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LAW OF THE ITERATED LOGARITHM FOR EMPIRICAL CUMULATIVE QUANTILE REGRESSION FUNCTIONS

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Abstract: Under some mild conditions we establish Strassen's law of the iterated logarithm for the empirical cumulative quantile regression function.

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1. Introduction

Let (X, Y) be a bivariate random vector with $E|Y|$ finite and denote by $m(x) = E(Y|X = x)$ the regression function of Y on X. Further let $F(x)$ be the marginal distribution function of X, taken to be left continuous, and let F^{-1} be the right continuous inverse of F. In Rao and Zhao (1993a) we defined the quantile regression (QR) function of Y on X as

$$r(u) = E(Y|X = F^{-1}(u)) = m(F^{-1}(u)), \quad 0 \le u \le 1 \tag{1.1}$$

and the cumulative $QR(CQR)$ function as

$$M(u) = \int_0^u m(F^{-1}(t))dt = \int_0^u r(t)dt, \quad 0 \le u \le 1. \tag{1.2}$$

Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be an i.i.d. sample on (X, Y) and $X_{(1)} \le \cdots \le X_{(n)}$ be the order statistics of $X_1, \ldots, X_n$. Denote the Y associated with $X_{(i)}$ by $Y_{[i]}$. Then an empirical version of $M(u)$ is

$$M_n(u) = \int_{(-\infty, F_n^{-1}(u)]} \int_{-\infty}^{\infty} y dP_n(x, y) = n^{-1} \sum_{i=1}^{[nu]+1} Y_{[i]}, \ 0 \le u < 1, M_n(1) = M_n(1-), \tag{1.3}$$

where P_n and F_n are the empirical distribution functions of (X, Y) and X respectively, both taken to be left continuous, and F_n^{-1} is the right continuous inverse of F_n. These curves are related to the usual Lorenz curves and the Mahalanobis fractile ordinates (see Rao and Zhao (1993a) for the details).

There is considerable literature on the usual Lorenz curves. See, for instance, papers by Gastwirth (1971, 1972), Kakwani and Podder (1973, 1976), Bishop, Chakraborti and Thistle (1989). Most of the papers deal with the asymptotic distribution of a fixed number of Lorenz ordinates. Goldie (1977) initiated a new line of investigation by establishing convergence theorems for the empirical Lorenz curve and its inverse. Rao and Zhao (1995b) proved the Strassen law of iterated logarithm for the empirical Lorenz curve.

Given the above definitions of QR and CQR curves, Rao and Zhao (1995a) established the uniform strong consistency and the functional limit theorem for the empirical $CQR's$. In this paper, it is desired to establish the almost sure convergence rate for the sequence

$$\sqrt{n}(M_n(\cdot) - M(\cdot))/b_n \tag{1.4}$$

with

$$b_n = (2\log\log n)^{\frac{1}{2}}. \tag{1.5}$$

For simplicity and without loss of generality, we need only consider the case when $Y \geq 0$. We assume that the following conditions hold:

(A) $Y \geq 0$ and $E|Y|^{2+\alpha} < \infty$ for some $\alpha > 0$.

(B) F has a continuous and positive density f on (a, b), where $-\infty \leq a = \sup\{x : F(x) = 0\}$ and $+\infty \geq b = \inf\{x : F(x) = 1\}$, and m has a continuous derivative function m' on (a, b).

(C) One of the following is true.

(C1) $r = m \circ F^{-1}$ is bounded on [0,1], where $\circ$ denotes a composite function.

(C2) If $r(t)$ is not bounded when $t \downarrow 0$ (resp. $t \uparrow 1$), then $r(t)$ is nonincreasing (resp. nondecreasing) and $\sqrt{t}\, r(t)$ (resp. $\sqrt{1-t}\, r(t)$) is nondecreasing (resp. nonincreasing) in the interval $(0, \delta]$ (resp. [1-δ,1)) for some $\delta \in (0, \frac{1}{2})$, and there exist constants $C_1 > 0$ and $\tau < 1$ such that for any $t_1, t_2 \in (0, \delta]$ (resp.$[1-\delta, 1)$),

$$\Big|\frac{r(t_1)}{r(t_2)}\Big| \leq C_1 \Big(\frac{(t_1 \vee t_2)(1 - y_1 \wedge t_2)}{(t_1 \wedge t_2)(1 - t_1 \vee t_2)}\Big)^{\tau}, \tag{1.6}$$

where $t_1 \wedge t_2 = \min(t_1, t_2)$ and $t_1 \vee t_2 = \max(t_1, t_2)$.

For convenience we write

$$V(x) = E((Y - m(X))^2 | X = x) \tag{1.7}$$

and

$$\zeta(u) = \int_0^u V(F^{-1}(t))dt \quad \text{with} \quad \zeta(1) \equiv \sigma_1^2. \tag{1.8}$$

Note that $\sigma_1^2 = E(Y - m(X))^2$.

Let $D \equiv D[0,1]$ be the space of functions on [0,1] that are right-continuous and have left-side limits, and $\mathcal{D}$ be the σ-field generated by all the cylinder sets of D induced by the maps $z \to z(t)$. For each $z \in D[0,1]$ we define the norm $\|z\| = \sup_{0\le t\le 1}|z(t)|$, and use $C[0,1]$ to denote the subset of $D[0,1]$ consisting of all continuous functions on [0,1]. Define

$$\mathcal{K} = \Big\{k: \quad k \quad \text{is absolutely continuous on [0,1]} \quad \text{with } k(0)=0 \quad \text{and} \quad \int_0^1 (k'(t))^2 dt \le 1\Big\}, \tag{1.9}$$

$$\mathcal{H} = \Big\{h: \quad h \quad \text{is absolutely continuous on [0,1]} \quad \text{with } h(0)=h(1)=0 \quad \text{and} \quad \int_0^1 (h'(t))^2 dt \le 1\Big\}, \tag{1.10}$$

and

$$\mathcal{G} = \Big\{g: g(u) = \sigma_1 k(\zeta(u)/\sigma_1^2) - \int_0^u h(t)dr(t),\ 0 \le u \le 1, \quad k \in \mathcal{K} \quad \text{and} \quad h \in \mathcal{H}\Big\}. \tag{1.11}$$

We establish the following theorem.

Theorem. *Suppose that Assumptions* (A), (B) *and* (C) *are satisfied. Then the sequence* (1.4) *is, with probability one, relatively compact in* $(D,\mathcal{D})$ *with respect to the metric determined by the sup-norm* $\|\ \ \|$, *and the set of its limit points coincides with* $\mathcal{G}$.

For simplicity, we write this fact as

$$\sqrt{n}(M_n - M)/b_n \rightsquigarrow \mathcal{G} \quad \text{a.s. w.r.t.} \quad \|\ \ \| \quad \text{on} \quad (D,\mathcal{D}) \tag{1.12}$$

using the notation of Shorack and Wellner (1986, p. 69).

2. Proof of the Theorem

First we review (see Kuelbs (1976), Ledoux and Talagrand (1991)) some relevant results on the LIL for $D[0,1]$ valued random variables. In this case, the set of limit points is uniquely determined by the covariance function. To be precise let $\{Z(t), 0 \le t \le 1\}$ be such a random element on $(D,\mathcal{D})$ with mean function identically zero and continuous covariance function $R(s,t) = E(Z(s)Z(t))$, $0 \le s,t \le 1$. Then, since $R(s,t)$ is symmetric, continuous, and nonnegative definite, by Mercer's theorem (see Riesz and Sz-Nagy (1955, p. 245)), it has the eigenfunction expansion $\sum_n \lambda_n \phi_n(s)\phi_n(t)$ which converges uniformly on $[0,1]\times[0,1]$, the eigenfunctions $\{\phi_n(t)\}$ are continuous orthonormal elements of $L^2[0,1]$, and the eigenvalues λ_n are positive numbers such that $\sum_n \lambda_n < \infty$.

Let H_R denote the set of elements in $L^2[0,1]$ which are in the closure of the span of $\{\phi_n, n \geq 1\}$ and such that

$$\sum_n \frac{(z,\phi_n)^2}{\lambda_n} < \infty,$$

where $(z_1, z_2) = \int_0^1 z_1(t)z_2(t)dt$. H_R is a Hilbert space with the inner product

$$(z_1, z_2)_{H_R} = \sum_n \frac{(z_1,\phi_n)(z_2,\phi_n)}{\lambda_n},$$

and $\{\lambda_n^{\frac{1}{2}}\phi_n, n \geq 1\}$ is a complete orthonormal set in H_R.

If K_R is the unit ball of H_R (in the H_R norm) then, since $R(s,t)$ is continuous, it is fairly easy to see that K_R is a compact subset of $C[0,1]$ in the sup-norm, and we shall see that K_R is the set of limit points of interest. Here, of course, we identify equivalence classes of H_R with their continuous representative.

The Hilbert space H_R is commonly called the reproducing kernel Hilbert space (RKHS) of the kernel R. We have the following

Lemma 1. *Let $Z_1, Z_2, \ldots$ be i.i.d. random elements of $(D, \mathcal{D})$ such that each $\{Z_i(t) : 0 \leq t \leq 1\}$ is a martingale. Further, assume there exists a constant $\alpha > 0$ such that*

$$EZ_i(t) = 0, \quad \textit{and} \quad E|Z_i(t)|^{2+\alpha} < \infty,\ 0 \leq t \leq 1,$$

and the covariance function

$$R(s,t) = E(Z_i(s)Z_i(t))$$

is continuous on $[0,1]\times[0,1]$. If K_R denotes the unit ball of the RKHS H_R, *then*

$$\sum_{i=1}^n Z_i/(\sqrt{n}b_n) \sim\!\!\rightarrow K_R \quad \textit{a.s. w.r.t.} \quad \|\ \| \quad \textit{on} \quad (D,\mathcal{D}),$$

with the notation of (1.12). Refer to Kuelbs (1976).

Let $\{U_n\}$ be a sequence of independent uniform (0,1) random variables with $U_n = F(X_n)$. Define the empirical process

$$\beta_n(t) = \sqrt{n}\Big(\frac{1}{n}\sum_{i=1}^n \chi(0 < U_i \leq t) - t\Big), \quad 0 \leq t \leq 1,$$

where $\chi(A)$ denotes the indicator function of a set A. We have

Lemma 2. (James (1975)) *Suppose that q is a continuous, nonnegative function on $[0,1]$ that is symmetric about $t=\frac{1}{2}$, and that*

$$q \quad \uparrow \quad \textit{and} \quad q(t)/\sqrt{t} \quad \downarrow \quad \textit{on} \quad \Big[0,\frac{1}{2}\Big].$$

If

$$\int_0^1 (q^2(t)\log\log[t(1-t)]^{-1})^{-1}dt < \infty,$$

then

$$\frac{\beta_n}{qb_n} \sim\!\!\rightarrow \mathcal{H}_q = \{h/q: \quad h\in\mathcal{H}\} \quad \text{a.s.} \quad \text{w.r.t.} \quad \|\ \ \| \quad \text{on} \quad (D,\mathcal{D}).$$

For a proof, see Shorack and Wellner (1986), Theorem 13.4.1, pp. 517-525.

Now we are in a position to prove the theorem.

Proof. Without loss of generality we can assume that $r(t)\downarrow \sqrt{t}\,r(t)\uparrow$ on $(0,\delta]$, and $r(t)\uparrow \sqrt{1-t}\,r(t)\downarrow$ on $[1-\delta,1), r(0+)=r(1-)=\infty$, and (1.6) holds. For the cases when r is bounded on [0,1], or r is bounded on $[0,\delta]$ or $[1-\delta,1]$, the proof may be easier. Write

$$\begin{aligned}
\alpha_{n1}(t) &= n^{-\frac{1}{2}}\sum_{i=1}^{n}(Y_i-m(X_i))\chi(X_i\le F^{-1}(t)),\\
\alpha_{n2}(t) &= n^{-\frac{1}{2}}\sum_{i=1}^{n}(m(X_i)\chi(X_i\le F^{-1}(t))-M(t)),\\
\beta_n(t) &= n^{\frac{1}{2}}(F_{nR}\circ F^{-1}(t)-t), \quad 0\le t\le 1,
\end{aligned}\tag{2.1}$$

where F_{nR} is the right-continuous version of the empirical distribution function F_n. Put $U_i=F(X_i)$,

$$\begin{aligned}
Z_i(t) &= (Y_i-m(X_i))\chi(U_i\le \zeta^{-1}(\sigma_1^2 t))/\sigma_1,\\
\xi_n(t) &= \alpha_{n1}(\zeta^{-1}(\sigma_1^2 t))/\sigma_1, \quad 0\le t\le 1.
\end{aligned}\tag{2.2}$$

It is easy to check that

$$EZ_i(t)=0,\ E(Z_i(s)Z_i(t))=\zeta(\zeta^{-1}(\sigma_1^2(t\wedge s)))/\sigma_1^2=t\wedge s, \quad 0\le s,t\le 1. \tag{2.3}$$

Now we proceed to show that for each $i, \{Z_i(t),\, 0\le t\le 1\}$ is a martingale. To this end, we need only show that for $0\le s<t\le 1$,

$$E(Z_i(t)-Z_i(s)|\mathcal{F}_s)=0 \quad \text{a.s.}, \tag{2.4}$$

where $\mathcal{F}_s=\sigma\{Z_i(u), 0\le u\le s\}$. Write $\eta=(Y_i-m(X_i))/\sigma_1$ and denote by $\mathcal{B}_1$ the σ-field of Borel subsets of $(-\infty,\infty)-\{0\}$. Then $\mathcal{F}_s$ is generated by the family $\mathcal{C}_s$ of all sets of the form $\{\eta \in B_1\} \cap \{U_i \le \zeta^{-1}(\sigma_1^2 u)\}$, where $B_1 \in \mathcal{B}_1$ and $0 \le u \le s$. Denote by $(\Omega, \mathcal{F}, P)$ the probability space and write

$$\mathcal{A} = \{A \in \mathcal{F} : E(Z_i(t) - Z_i(s))\chi(A) = 0\}.$$

Since $E(Z_i(t) - Z_i(s)) = 0$, we have $\Omega \in \mathcal{A}$. By the fact that $(Z_i(t) - Z_i(s))\chi(U_i \le \zeta^{-1}(\sigma_1^2 u)) = 0$ for any $u \le s$, we see that $\mathcal{A} \supset \mathcal{C}_s$. Now it follows that $\mathcal{A}$ is a σ-field and $\mathcal{A} \supset \mathcal{F}_s$, and (2.4) is proved.

Denote by $W(t)$, $0 \le t < \infty$, the standard Brownian motion on $[0, \infty)$. By the well known Strassen's LIL on Brownian motion (refer to Shorack and Wellner (1986), Theorem 2.9.1, p. 80),

$$W(nI)/(\sqrt{n} b_n) \sim\!\!\rightarrow \mathcal{K} \quad \text{a.s.} \quad \text{w.r.t.} \quad \|\ \ \| \quad \text{on} \quad (D, \mathcal{D}) \tag{2.5}$$

as $n \to \infty$, where I is the identity mapping on [0,1].

Noting that

$$\xi_n(t) = n^{-\frac{1}{2}} \sum_{i=1}^n Z_i(t), \quad W(nt)/\sqrt{n} = n^{-\frac{1}{2}} \sum_{i=1}^n (W(it) - W((i-1)t))$$

and that $\{Z_i(t),\ 0 \le t \le 1\}$ and $\{W(it) - W((i-1)t), 0 \le t \le 1\}$ have the same covariance functions, by Lemma 1 we get

$$\xi_n/b_n \quad \sim\!\!\rightarrow \quad \mathcal{K} \quad \text{a.s.} \quad \text{w.r.t.} \quad \|\ \ \| \quad \text{on} \quad (D, \mathcal{D}), \tag{2.6}$$

and

$$\begin{aligned} \alpha_{n1}/b_n \quad \sim\!\!\rightarrow \quad & \mathcal{G}_1 = \{g_1 : g_1(u) = \sigma_1 k(\zeta(u)/\sigma_1^2), k \in \mathcal{K}\} \\ & \text{a.s.} \quad \text{w.r.t.} \quad \|\ \ \| \quad \text{on} \quad (D, \mathcal{D}). \end{aligned} \tag{2.7}$$

Now

$$\alpha_{n2}(u) = \int_0^u r(t) d\beta_n(t) = r(u)\beta_n(u) - \int_0^u \beta_n(t) dr(t). \tag{2.8}$$

Write

$$\nu_n = n^{\frac{1}{2}}(M \circ \theta_n - M) \quad \text{with} \quad \theta_n = F_{nR} \circ F^{-1} = I + n^{-\frac{1}{2}}\beta_n. \tag{2.9}$$

We proceed to prove that

$$\begin{aligned} (\alpha_{n2} - \nu_n)/b_n \quad \sim\!\!\rightarrow \quad & \mathcal{G}_2 = \{g_2 : g_2(u) = -\int_0^u h(t) dr(t),\ h \in \mathcal{H}\} \\ & \text{a.s.} \quad \text{w.r.t.} \quad \|\ \ \| \quad \text{on} \quad (D, \mathcal{D}). \end{aligned} \tag{2.10}$$

Take a small constant $C_2 > 0$ and write $q(t) = 1/(r(t) \vee C_2)$. By the monotonicity of $r(t)$ on $(0, \delta]$ and $E(m(X))^{2+\alpha} < \infty$, we have

$$(r(u))^{2+\alpha} \cdot u \le \int_0^u (r(t))^{2+\alpha} dt \to 0 \quad \text{as} \quad u \to 0,$$

and

$$r(u) = o(u^{-1/(2+\alpha)}) \quad \text{as} \quad u \downarrow 0. \tag{2.11}$$

In the same way,

$$r(u) = o((1-u)^{-1/(2+\alpha)}) \quad \text{as} \quad u \uparrow 0. \tag{2.12}$$

By (2.11) and (2.12), it is easily seen that

$$\int_0^1 (q^2(t) \log\log[t(1-t)]^{-1})^{-1} dt < \infty. \tag{2.13}$$

From the behavior of $r(t)$ on $(0,\delta]$ and $[1-\delta, 1)$, (2.13) and Lemma 2, it is easily shown that, with probability one,

$$\frac{\beta_n}{qb_n} \sim\!\!\rightarrow \mathcal{H}_q = \{h/q : h \in \mathcal{H}\} \quad \text{w.r.t.} \quad \| \quad \| \quad \text{on} \quad (D, \mathcal{D}). \tag{2.14}$$

For any fixed $\lambda \in (0,1)$, by Lemma 2, with probability one we have

$$\frac{\beta_n}{b_n(I(1-I))^{(1-\lambda)/2}} \quad \sim\!\!\rightarrow \quad \mathcal{H}_0 = \Big\{\frac{h}{(I(1-I))^{(1-\lambda)/2}} : h \in \mathcal{H}\Big\} \quad \text{w.r.t.} \quad \| \quad \| \quad \text{on} \quad (D, \mathcal{D}). \tag{2.15}$$

It means that there is a event N with $P(N) = 0$ such that (2.14) and (2.15) hold for $\omega \notin N$. In the following we always assume that $\omega \notin N$.

By the formula $\theta_n = I + n^{-\frac{1}{2}}\beta_n$ and (2.14), there exists a constant $\delta_1 \in (0,\delta)$ such that for n large, $u \in (0,\delta_1)$ implies $\theta_n(u) \in (0,\delta)$ and $u \in (1-\delta_1, 1)$ implies $\theta_n(u) \in (1-\delta, 1)$. By the monotonicity of $r(t)$ on $(0,\delta]$ and (1.6), for n large and $u \in (0,\delta_1)$,

$$\begin{aligned} 0 &\le q(u) \cdot \frac{M(\theta_n(u)) - M(u)}{\theta_n(u) - u} \le q(u) \cdot u^{-1} \int_0^u r(t) dt \\ &\le C_3 u^{-1} \int_0^u (u/t)^\tau dt \le C_3/(1-\tau), \end{aligned}$$

where C_3 is a constant. Now we write

$$\nu_n = (\beta_n/q) \cdot q \cdot (M \circ \theta_n - M)/(\theta_n - I).$$

If for some $h \in \mathcal{H}$,

$$\|(\beta_{n'}/b_{n'} - h)/q\| \to 0 \quad \text{as the subsequence} \quad n' \to \infty, \tag{2.16}$$

then by using $(h/q)(0) = 0 = (h/q)(0+)$, for any given $\epsilon > 0$, we may find a constant $\delta_2 \in (0,\delta_1)$ such that for n' large,

$$\sup_{0 \le u \le \delta_2} |\nu_{n'}(u)/b_{n'}| < \epsilon/2$$

and

$$\sup_{0\le u\le \delta_2} |(\nu_{n'}(u) - r(u)\beta_{n'}(u))/b_{n'}| < \epsilon. \tag{2.17}$$

Similarly we may find a constant $\delta_3 \in (0, \delta_1)$ such that for n' large,

$$\sup_{1-\delta_3\le u\le 1} |(\nu_{n'}(u) - r(u)\beta_{n'}(u))/b_{n'}| < \epsilon. \tag{2.18}$$

For $\delta_2 \le u \le 1-\delta_3$, by the uniform continuity of $r(t)$ on $[\delta_2, 1-\delta_3]$ and (2.16),

$$\begin{aligned} |(\nu_{n'}(u) - r(u)\beta_{n'}(u))/b_{n'}| &= \frac{(n')^{\frac{1}{2}}}{b_{n'}}\Big|\int_u^{u+(n')^{-\frac{1}{2}}\beta_{n'}(u)} (r(t)-r(u))dt\Big| \\ &\to 0 \quad \text{uniformly on} \quad [\delta_2, 1-\delta_3] \end{aligned} \tag{2.19}$$

as the subsequence $n' \to \infty$. From (2.14) and (2.16)-(2.19), it follows that

$$\|(\nu_n - r\beta_n)/b_n\| \to 0 \quad \text{a.s.} \tag{2.20}$$

Take $\lambda \in (0, \frac{\alpha}{2+\alpha})$, then $\frac{1-\lambda}{2} > \frac{1}{2+\alpha}$. By (2.11),

$$\lim_{t\to 0+} (t(1-t))^{(1-\lambda)/2} r(t) = \lim_{t\to 0+} t^{(1-\lambda)/2}\cdot o(t^{-1/(2+\alpha)}) = 0.$$

In the same way

$$\lim_{t\to 1-} (t(1-t))^{(1-\lambda)/2} r(t) = 0.$$

From these, (2.11) and (2.12), and noting that $2^{-1}(1-\lambda) > 2+\alpha^{-1}$, we have

$$\begin{aligned} \Big|\int_0^1 (t(1-t))^{(1-\lambda)/2} dr(t)\Big| &= \Big|\int_0^1 r(t) d(t(1-t))^{(1-\lambda)/2}\Big| \\ &\le C_4 \int_0^1 (t(1-t))^{-\frac{1}{2+\alpha}+\frac{1-\lambda}{2}-1} dt < \infty, \end{aligned} \tag{2.21}$$

where C_4 is a constant. By Assumption (B), the monotonicity of $r(t)$ on $(0,\delta]$ and $[1-\delta, 1)$, (2.21) implies that

$$\int_0^1 (t(1-t))^{(1-\lambda)/2} \frac{|m'(F^{-1}(t))|}{f(F^{-1}(t))} dt < \infty. \tag{2.22}$$

If for some $h \in \mathcal{H}$,

$$\begin{aligned} \epsilon(n') &\triangleq \Big\|(\frac{\beta_{n'}}{b_{n'}} - h)/(I(1-I))^{(1-\lambda)/2}\Big\| \to 0 \\ &\quad \text{as the subsequence} \quad n' \to \infty, \end{aligned} \tag{2.23}$$

then by (2.22),

$$\begin{aligned} &\sup_{0\le u\le 1}\Big|\int_0^u (\frac{\beta_{n'}(t)}{b_{n'}} - h(t))\frac{m'(F^{-1}(t))}{f(F^{-1}(t))} dt\Big| \\ \le\ &\epsilon(n')\int_0^1 (t(1-t))^{(1-\lambda)/2}\frac{|m'(F^{-1}(t))|}{f(F^{-1}(t))} dt \to 0. \end{aligned} \tag{2.24}$$

By (2.15), (2.23) and (2.24),

$$-\frac{1}{b_n}\int_0^{\cdot} \beta_n(t)dr(t) \sim\!\rightarrow \mathcal{G}_2 \quad \text{a.s.} \quad \text{w.r.t.} \quad \| \quad \| \quad \text{on} \quad (D,\mathcal{D}). \tag{2.25}$$

Now (2.10) follows from (2.8), (2.20) and (2.25).

By (2.7) and (2.10), we have

$$\begin{aligned}\frac{n^{\frac{1}{2}}}{b_n}(G_n \circ F^{-1} - M \circ F_{nR} \circ F^{-1}) &= (\alpha_{n1} + \alpha_{n2} - \nu_n)/b_n \\ &\sim\!\rightarrow \mathcal{G} \quad \text{a.s.} \quad \text{w.r.t.} \quad \| \quad \| \quad \text{on} \quad (D,\mathcal{D}),\end{aligned} \tag{2.26}$$

where $G_n(x) = n^{-1}\sum_1^n Y_i\chi(X_i \le x)$.

Let $U_{(1)} \le \cdots \le U_{(n)}$ be the order statistics of $U_1 = F(X_1), \ldots, U_n = F(X_n)$. For $0 \le u < 1$,

$$F \circ F_n^{-1}(u) = F(X_{([nu]+1)}) = U_{([nu]+1)}.$$

By the LIL of the quantile processes (the Smirnov theorem, refer to Shorack and Wellner (1986), Theorem 13.1.1, p. 504),

$$\|F \circ F_n^{-1} - I\| = 0(b_n/\sqrt{n}) \quad \text{a.s.} \tag{2.27}$$

By (2.26) and (2.27), and noting that $\mathcal{G} \subset C[0,1]$, we have

$$\begin{aligned}&\frac{n^{\frac{1}{2}}}{b_n}(G_n \circ F_n^{-1} - M \circ F_{nR} \circ F_n^{-1}) \sim\!\rightarrow \mathcal{G} \quad \text{a.s.} \\ &\qquad \text{w.r.t.} \quad \| \quad \| \quad \text{on} \quad (D,\mathcal{D}).\end{aligned} \tag{2.28}$$

For any $0 \le u \le 1, F_{nR} \circ F_n^{-1}(u) = ([nu]+1)/n$. Since $r(u) \ge 0, r(u) \downarrow$ for $u \in (0,\delta)$, we have for n large and $0 \le u < \delta_1$,

$$\begin{aligned}0 &\le \frac{n^{\frac{1}{2}}}{b_n}(M(u) - M \circ F_{nR} \circ F_n^{-1}(u)) \le \frac{n^{\frac{1}{2}}}{b_n}\int_0^{\frac{1}{n}} r(t)dt \\ &\le \frac{n^{\frac{1}{2}}}{b_n} \cdot n^{-\frac{1}{2}}\Big(\int_0^{\frac{1}{n}} r^2(t)dt\Big)^{\frac{1}{2}} \rightarrow 0.\end{aligned} \tag{2.29}$$

Similarly, we have for n large and $1 - \delta_1 < u \le 1$,

$$0 \le \frac{n^{\frac{1}{2}}}{b_n}(M \circ F_{nR} \circ F_n^{-1}(u) - M(u)) \le \frac{n^{\frac{1}{2}}}{b_n}\int_{1-\frac{1}{n}}^1 r(t)dt \rightarrow 0. \tag{2.30}$$

Write $K = \max\{\,|r(u)| : \quad \delta_1/2 \le u \le 1 - \delta_1/2\}$. Then

$$\begin{aligned}&\sup_{\delta_1 \le n \le 1-\delta_1} \frac{n^{\frac{1}{2}}}{b_n}|(M \circ F_{nR} \circ F_n^{-1}(u) - M(u))| \\ &\le \quad K/(n^{\frac{1}{2}}b_n) \rightarrow 0.\end{aligned} \tag{2.31}$$

By (2.29)-(2.31),

$$\left\| \frac{\sqrt{n}}{b_n}(M \circ F_{nR} \circ F_n^{-1} - M) \right\| \to 0 \quad \text{a.s.} \tag{2.32}$$

From (2.28) and (2.32), it follows that

$$n^{\frac{1}{2}}(M_n - M)/b_n \sim\!\!\to \mathcal{G} \quad \text{a.s.} \quad \text{w.r.t.} \quad \|\ \ \| \quad \text{on} \quad (D, \mathcal{D}) \tag{2.33}$$

and the theorem is proved.

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References

Bishop, J. A., Chakraborti, S. and Thistle, P. D. (1989). Asymptotically distribution-free statistical inference for generalized Lorenz curves. *Rev. Econom. Statist.* **71**, 725-727.

Gastwirth, J. L. (1971). A general definition of the Lorenz curve. *Econometrika* **39**, 1037-1039.

Gastwirth, J. L. (1972). The estimation of the Lorenz curve and Gini index. *Rev. Econom. Statist.* **54**, 306-316.

Goldie, C. M. (1977). Convergence theorems for empirical Lorenz curves and their inverses. *Adv. in Appl. Probab.* **9**, 765-791.

James, B. R. (1975). A functional law for the iterated logarithm for weighted empirical distributions. *Ann. Probab.* **3**, 762-772.

Kakwani, N. C. and Podder, N. (1973). On the estimation of Lorenz curves from grouped observations. *Internat. Econom. Rev.* **14**, 278-292.

Kakwani, N. C. and Podder, N. (1976). Efficient estimation of the Lorenz curve and associated inequality measures from grouped observations. *Econometrika* **44**, 137-148.

Kuelbs, J. (1976). A strong convergence theorem for Banach space valued random variables. *Ann. Probab.* **4**, 744-771.

Ledoux, M. and Talagrand, M. (1991). *Probability in Banach Space.* Springer-Verlag, New York.

Rao, C. R. and Zhao, L. C. (1995a). Convergence theorems for empirical cumulative quantile regression function. *Mathematical Methods of Statistics* **4**, 81-91.

Rao, C. R. and Zhao, L. C. (1995b). Strassen's law of iterated logarithms for Lorenz curves. *J. Multivariate Anal.* **54**, 239-252.

Riesz, F. and Sz-Nagy, B. (1955). *Functional Analysis.* Ungar, New York.

Shorack, G. R. and Wellner, J. A. (1986). *Empirical Processes with Applications to Statistics.* John Wiley, New York.

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