n)$. The corresponding cdf's will be denoted by $H(x|n)$ and $H(x|n,\xi)$, respectively. With $c_j = [\exp(-\xi/2)](\xi/2)^j/j!$ formula (3.2) can be written as $h(x|n,\xi) = \sum_{j=0}^{\infty} c_j h(x|n+2j)$. A random variable having a χ^2_n distribution will be denoted by X_n (short: $X_n \sim \chi^2_n$) or by Y_n. In the non-central case we adopt the notation $X_{n,\xi} \sim \chi^2_{n,\xi}$ or $Y_{n,\xi} \sim \chi^2_{n,\xi}$. According to Siegel (1979), p. 382, a $\chi^2_{n,\xi}$-variable $X_{n,\xi}$ can be decomposed as $X_{n,\xi} = X_n + X_{0,\xi}$, where $X_{n,\xi}$ and $X_{0,\xi}$ are independent, $X_n \sim \chi^2_n$, and $X_{0,\xi}$ can be considered as a Chi-square variable with non-centrality parameter $\xi \geq 0$ and zero degrees of freedom. The distribution of $X_{0,\xi}$ (called the eccentric part by Hjort (1988)) has the point-mass $c_0 = \exp(-\xi/2)$ at 0 and, if $\xi > 0$, a continuous part concentrated on $(0,\infty)$ characterized by the Lebesgue density $g(x|\xi) = \sum_{j=1}^{\infty} c_j h(x|2j)$. The next lemma is the basis for all that follows.

Lemma 3.1. *For all $\xi \geq 0$, the density $g(x|\xi)$ of the continuous part of the distribution of $X_{0,\xi}$ is* lcc(x).

Proof. For $\alpha, \xi > 0$ we consider the auxiliary function

$$g(x|\alpha,\xi) \quad = \quad u(x) \sum_{j=0}^{\infty} d_j \frac{x^{\alpha+j}}{\Gamma(\alpha+j+1)} I_{(0,\infty)}(x) = u(x)w(x) \ \ (\text{say}),$$

where $u(x) = c_0 \exp(-x/2)$ and $d_j = (\xi/4)^{j+1}/\Gamma(j+2)$ for $j \in \mathbf{N}_0$. Noting that d_{r-s} is $\mathrm{TP}_2(r,s)$ and that $0 < \sum_{j=0}^{\infty} d_j < \infty$, we obtain from Theorem 2.1 in Karlin (1968), p. 107, that $w(x-y)$ is $\mathrm{TP}_2(x,y)$ and hence that $w(x)$ is lcc(x). But $u(x)$ is also lcc(x), thus $g(x|\alpha,\xi)$ is lcc(x). Letting $\alpha \to 0$ continuity arguments yield that $g(x|\xi)$ is lcc(x) for all $\xi > 0$.

It is well-known that the log-concavity of a density implies that the corresponding cdf $F(x)$ and the related functions $\overline{F}(x) = 1 - F(x)$ and $J_c(x) = F(x+c) - F(x)$ are lcc(x). We now discuss log-concavity properties of the corresponding functions with respect to the distribution of $X_{0,\xi}$. As this distribution is a mixture of a discrete and an absolutely continuous distribution the usual methods for proving log-concavity results fail and a more refined technique is required.

Lemma 3.2. *Let $G(x|\xi)$ denote the cdf of $X_{0,\xi}$. Then $G(x|\xi)$ is* lcc(x) *on* **R**, *and $\overline{G}(x|\xi) = 1 - G(x|\xi)$, $J_c(x|\xi) = G(x+c|\xi) - G(x|\xi)$ are* lcc(x) *for $x \geq 0$.*

Proof. Let $u(x|\xi) = (\xi/4)\exp(\xi x/4 - \xi/2) I_{(-\infty,0]}(x)$ and define $v(x|\xi) = u(x|\xi) + g(x|\xi)$, $x \in \mathbf{R}$. Then $v(x|\xi)$ is a pdf with respect to Lebesgue measure, which is lcc(x) on **R**. This follows easily by verifying that
(i) $\int_{\mathbf{R}} u(x|\xi) d\lambda(x) = \exp(-\xi/2) = c_0$,
(ii) $u(x|\xi)$ is lcc(x),
(iii) $u(0|\xi) = \lim_{x\downarrow 0} g(x|\xi) = (\xi/4)\exp(-\xi/2)$,
(iv) $\lim_{x\uparrow 0} u'(x|\xi) = (\xi/4)^2 \exp(-\xi/2) > (\xi/8)(\xi/4 - 1)\exp(-\xi/2) = \lim_{x\downarrow 0} g'(x|\xi)$.

Noting that $G(x|\xi)$, $\overline{G}(x|\xi)$, and $J_c(x|\xi)$ coincide for $x \geq 0$ with the corresponding functions belonging to the pdf $v(x|\xi)$, the assertion follows.

Now we consider the cdf $H(x|n,\xi)$ of the non-central Chi-square distribution as a function of x. To prove that $H(x|n,\xi)$ is lcc(x) we need the following result which can easily be proved by using the definition of log-concavity.

Lemma 3.3. *Let $u(x,y)$ be* lcc(x,y) *and non-decreasing in y for each x, and let $v(z)$ be concave. Then $w(x,z) = u(x, v(z))$ is* lcc(x,z).

Theorem 3.4. *For all $n, \xi \geq 0$, $H(x|n,\xi)$ is* lcc(x).

Proof. For $n = 0$ we refer to Lemma 3.2, so let $n > 0$ and let $X_{0,\xi}$ and X_n be defined as before. Then

$$\begin{aligned} H(x|n,\xi) &= P(X_{0,\xi} + X_n \leq x) \\ &= \int_{\mathbf{R}} P(X_{0,\xi} \leq x - y) h(y|n) d\lambda(y) \\ &= \int_{\mathbf{R}} G(x - y|\xi) h(y|n) d\lambda(y) \\ &= \int_{\mathbf{R}} G(x - \exp(y)|\xi) \exp(y) h(\exp(y)|n) d\lambda(y). \end{aligned}$$

From Lemma 3.2 we obtain that $G(x|\xi)$ is lcc(x), which implies that $G(x + y|\xi)$ is lcc(x,y). With Lemma 3.3 it follows that $G(x - \exp(y)|\xi)$ is lcc(x,y).

Furthermore, $h(\exp(y)|n)$ and $\exp(y)$ are $\text{lcc}(y)$, and the product of all these functions is $\text{lcc}(x, y)$. So Prekopa's theorem (Prekopa (1973), cf. Eaton (1987), p. 79) yields that $H(x|n, \xi)$ is $\text{lcc}(x)$.

Remark 3.5. If $0 < n \leq 2,\ 0 \leq \xi \leq 2$, then it can be shown by calculating derivatives that $h(x|n, \xi)$ is non-increasing in x. This implies that $H(x|n, \xi)$ is even concave for $x \geq 0$ and consequently log-concave in $x \in \mathbf{R}$.

Remark 3.6. For $n \geq 2,\ \xi \geq 0,\ h(x|n, \xi)$ is $\text{lcc}(x)$, hence, e.g. by virtue of Proposition 2.1, $\overline{H}(x|n, \xi) = 1 - H(x|n, \xi)$ and $J_c(x|n, \xi) = H(x + c|n, \xi) - H(x|n, \xi)$ are $\text{lcc}(x)$ as well. For $n < 2$, neither log-concavity nor log-convexity properties in x hold for $\overline{H}(x|n, \xi)$ and $J_c(x|n, \xi)$ in general. An exception is the case $0 < n < 2,\ \xi = 0$, where $h(x|n, \xi)$ is log-convex in $x > 0$ so that $\overline{H}(x|n, \xi)$ is log-convex in $x \geq 0$ (cf. Finner and Roters (1993a)).

Remark 3.7. Apparently only parts of the assertions of Theorem 3.4 have been known hitherto, namely that $H(x, |n, \xi)$ is $\text{lcc}(x)$ for $n \geq 2,\ \xi \geq 0$ or $n > 0,\ \xi = 0$, and that $\overline{H}(x, |n, \xi)$ is $\text{lcc}(x)$ for $n \geq 2,\ \xi \geq 0$ (cf. Das Gupta and Sarkar (1984), p. 57).

Before we state the log-concavity results in n and ξ, some preliminary remarks are helpful. First, note that the well-known convolution property $\chi^2_{n_1+n_2,\ \xi_1+\xi_2} = \chi^2_{n_1,\xi_1} * \chi^2_{n_2,\xi_2}$ (cf. e.g. Johnson and Kotz (1970), p. 135) which was stated there only for $n_i > 0, \xi_i \geq 0,\ i = 1, 2$ is even valid for $n_i, \xi_i \geq 0,\ i = 1, 2$ (cf. Siegel (1979), p. 382). Special cases are

$$\chi^2_{n_1+n_2,\xi} = \chi^2_{n_1,\xi} * \chi^2_{n_2} \quad \text{for all} \quad n_1, n_2 \geq 0, \tag{3.3}$$

and

$$\chi^2_{n,\xi_1+\xi_2} = \chi^2_{n,\xi_1} * \chi^2_{0,\xi_2} \quad \text{for all} \quad n, \xi_1, \xi_2 \geq 0. \tag{3.4}$$

In the terminology of Section 2, (3.3) states that $H(x|n, \xi)$ has the reproductive property in $n \geq 0$, while (3.4) means that $H(x|n, \xi)$ has $\text{RP}(\xi)$.

The following proposition states some valuable facts concerning the non-central Chi-square density $h(x|n, \xi)$.

Proposition 3.8. *For all $x \in \mathbf{R}$, the pdf $h(x|n, \xi)$ is*
(i) $\text{lcc}(x)$ *for all* $n \geq 2,\ \xi \geq 0$,
(ii) $\text{TP}_2(x, n)$ *for all* $n > 0,\ \xi = 0$ *or* $n \geq 1,\ \xi > 0$,
(iii) $\text{TP}_2(x, \xi)$ *for all* $n > 0,\ \xi \geq 0$.

Proof. (i) is stated in Das Gupta and Sarkar (1984), the first part of (ii) is easily established, whereas the second part of (ii) is implicitly proved in Ghosh (1973), p. 490, (we note that it is easy to find numerical examples revealing that the TP_2-property of $h(x|n, \xi)$ in x and n no longer obtains for $x \in \mathbf{R},\ 0 < n < 1,\ \xi \geq 0$), and (iii) is essentially proved in Witting (1985), pp. 218-219. The proof given there for integer values of n is also valid for arbitrary $n > 0$.

As a consequence we now easily obtain, by applying the results of Section 2, the following

Theorem 3.9. *For all $x \in \mathbf{R}$ the cdf $H(x|n,\xi)$ of the non-central Chi-square distribution is*
(i) $\mathrm{lcc}(n)$ *for all* $n, \xi \geq 0$,
(ii) $\mathrm{lcc}(\xi)$ *for all* $n, \xi \geq 0$.
Furthermore, the corresponding functions $\overline{H}(x|n,\xi)$ and $J_c(x|n,\xi)$ are
(iii) $\mathrm{lcc}(n)$ *for all* $n \geq 1,\ \xi \geq 0$, *or* $n \geq 0,\ \xi = 0$,
(iv) $\mathrm{lcc}(\xi)$ *for all* $n, \xi \geq 0$.

Remark 3.10. Main parts of Theorem 3.9 are available in Das Gupta and Sarkar (1984) who utilized a so-called mixture property in order to obtain the log-concavity results in n, ξ for $H(x|n,\xi)$ and $\overline{H}(x|n,\xi)$. However, the reproductive properties (3.3) and (3.4) allow a simple and efficient proof without making use of the mixture property. The extension to values $n \leq 2$ in (i) and $1 \leq n < 2$ in (iii) as well as the assertions for $J_c(x|n,\xi)$ supplement the results obtained by Das Gupta and Sarkar (1984). It is also worth noting that the proofs in this section only require the use of the original reproductive property, whereas the generalized reproductive property will substantially be exploited in Section 4.

It follows from Theorem 3.4, Proposition 2.5 and the reproductive properties of $H(x|n,\xi)$ in n and ξ, respectively, that $H(x|n,\xi)$ is $\mathrm{TP}_2(x,n)$ and $\mathrm{TP}_2(x,\xi)$ for $x \in \mathbf{R},\ n,\xi \geq 0$. As a consequence, $H(x|n,\xi)$ is non-increasing in $n \geq 0$ for fixed $\xi \geq 0,\ x \in \mathbf{R}$, and likewise non-increasing in $\xi \geq 0$ for fixed $n \geq 0,\ x \in \mathbf{R}$. Obviously, $\overline{H}(x|n,\xi)$ is non-decreasing in these cases. The result for n is due to Ghosh (1973), while the result for ξ is a consequence of the fact that the pdf $h(x|n,\xi)$ is $\mathrm{TP}_2(x,\xi)$ for $x \in \mathbf{R},\ \xi \geq 0,\ n > 0$, i.e., the family of non-central Chi-square distributions possesses a monotone likelihood ratio and is hence stochastically increasing. It should be noted that the proofs of these two monotonicity results are based on properties of the pdf $h(x|n,\xi)$. However, these results can be obtained merely from the corresponding TP_2-properties of the cdf $H(x|n,\xi)$, which is of course the weaker assumption in comparison with the TP_2-properties of the pdf $h(x|n,\xi)$.

For, if $F(x|\vartheta)$ is $\mathrm{TP}_2(x,\vartheta)$, i.e., $F(x+h|\vartheta_2)F(x|\vartheta_1) \geq F(x|\vartheta_2)F(x+h|\vartheta_1)$ for all $x \in \mathbf{R},\ h > 0$ and $\vartheta_1 < \vartheta_2$, then, by letting $h \to \infty$, it follows with $\lim_{h\to\infty} F(x+h|\vartheta) = 1$ for all ϑ, that $F(x|\vartheta_1) \geq F(x|\vartheta_2)$.

If $J_c(x|n,\xi)$ is log-concave in n or ξ, it is unimodal (and non-monotonic) or monotonic in these variables, respectively. There exist numerical examples for either shape behaviour. However, in case of $\xi = 0$ and $x > 0$, $J_c(x|n,\xi)$ is strictly unimodal in n since the family of central Chi-square distributions constitutes a one-parameter exponential family with open natural parameter space and $J_c(x|n,\xi)$ as a function of $n > 0$ can be considered as the power function of a two-sided test (cf. Finner and Roters (1993b)).

4. Results for F and Beta Distributions

In this section we intend to study properties of $F(x|\vartheta)$, $\overline{F}(x|\vartheta)$, and $J_c(x|\vartheta)$ for Beta and F distributions by considering in more generality the behaviour of the corresponding functions belonging to the ratio of two independent non-central Chi-square variables both with non-negative and real degrees of freedom and non-centrality parameters.

We start this section with a log-concavity result for the cdf of the central $\mathrm{F}_{n,m}$ distribution. Though not needed in the sequel it is of independent interest.

Theorem 4.1. *The cdf* $F(x|n,m)$ *(say) of the central* $\mathrm{F}_{n,m}$ *distribution with* $n,m > 0$ *is* $\mathrm{lcc}(x)$.

Proof. Since for $n,m > 0$ the pdf $f(x|n,m)$ of $\mathrm{F}_{n,m}$ is given by

$$f(x|n,m) = \frac{\Gamma((n+m)/2)}{\Gamma(n/2)\Gamma(m/2)}(\frac{n}{m})^{n/2}\frac{x^{n/2-1}}{(1+nx/m)^{(n+m)/2}},\ x \geq 0,$$

it obviously has the property that $f(\exp(y)|n,m)$, $y \in \mathbf{R}$, is $\mathrm{lcc}(y)$. Hence the desired assertion follows directly from Lemma 2.5 in Finner and Roters (1993a).

Remark 4.2. The assertion of Theorem 4.1 obviously continues to hold for the cdf $F^u(x|n,m)$ (say) of the unnormed central F distribution $\mathrm{F}^u_{n,m}$ (say) of X_n/Y_m for X_n, Y_m independent and (centrally) Chi-square distributed. This is true because the cdf of $\mathrm{F}^u_{n,m}$ is given by $F(mx/n|n,m)$, where $F(x|n,m)$ is the cdf of the central $\mathrm{F}_{n,m}$ distribution as defined above.

In the sequel we concentrate on the investigation of parameter log-concavity and log-convexity properties of the unnormed doubly non-central F distribution, which shall be defined as the distribution of the ratio $X_{n,\xi}/Y_{m,\delta}$ and be denoted by $\mathrm{F}^u_{n,m,\xi,\delta}$, where $X_{n,\xi}$ and $Y_{m,\delta}$ are independent non-central Chi-square variables. Let, further, $F^u(x|n,m,\xi,\delta)$ denote the cdf of $\mathrm{F}^u_{n,m,\xi,\delta}$ and set $\overline{F}^u(x|n,m,\xi,\delta) = 1 - F^u(x|n,m,\xi,\delta)$. Then we can prove the following

Theorem 4.3. *The cdf* $F^u(x|n,m,\xi,\delta)$ *of the unnormed doubly non-central* F *distribution is*

(i) $\mathrm{lcc}(n)$ *for all* $n \geq 0$, $m \geq 2$, $\xi = 0$, $\delta \geq 0$ *or all* $n \geq 1$, $m \geq 2$, $\xi > 0$, $\delta \geq 0$,

(ii) $\mathrm{lcc}(\xi)$ *for all* $n \geq 0$, $m \geq 2$, $\xi,\delta \geq 0$,

(iii) $\operatorname{lcc}(m)$ *for all* $n \geq 0,\ m > 0,\ \xi \geq 0,\ \delta = 0$ *or all* $n \geq 0,\ m \geq 1,\ \xi \geq 0,\ \delta > 0$,
(iv) $\operatorname{lcc}(\delta)$ *for all* $n \geq 0,\ m \geq 1,\ \xi, \delta \geq 0$.

Proof. For every $x > 0$ fixed, the cdf of $\mathrm{F}^u_{n,m,\xi,\delta}$ is given by

$$F^u(x|n, m, \xi, \delta) = P(X_{n,\xi}/Y_{m,\delta} \leq x) = P(X_{n,\xi} - xY_{m,\delta} \leq 0).$$

Now, for all $z \in \mathbf{R}$ we consider the cdf of $X_{n,\xi} - xY_{m,\delta}$, which can be written as

$$\begin{aligned} P(X_{n,\xi} - xY_{m,\delta} \leq z) &= \int_{\mathbf{R}} P(y - xY_{m,\delta} \leq z) dP^{X_{n,\xi}}(y) \\ &= \int_{\mathbf{R}} P(Y_{m,\delta} \geq (y - z)/x) h(y|n, \xi) d\lambda(y) \\ &= \int_{\mathbf{R}} \overline{H}((y - z)/x|m, \delta) h(y|n, \xi) d\lambda(y). \qquad (4.1) \end{aligned}$$

(i) Since, due to Remark 3.6, $\overline{H}(t|m, \delta)$ is $\operatorname{lcc}(t)$ for all $m \geq 2,\ \delta \geq 0$, it follows that $\overline{H}((y-z)/x|m, \delta)$ is $\mathrm{TP}_2(y, z)$, and as $h(y|n, \xi)$ is $\mathrm{TP}_2(y, n)$ for $n > 0,\ \xi = 0$ or $n \geq 1,\ \xi > 0$ by virtue of Proposition 3.8 (ii), the basic composition formula and a continuity argument for $n = 0$ yield that $P(X_{n,\xi} - xY_{m,\delta} \leq z)$ is $\mathrm{TP}_2(z, n)$ for $z \in \mathbf{R}$ and $n \geq 0,\ \xi = 0$ or $n \geq 1,\ \xi > 0$.

Defining $\Theta = [0, \infty)$ and the probability measures $P_\eta,\ \eta \in \Theta$, as the central Chi-square distribution with η degrees of freedom it is obvious that for all $\vartheta, \eta \in \Theta$ and $z \in \mathbf{R}$

$$P(X_{\vartheta+\eta,\xi} - xY_{m,\delta} \leq z) = \int_{\mathbf{R}} P(X_{\vartheta,\xi} - xY_{m,\delta} \leq z - y) dP_\eta(y)$$

holds, i.e., both $P(X_{\vartheta,\xi} - xY_{m,\delta} \leq z)$ and $P(X_{\vartheta+1,\xi} - xY_{m,\delta} \leq z)$ have $\mathrm{RP}(\vartheta)$.

Since, in addition, $P(X_{\vartheta+k,\xi} - xY_{m,\delta} \leq z),\ k \in \{0, 1\}$, is non-increasing, hence Borel-measurable in $\vartheta \in \Theta$, Proposition 2.5 (ii) implies that $P(X_{n,\xi} - xY_{m,\delta} \leq z)$ is $\operatorname{lcc}(n)$ for $n \geq 0,\ \xi = 0$ or $n \geq 1,\ \xi > 0$, and all $z \in \mathbf{R}$. Setting $z = 0$ we finally obtain that $F^u(x|n, m, \xi, \delta) = P(X_{n,\xi}/Y_{m,\delta} \leq x)$ is $\operatorname{lcc}(n)$ for all $x \in \mathbf{R}$ and the parameter configuration specified in (i).

(ii) Using as in (i) the log-concavity of $\overline{H}(t|m, \delta)$ in $t \in \mathbf{R}$ for $m \geq 2,\ \delta \geq 0$, and the TP_2-property of $h(y|n, \xi)$ in (y, ξ) for $y \in \mathbf{R}$, $\xi \geq 0$ and $n > 0$, we obtain from (4.1) and a continuity argument for $n = 0$ that $P(X_{n,\xi} - xY_{m,\delta} \leq z)$ is $\mathrm{TP}_2(z, \xi)$ for $z \in \mathbf{R},\ \xi \geq 0,\ n \geq 0$, and $n \geq 0$, and by defining P_η as the Chi-square distribution with 0 degrees of freedom and non-centrality parameter $\eta \geq 0$ we may conclude as in (i) that $P(X_{n,\xi} - xY_{m,\delta} \leq z)$ has $\mathrm{RP}(\xi)$, is non-increasing in $\xi \geq 0$ and hence $\operatorname{lcc}(\xi)$ for all $z \in \mathbf{R}$ and $n \geq 0,\ m \geq 2,\ \xi, \delta \geq 0$. Setting again $z = 0$ yields the assertion of (ii).

(iii) + (iv) We consider again the expression

$$\begin{aligned} P(X_{n,\xi} - xY_{m,\delta} \le z) &= \int_{\mathbf{R}} P(X_{n,\xi} \le z + xy) dP^{Y_{m,\delta}}(y) \\ &= \int_{\mathbf{R}} H(xy - (-z)|n,\xi) h(y|m,\delta) d\xi(y) \end{aligned}$$

for all $z \in \mathbf{R}$. Theorem 3.4 implies that $H(xy - (-z)|n,\xi)$ is $\mathrm{TP}_2(y,-z)$ for all $y, z \in \mathbf{R}$ and $n, \xi \ge 0$, so that by using the same argumentation as before the above expression $P(xY_{m,\delta} - X_{n,\xi} \ge -z) = P(X_{n,\xi} - xY_{m,\delta} \le z)$ is seen to be $\mathrm{TP}_2(-z, m)$ for the parameter configuration specified in (iii) and $\mathrm{TP}_2(-z,\delta)$ for the configuration in (iv).

Analogously as in (i) and (ii) we may conclude that $P(xY_{m,\delta} - X_{n,\xi} \ge -z)$ has $\mathrm{RP}(m)$ in (iii) and $\mathrm{RP}(\delta)$ in (iv), so that finally the assertions of (iii) and (iv) follow.

Due to the special structure of the F distribution it is possible to formulate a result for $\overline{F}^u(x|n,m,\xi,\delta) = 1 - F^u(x|n,m,\xi,\delta)$ as a corollary to the last theorem.

Corollary 4.4. *The function* $\overline{F}^u(x|n,m,\xi,\delta)$ *is*
(i) $\mathrm{lcc}(n)$ *for all* $n \ge 0,\ m > 0,\ \xi = 0,\ \delta \ge 0$ *or all* $n \ge 1,\ m,\xi > 0,\ \delta \ge 0$,
(ii) $\mathrm{lcc}(\xi)$ *for all* $n \ge 1,\ m > 0,\ \xi,\delta \ge 0$,
(iii) $\mathrm{lcc}(m)$ *for all* $n \ge 2,\ m > 0,\ \xi \ge 0,\ \delta = 0$ *or all* $n \ge 2,\ m \ge 1,\ \xi \ge 0,\ \delta > 0$,
(iv) $\mathrm{lcc}(\delta)$ *for all* $n \ge 2,\ m > 0,\ \xi,\delta \ge 0$.

Proof. Everything follows from the relation

$$\overline{F}^u(x|n,m,\xi,\delta) = P(X_{n,\xi}/Y_{m,\delta} \ge x) = P(Y_{m,\delta}/X_{n,\xi} \ge 1/x) = F^u(1/x|m,n,\delta,\xi)$$

for $x > 0,\ n,m > 0,\ \xi,\delta \ge 0$ and from continuity arguments.

For the next result the following relationship between the cdf of the Beta distribution $B(x|\alpha,\beta)$ (say) and the cdf $F^u(x|n,m)$ of the unnormed central F distribution $\mathrm{F}^u_{n,m}$ is important. For all $x \ge 0,\ n,m > 0$,

$$F^u(x|n,m) = B(\frac{x}{1+x}|\frac{n}{2},\frac{m}{2}). \tag{4.2}$$

This equation can be found for instance in Patel, Kapadia and Owen (1976), p. 217. Since $B(x|\alpha,\beta)$ is also known as the incomplete Beta function ratio (denoted by $I_x(\alpha,\beta)$) a well-known series representation of $I_x(\alpha,\beta)$ can be used to express $B(x|\alpha,\beta)$ and by (4.2) also $F^u(x|n,m)$ (cf. Patel, Kapadia and Owen (1976), p. 247).

For $0 \le x < 1,\ \alpha,\beta > 0$ we have

$$I_x(\alpha,\beta) = (x^\alpha (1-x)^\beta / \Gamma(\beta)) \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+\beta+n)}{\Gamma(\alpha+1+n)} x^n. \tag{4.3}$$

Theorem 4.5. *For every $x \in \mathbf{R}$ the following assertions concerning the unnormed non-central* F *distribution $F^u_{n,m,\xi}$ hold:*
(i) $F^u(x|n,m,\xi)$ $(= F^u(x|n,m,\xi,0)$ *(say))* *is* $\text{lcx}(n)$ *for* $n,\xi \geq 0,\ 0 < m \leq 2$.
(ii) $\overline{F}^u(x|n,m,0,\delta)$ *is* $\text{lcx}(m)$ *for* $0 \leq n \leq 2,\ m > 0,\ \delta \geq 0$.
(iii) $\overline{F}^u(x|n,m,\xi) = 1 - F^u(x|n,m,\xi)$ *is* $\text{cc}(n)$ *and hence* $\text{lcc}(n)$ *for* $n,\xi \geq 0$, $0 < m \leq 2$.

Proof. (i) Let $x \geq 0,\ 0 < m \leq 2$ and $\xi \geq 0$ be fixed. Then by (4.2) and (4.3) it follows for $n \geq 0$

$$\begin{aligned} F^u(x|n,m,\xi) &= P(X_{n,\xi}/Y_m \leq x) = \sum_{j=0}^{\infty}(\frac{\xi}{2})^j\frac{\exp(-\xi/2)}{j!}P(X_{n+2j}/Y_m \leq x) \\ &= \sum_{j=0}^{\infty}(\frac{\xi}{2})^j\frac{\exp(-\xi/2)}{j!}F^u(x|n+2j,m), \end{aligned}$$

where $F^u(x|n+2j,m)$ is equal to

$$\Big((\frac{x}{1+x})^{n/2+j}(\frac{1}{1+x})^{m/2}/\Gamma(\frac{m}{2})\Big)\sum_{k=0}^{\infty}\frac{\Gamma(n/2+j+m/2+k)}{\Gamma(n/2+j+1+k)}(\frac{x}{1+x})^k.$$

Now, since $f(n) = y^n$ (say), $y > 0$ fixed, is $\text{lcx}(n)$ (even log-linear in n) and

$$g(n) = \Gamma(n+c_1)/\Gamma(n+c_2) = \int_0^1 t^{n+c_1-1}(1-t)^{c_2-c_1-1}dt/\Gamma(c_2-c_1),$$

$c_2 > c_1 > 0$, is $\text{lcx}(n)$ due to Artin's theorem (cf. Marshall and Olkin (1979), Chapter 16), since, furthermore, series and products of log-convex functions are again log-convex, the above series expansion of $F^u(x|n,m,\xi)$ reveals that it is $\text{lcx}(n)$ for $n \geq 0$, as was to be shown in (i).
(ii) This can be proved by using the same argument as in Corollary 4.4.
(iii) From (i) we know that $F^u(x|n,m,\xi)$ is $\text{lcx}(n)$ and hence $\text{cx}(n)$. So $\overline{F}^u(x|n,m,\xi) = 1 - F^u(x|n,m,\xi)$ is $\text{cc}(n)$ and hence $\text{lcc}(n)$ for $n \geq 0,\ 0 < m \leq 2,\ \xi \geq 0$.

As a corollary of all that is proved above we are now able to conclude a result for the Beta distribution which seems to be very hard to obtain by usual analytical methods.

Corollary 4.6. *For every $0 \leq x \leq 1$ we obtain that*
(i) $B(x|\alpha,\beta)$ *is* $\text{lcc}(\alpha)$ *for all* $\alpha > 0,\ \beta \geq 1$,
(ii) $B(x|\alpha,\beta)$ *is* $\text{lcx}(\alpha)$ *for all* $\alpha > 0,\ 0 < \beta \leq 1$,
(iii) $B(x|\alpha,\beta)$ *is* $\text{lcc}(\beta)$ *for* $\alpha,\beta > 0$ *and even* $\text{cc}(\beta)$ *for* $0 < \alpha \leq 1,\ \beta > 0$,
(iv) $\overline{B}(x|\alpha,\beta)$ $(= 1 - B(x|\alpha,\beta)$ *(say))* *is* $\text{lcc}(\beta)$ *for all* $\alpha \geq 1,\ \beta > 0$,
(v) $\overline{B}(x|\alpha,\beta)$ *is* $\text{lcx}(\beta)$ *for all* $0 < \alpha \leq 1,\ \beta > 0$,

(vi) $\overline{B}(x|\alpha,\beta)$ *is* lcc(α) *for* $\alpha,\beta>0$ *and even* cc(α) *for* $\alpha>0,\ 0<\beta\leq 1$.

Proof. Invoking the relation (4.2), part (i) can be concluded from Theorem 4.3 (i). Part (ii) follows from Theorem 4.5 (i), and the log-concavity part of (iii) is a consequence of Theorem 4.3 (iii). Moreover, (iv), (v) and the log-concavity part of (vi) result from what has just been shown in this proof and the relation $B(x|\alpha,\beta)=\overline{B}(1-x|\beta,\alpha)$ for all $0\leq x\leq 1,\ \alpha,\beta>0$ (cf. Patel, Kapadia and Owen (1976), p. 246). It only remains to prove the concavity parts of (iii) and (vi). However, the corresponding assertion in (iii) follows directly from (v), and the concavity assertion in (vi) results from (ii) by using the definition of $\overline{B}(x|\alpha,\beta)$ (cf. (iv)). Now the proof is complete.

Remark 4.7. The log-concavity part of (iii) was proved by Das Gupta and Sarkar (1984) by studying a so-called restricted reproductive property. The assertion of (i) was proved in Finner and Roters (1993a) only for integer-valued $\beta\geq 1$, but conjectured to the extent proved in this paper. In addition, the result of (ii) was mentioned there without proof.

Remark 4.8. The parameter log-concavity results for the unnormed doubly non-central F distribution carry over immediately to the doubly non-central Beta distribution, which is defined as the distribution of the ratio $X_{n,\xi}/(X_{n,\xi}+Y_{m,\delta})$, where $X_{n,\xi}$ and $Y_{m,\delta}$ are independent non-central Chi-square variables.

5. Solution of a Monotonicity Problem for a Step-Down Multiple F-Test Procedure

The last part of this work treats an application of one of the results for the central F distribution to a monotonicity problem occurring in a step-down multiple F-test procedure for all pairwise comparisons between k means in an ANOVA-setup (cf. Finner (1993)). For the sake of simplicity we consider a one-way ANOVA-model with possibly unequal sample sizes given by

$$X_{ij}\sim N(\vartheta_i,\sigma^2),\ j=1,\ldots,n_i,\ i\in I=\{1,\ldots,k\},\ k\geq 3,$$

where the X_{ij} are assumed to be independent random variables, and let $\nu=n_{.}-k\geq 2$ with $n_{.}=\sum_{i\in I}n_i$. Suppose that we are mainly interested in testing all so-called pair hypotheses

$$H_{ij}:\vartheta_i=\vartheta_j\ \text{ vs. }\ K_{ij}:\vartheta_i\neq\vartheta_j,\ 1\leq i<j\leq k.$$

Besides range test procedures a popular choice of a test procedure for this problem is a stepwise multiple F-test procedure based on the family of all so-called homogeneity hypotheses $H_Q : \vartheta_i = \vartheta_j$ for all $i, j \in Q,\ i \neq j$, with $Q \subseteq I,\ |Q| \geq 2$, by using the unnormed F-statistics

$$T_Q(X) = \sum_{i \in Q} n_i(\overline{X}_{i\cdot} - \overline{X}_{\cdot\cdot}^Q)^2/W^2, \tag{5.1}$$

where $\overline{X}_{i\cdot} = \sum_{j=1}^{n_i} X_{ij}/n_i$, $\overline{X}_{\cdot\cdot}^Q = \sum_{i \in Q}\sum_{j=1}^{n_i} X_{ij}/n_\cdot^Q$ with $n_\cdot^Q = \sum_{i\in Q} n_i$, and $W^2 = \sum_{i=1}^{k}\sum_{j=1}^{n_i} (X_{ij} - \overline{X}_{i\cdot})^2$. Then, under H_Q, $T_Q(X)$ has an unnormed F distribution with degrees of freedom $m = |Q| - 1$ and ν. The statistics defined in (5.1) possess the desirable monotonicity property $T_Q \geq T_R$ if $H_Q \subset H_R$. This follows immediately from the fact that T_Q is a likelihood ratio statistic for testing H_Q, or alternatively, by verifying that

$$\sum_{i \in Q} n_i(\overline{x}_{i\cdot} - \overline{x}_{\cdot\cdot}^Q)^2 = \sum_{i \in R} n_i(\overline{x}_{i\cdot} - \overline{x}_{\cdot\cdot}^R)^2 + n_q \frac{n_\cdot^R}{n_\cdot^Q}(\overline{x}_{q\cdot} - \overline{x}_{\cdot\cdot}^R)^2,$$

where $R \subset Q \subseteq I$ with $2 \leq |R| = |Q| - 1$ and q is the unique element of $Q \setminus R$ (say).

The critical values for the stepwise F-test procedure are based on so-called adjusted significance levels which ensure that the test controls a multiple level α (also known as familywise error rate (FWE) α), $\alpha \in (0, 1)$. A popular choice of significance levels is given by $\alpha_p = 1 - (1 - \alpha)^{p/k}$ for $p = 2, \ldots, k-2, k$, and $\alpha_{k-1} = \alpha$. The critical values are chosen as upper α-points of the corresponding unnormed F distribution with degrees of freedom $p-1$ and ν, i.e., $F^u(c_p(\alpha_p)|p-1, \nu) = 1 - \alpha_p,\ p = 2, \ldots, k$. Then a hypothesis $H_{ij} : \vartheta_i = \vartheta_j$ is rejected if $T_Q > c_{|Q|}(\alpha_{|Q|})$ for all Q with $\{i, j\} \subseteq Q \subseteq I$. Moreover, a hypothesis H_R can be rejected if $T_Q > c_{|Q|}(\alpha_{|Q|})$ for all Q with $R \subseteq Q \subseteq I$.

In view of the structure of the test procedure a desirable property which also facilitates the determination of the test results for the pair hypotheses is the monotonicity of the critical values (cf. e.g. Finner (1990, 1993)), i.e., $c_k(\alpha_k) \geq \cdots \geq c_2(\alpha_2)$. This is clearly fulfilled in case $k = 3$ for the choice $\alpha_3 = \alpha_2 = \alpha$. Therefore, take $k \geq 4$ in the sequel. If $\alpha_{k-1} = \alpha$ is replaced by $\alpha_{k-1} = 1 - (1 - \alpha)^{(k-1)/k}$, the monotonicity of the corresponding critical values can then be proved by noting that $F^u(x|p, \nu)$ is lcc(p) for $\nu \geq 2$. First, this log-concavity property implies, as noted in the introduction, the inequalities

$$\forall\ 0 < p < q : \ \forall\ \nu \geq 2 : \ \forall\ c \geq 0 : \ F^u(c|p, \nu)^{1/p} \geq F^u(c|q, \nu)^{1/q}. \tag{5.2}$$

But then we also obtain

$$\forall\ p \in \{2, \ldots, k-1\} : \ \forall\ \nu \geq 2 : \ \forall\ c \geq 0 : \ F^u(c|p-1, \nu)^{1/p} \geq F^u(c|p, \nu)^{1/(p+1)}, \tag{5.3}$$

which implies for $\alpha_p = 1-(1-\alpha)^{p/k}$, $p=2,\ldots,k$, by using the definition of $c_p(\alpha_p)$, that

$$(1-\alpha)^{(p+1)/k} = F^u(c_p(\alpha_p)|p-1,\nu)^{(p+1)/p} \geq F^u(c_p(\alpha_p)|p,\nu).$$

Thus, the desired monotonicity $c_{p+1}(\alpha_{p+1}) \geq c_p(\alpha_p)$ for $p=2,\ldots,k-1$ is proved. One can easily find (numerical) examples, where the choice $\alpha_{k-1}=\alpha$ destroys the monotonicity of the corresponding critical values. However, in practice, one can use without loss,

$$c'_q = \begin{cases} c_q(\alpha_q), & q=2,\ldots,k-2,k, \\ \max\{c_{k-2}(\alpha_{k-2}), c_{k-1}(\alpha)\}, & q=k-1, \end{cases}$$

which are monotonic and lead to the same test results for all pair hypotheses as the c_q. If one is interested in the test results for all intersection hypotheses one should use the original (possibly non-monotonic) critical values.

We note that for $\nu=2$ we have equality in (5.2), and the inequality sign is reversed for $\nu=1$, since $F^u(x|n,m)$ is lcx(n) for all $n\geq 0$ and $0<m\leq 2$ (cf. Theorem 4.5 (i)). However, (5.3) can be valid for some values ν slightly smaller than 2. But the case $\nu=1$ seems of less practical interest so that we relinquish a detailed discussion of the monotonicity of the critical values in this case.

Finally, we point out that an inequality of the type (5.2) does not hold in general for the cdf of the studentizedz range distribution if $\nu>2$. As a consequence (cf. Finner (1990, 1993)) the corresponding multiple range test procedure may have non-monotonic critical values.

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