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QUALITATIVE EVALUATION OF ASSOCIATIONS BY THE TRANSITIVITY OF THE ASSOCIATION SIGNS

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Abstract: We say that the signs of association measures among three variables $\{X, Y, Z\}$ are transitive if a positive association measure between X and Y and a positive association measure between Y and Z imply a positive association measure between X and Z. We introduce four association measures with different stringencies, and discuss conditions for the transitivity of the signs of these association measures. When the variables follow exponential family distributions, the conditions become simpler and more interpretable. Applying our results to two data sets from an observational study and a randomized experiment, we demonstrate that the results can help us draw conclusions about the signs of the association measures between X and Z based only on separate studies about $\{X, Y\}$ and $\{Y, Z\}$.

Key words and phrases: Association measure, causal inference, Prentice's criterion, surrogate endpoint, Yule–Simpson Paradox.

1. Introduction

Reasoning by transitivity is commonly, at least implicitly, applied to statistical results from different studies. For association measures, however, transitivity is not guaranteed without conditions. If an epidemiologic study found that irregular heart beat had a positive association with sudden death, and a clinical trial found that a certain drug significantly corrected irregular heart beat, might one conclude that the drug can reduce the rate of sudden death? What conditions would one need to reason from the known statistical results? Suppose, in a meta-analysis, some associations between variable pairs are obtained from published papers, but the original data are not available. What conditions are required to qualitatively evaluate an association between another pair of variables based on these known associations? There are a few statistical approaches for such qualitative reasonings. Vanderweele and Robins (2010) propose the signed directed acyclic graph (DAG) to qualitatively reason the sign of association or causal measure between two variables by the signs of directed edges on the paths between the two variables. Their approach requires a whole DAG over all variables and the signs of all edges on each path between the two variables. Vanderweele and Tan (2012) propose an approach for propagation of bounds within the DAG framework, which requires a known DAG with edge-specific bounds. Rubin (2004) discusses how to combine the results from two randomized trials of anthrax vaccine on human volunteers and macaques, where the outcome of interest, survival when challenged with lethal doses of anthrax, is available only from macaques. Pearl and Bareinboim (2011) discuss the transportability of causal effects across different studies based on the DAG framework. Prentice (1989) proposes a criterion for surrogate endpoints in clinical trials so that a null effect of treatment on a surrogate implies a null effect of treatment on the endpoint. Chen, Geng, and Jia (2007) and Ju and Geng (2010) discuss the transitivity of causal effects between variable pairs of treatment, surrogate and endpoint.

We focus on the transitivity of the signs of association measures. We introduce four association measures with different stringency levels: density, cumulative distribution, expectation, and correlation levels (Cox and Wermuth (2003); Whittaker (1990)). We say that the signs of association measures are transitive if a non-negative (or positive) association between X and Y and a non-negative (or positive) association between Y and Z imply a non-negative (or positive) association between X and Z. We discuss the transitivity of the signs of association measures, and present conditions and assumptions (or prior knowledge) required for the transitivity. We show that a more stringent association measure has stronger transitivity. We focus on the transitivity of the signs of association measures among three variables, and these results can be easily extended to cases with more variables. We discuss conditions for the transitivity of these association measures separately for two cases, with and without the conditional independence of X and Z given Y. Conditional independence is one condition of Prentice's criterion for evaluating a surrogate Y for the endpoint Z (Prentice (1989)). The conditions for transitivity proposed here allow for qualitative assessment of the association between two variables X and Z, by the data sampled from the marginal distributions of (X, Y) and (Y, Z) or from the conditional distribution of (X, Z) given Y.

The remainder of the paper is organized as follows. Section 2 presents the definitions of four association measures and discusses their stringencies. In Section 3, we consider the transitivity of the signs of these association measures under the conditional independence of X and Z given Y, and give results for an exponential family distribution. We generalize the results about transitivity without conditional independence in Section 4. We apply our theoretical results to two data sets in Section 5 and conclude with a discussion in Section 6. Proofs of the theorems are in the web-based supporting material.

2. Association Measures and Their Stringencies

We introduce four commonly-used association measures and show their relative stringencies for depicting associations.

Definition 1. Association measures between X and Y are

(1) density association: $\partial^2 \ln f(x,y)/\partial x \partial y$ (Whittaker (1990));

(2) distribution association: $\partial F(y|x)/\partial x$ (Cox and Wermuth (2003));

(3) expectation association: $\partial E(Y|x)/\partial x$;

(4) correlation coefficient: $r(X,Y)$.

For these measures, X and Y may be continuous, discrete or mixed random variables. For a discrete variable, one can replace the partial derivative by the difference between two adjacent levels. For instance, when X and Y are both binary variables, the density association is the log odds ratio

$$\ln \frac{P(X=1,Y=1)P(X=0,Y=0)}{P(X=1,Y=0)P(X=0,Y=1)},$$

and the distribution association and expectation association are both equal to the risk difference $P(Y=1|X=1) - P(Y=1|X=0)$. If we are concerned only about the signs of the association measures, a non-negative expectation association implies that the risk difference is greater than or equal to zero.

The density association $\partial^2 \ln f(x,y)/\partial x \partial y$ depicts a local dependence between X and Y around the point (x,y). An important property is that

$$\frac{\partial^2 \ln f(x,y)}{\partial x \partial y} = \frac{\partial^2 \ln f(x|y)}{\partial x \partial y} = \frac{\partial^2 \ln f(y|x)}{\partial x \partial y},$$

which can be identified by sampling conditionally on X or Y, such as a prospective study (conditionally on X) or a retrospective study (conditionally on Y). The distribution association $\partial F(y|x)/\partial x$ depicts the dependence of a global $Y \le y$ on a local $X = x$, and $\partial F(y|x)/\partial x \le 0$ means that Y given X is stochastically increasing in X (Cox and Wermuth (2003)). The expectation association $\partial E(Y|x)/\partial x$ depicts the overall dependence of Y on X, and correlation coefficient $r(X,Y)$ depicts a linear association. We say that X and Y are non-negatively associated with respect to a measure, say $\partial^2 \ln f(x,y)/\partial x \partial y$, if $\partial^2 \ln f(x,y)/\partial x \partial y \ge 0$ for all x and y, and we say that X and Y are positively associated with respect to a measure if further strict inequality holds for some x or y. Let $A \perp\!\!\!\perp B$ denote independence between A and B, and let $A \perp\!\!\!\perp B|C$ denote conditional independence of A and B given C. Generally, a non-negative association measure at a more stringent level implies a non-negative association measure at a less stringent level, as summarized in the following properties.

Property 1. The implication relationship (Xie, Ma, and Geng (2008)) holds:

$$\frac{\partial^2 \ln f(x,y)}{\partial x \partial y} \ge 0, \forall\, x, y, \Longrightarrow \frac{\partial F(y|x)}{\partial x} \le 0, \forall\, x, y, \Longrightarrow \frac{\partial E(Y|x)}{\partial x} \ge 0, \forall\, x,$$
$$\Longrightarrow r(X,Y) \ge 0;$$

Property 2. The equivalence relationship among null association measures holds:

$$\frac{\partial^2 \ln f(x,y)}{\partial x \partial y} = 0, \forall\ x, y, \Longleftrightarrow \frac{\partial F(y|x)}{\partial x} = 0, \forall\ x, y, \Longleftrightarrow X \perp\!\!\!\perp Y;$$

Property 3. For a bivariate normal vector (X, Y), the equivalence relationship holds:

$$\frac{\partial^2 \ln f(x,y)}{\partial x \partial y} \geq 0, \forall\ x, y, \Longleftrightarrow \frac{\partial F(y|x)}{\partial x} \leq 0, \forall\ x, y, \Longleftrightarrow \frac{\partial E(Y|x)}{\partial x} \geq 0, \forall\ x, \\ \Longleftrightarrow r(X,Y) \geq 0;$$

Property 4. For a binary Y, the equivalence relationship holds:

$$\frac{\partial^2 \ln f(x,y)}{\partial x \partial y} \geq 0, \forall\ x, y, \Longleftrightarrow \frac{\partial F(y|x)}{\partial x} \leq 0, \forall\ x, y, \Longleftrightarrow \frac{\partial E(Y|x)}{\partial x} \geq 0, \forall\ x;$$

Property 5. For a binary X, the equivalence relationship holds:

$$\frac{\partial E(Y|x)}{\partial x} \geq 0, \forall\ x, \Longleftrightarrow r(X,Y) \geq 0.$$

These relationships above also hold if "$\geq$" and "$\leq$" are replaced by "$>$" and "$<$" or "$=$" and "$=$".

By the implication relationship in Property 1, density association is the most stringent, and correlation coefficient is the least stringent. For the case of two normal variables or two binary variables, these association measures have the same sign (non-negative, null, or positive).

3. Transitivity of Association Signs with Conditional Independence

We consider three variables $\{X, Y, Z\}$, and discuss the conditions for the transitivity of association signs among them. Prentice (1989) uses conditional independence as a criterion for validating a surrogate Y, when evaluating the effect of treatment X on the endpoint Z. We call $X \perp\!\!\!\perp Z|Y$ the conditional independence assumption, where Y breaks the dependence between X and Z. Note that we can have both $X \perp\!\!\!\perp Z|Y$ and $X \perp\!\!\!\perp Z$ even when both pairs (Y, Z) and (X, Y) are associated, and Table 1 is an example of this given in Birch (1963). This is called a violation of weak transitivity or of singleton transitivity (Lněnička and Matúš (2007)), in which case we may not infer the transitivity of associations only under the conditional independence assumption. This may happen because the associations within different level sets have different signs. In this section, we discuss the transitivity of the association signs under the conditional independence assumption, and we discuss the case without the conditional independence assumption in Section 4.

Table 1. Example of violation of weak transitivity.

	$Y=0$		$Y=1$		$Y=2$		all Y	
	$Z=1$	$Z=0$	$Z=1$	$Z=0$	$Z=1$	$Z=0$	$Z=1$	$Z=0$
$X=1$	4	2	2	1	1	4	7	7
$X=0$	2	1	4	2	1	4	7	7
all X	6	3	6	3	2	8	14	14

3.1. Transitivity without distributional assumptions

We show that, under the conditional independence assumption, the density and distribution associations are transitive, but the expectation association and correlation are not transitive without additional conditions or assistance from the more stringent measures.

Theorem 1 (Density association). *If $X \perp\!\!\!\perp Z|Y$,*

$$(1) \quad \frac{\partial^2 \ln f(x,y)}{\partial x \partial y} \geq 0, \quad \forall\ x,y, \qquad \textit{and} \qquad (2) \quad \frac{\partial^2 \ln f(y,z)}{\partial y \partial z} \geq 0, \quad \forall\ y,z,$$

then $\partial^2 \ln f(x,z)/\partial x \partial z \geq 0, \forall\ x,z$.

We can thus conclude a non-negative (positive) density association between X and Z from the prior knowledge of a non-negative (positive) density association between X and Y and a non-negative (positive) density association between Y and Z. Notice that the transitivity of the signs is also applicable to non-positive association measures if we define $X' = -X$ or $Y' = -Y$.

Even if the conditions in Theorem 1 are satisfied, we may have $\partial^2 \ln f(x,z)/\partial x \partial z = 0$ for all x and z, i.e., $X \perp\!\!\!\perp Z$. If strict inequalities hold in (1) and (2) in Theorem 1 for any (x,y,z) in a set of nonzero measure, then $\partial^2 \ln f(x,z)/\partial x \partial z > 0$ in this set, i.e., the density association measure between X and Z is positive. If all the variables are discrete, then we have strict inequality unless $X \perp\!\!\!\perp Y$ or $Y \perp\!\!\!\perp Z$.

Theorem 2 (Distribution association). *If $X \perp\!\!\!\perp Z|Y$,*

$$(1) \quad \frac{\partial F(y|x)}{\partial x} \leq 0, \quad \forall\ x,y, \qquad \textit{and} \qquad (2) \quad \frac{\partial F(z|y)}{\partial y} \leq 0, \quad \forall\ y,z,$$

then $\partial F(z|x)/\partial x \leq 0, \forall\ x,z$.

The expectation associations are not themselves transitive, they require assistance from more stringent measures.

Theorem 3 (Expectation association). *If $X \perp\!\!\!\perp Z|Y$,*

$$(1) \quad \frac{\partial F(y|x)}{\partial x} \leq 0, \quad \forall\ x,y, \qquad \textit{and} \qquad (2) \quad \frac{\partial E(Z|y)}{\partial y} \geq 0, \quad \forall\ y,$$

then $\partial E(Z|x)/\partial x \geq 0, \forall\ x$.

Table 2. Transitivity of association signs under conditional independence.

Association between X and Y	Association between Y and Z		
	$\frac{\partial^2 \ln f(y,z)}{\partial y \partial z} \geq 0, \forall\, y, z$ $\Rightarrow$	$\frac{\partial F(z\mid y)}{\partial y} \leq 0, \forall\, y, z$ $\Rightarrow$	$\frac{\partial E(Z\mid y)}{\partial y} \geq 0, \forall\, y$
$\frac{\partial^2 \ln f(x,y)}{\partial x \partial y} \geq 0, \forall\, x, y$	$\frac{\partial^2 \ln f(x,z)}{\partial x \partial z} \geq 0, \forall\, x, z$	$\frac{\partial F(z\mid x)}{\partial x} \leq 0, \forall\, x, z$	$\frac{\partial E(Z\mid x)}{\partial x} \geq 0, \forall\, x$
$\Downarrow$ $\frac{\partial F(y\mid x)}{\partial x} \leq 0, \forall\, x, y$	$\frac{\partial F(z\mid x)}{\partial x} \leq 0, \forall\, x, z$	$\frac{\partial F(z\mid x)}{\partial x} \leq 0, \forall\, x, z$	$\frac{\partial E(Z\mid x)}{\partial x} \geq 0, \forall\, x$
$\Downarrow$ $\frac{\partial E(Y\mid x)}{\partial x} \geq 0, \forall\, x$	Under $E(Z\mid y) = \alpha + \beta y$, $\frac{\partial E(Z\mid x)}{\partial x} = \beta \frac{\partial E(Y\mid x)}{\partial x} \geq 0, \forall\, x$.		

For Theorems 2 and 3, if there exists a set of nonzero measure in which strict inequalities hold in (1) and (2), then the corresponding association measures between X and Z are positive.

If the conditions in Theorems 1 to 3 are conditional on another variable vector V, then we can obtain the association signs of X and Z conditional on V. This is useful for models with a covariate vector V.

In Theorems 1 to 3, if we have either $X \perp\!\!\!\perp Y$ or $Y \perp\!\!\!\perp Z$, we obtain $X \perp\!\!\!\perp Z$. Condition (1) of Theorem 3 is a distribution association but not an expectation association of Y on X. The expectation $E(Y|x)$ cannot replace $F(y|x)$ in (1) of Theorem 3; a counterexample is given in the web-based online material.

For a linear model of Z given Y, the expectation association measures are transitive, and the transitivity can be represented by an equation of expectation associations as follows.

Corollary 1. *Under the assumption* $X \perp\!\!\!\perp Z|Y$, *if* $E(Z|y) = \alpha + \beta y$, *then* $\partial E(Z|x)/\partial x = \beta \partial E(Y|x)/\partial x$.

We summarize the transitivity of association signs in Table 2. We exhibit the transitivity of non-negative association measures between X and Y and between Y and Z to a non-negative association measure between X and Z. Here we see that a non-negative association measure between X and Z requires the same or more stringent non-negative association measures between X and Y and between Y and Z. Table 2 is not symmetric, and the expectation association of Y on X does not have any implication for the sign of an association measure between X and Z, with an exception when Z follows a linear model.

A non-negative expectation association of Y on X and even the most stringent non-negative density association between Y and Z do not imply a non-negative expectation association of Z on X. Similarly, a non-negative correlation between X and Y (Y and Z) and another non-negative association measure between Y and Z (X and Y) do not imply a non-negative correlation between X and Z. We give counterexamples in the web-based online material.

3.2. Transitivity in the exponential family

The sufficient conditions for non-negative and positive association measures between X and Z, may not be necessary. We show the equivalence relationship between the sign of association measure between X and Y and that between X and Z, under the assumptions that Y follows an exponential family distribution and the association between Y and Z is non-negative.

Definition 2. We say that Y given X follows an exponential family distribution if its density (or its probability mass function for a discrete Y) has the form

$$f(y|x;\theta,\phi) = \exp\left\{\frac{y\theta_x - b(\theta_x)}{a(\phi)} + c(y,\phi)\right\},$$

where θ_x is a function of x and $a(\phi) > 0$.

Theorem 4. *If Y given X follows an exponential family distribution, then*

$$\frac{\partial^2 \ln f(x,y)}{\partial x \partial y} \geq 0, \quad \forall\ x,y \Longleftrightarrow \frac{\partial F(y|x)}{\partial x} \leq 0, \quad \forall\ x,y \Longleftrightarrow \frac{\partial E(Y|x)}{\partial x} \geq 0, \quad \forall\ x.$$

Equivalence relationship also holds if ("$\geq$", "$\leq$") in the inequalities are changed to ("$>$", "$<$") or ("$=$", "$=$").

Suppose below that we have prior knowledge of the sign of association between Y and Z, and we discuss the equivalence relationships between the signs of associations between X and Y and between X and Z.

Corollary 2. *Suppose $X \perp\!\!\!\perp Z|Y$ and that Y given X follows an exponential family distribution.*

(1) *If $\partial E(Z|y)/\partial y \geq 0, \forall\ y$, and strict inequality holds for a nonzero measure set, then*

$$\frac{\partial E(Y|x)}{\partial x} \geq 0, \quad \forall\ x \Longleftrightarrow \frac{\partial E(Z|x)}{\partial x} \geq 0, \quad \forall\ x.$$

(2) *If $\partial F(z|y)/\partial y \leq 0, \forall\ y,z$, and strict inequality holds for a nonzero measure set, then*

$$\frac{\partial F(y|x)}{\partial x} \leq 0, \quad \forall\ x,y \Longleftrightarrow \frac{\partial F(z|x)}{\partial x} \leq 0, \quad \forall\ x,z.$$

(3) *If $\partial^2 \ln f(y,z)/\partial y \partial z \geq 0, \forall\ y,z$, and strict inequality holds for a nonzero measure set, then*

$$\frac{\partial^2 \ln f(x,y)}{\partial x \partial y} \geq 0, \quad \forall\ x,y \Longleftrightarrow \frac{\partial^2 \ln f(x,z)}{\partial x \partial z} \geq 0, \quad \forall\ x,z.$$

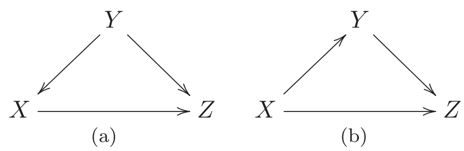

Figure 1. Two DAGs for (X, Y, Z).

Equivalence relationships also hold if the inequalities ("$\geq$", "$\leq$") above are changed to strict inequalities ("$>$", "$<$") or equalities ("$=$", "$=$").

Our results can be extended to transitivity of causal measures. If X is randomized or is conditionally independent of all potential outcomes given some covariates, then the associations between X and Y and between X and Z are also the causal effects of X on Y and X on Z, respectively. If we have prior knowledge that the conditional independence assumption holds and the association between Y and Z is non-negative before then, according to Corollary 2, the sign of the treatment effect of X on Z is the same as the sign of the treatment effect of X on Y.

4. Transitivity of Association Signs without Conditional Independence

In many applications, the conditional independence assumption $X \perp\!\!\!\perp Z|Y$ does not hold. In this section, we remove the conditional independence assumption, and discuss conditions for the transitivity of these association signs.

4.1. Motivating examples

We consider two cases: in the first case, as shown in Figure 1(a), Y is a confounder between X and Z; in the second case, as shown in Figure 1(b), X has a direct path to Z and an indirect path to Z through an intermediate variable Y.

Figure 1(a) can be tied to the Yule–Simpson Paradox, since we can evaluate whether the conditional association sign is the same as the marginal association sign between X and Z. Appleton, French, and Vanderpump (1996) give an example of the Yule–Simpson Paradox with a condensed form shown in Table 3. As seen there, the association between X and Z is positive conditional on Y but negative marginally, thus the Yule–Simpson Paradox occurs.

With Figure 1(b), we can draw conclusions about the effect of X on Z based on some assumptions about the pairs (X, Y) and (Y, Z). As shown in Table 4, using results in Section 4 which generalize the results in Section 3 to allow a directed arrow from X to Z, we have that $E(Z|Y = 1) - E(Z|Y = 0) = 0.08$ and $E(Y|X = 1) - E(Y|X = 0) = 0.2$, but $E(Z|X = 1) - E(Z|X = 0) = -0.08$. Without the conditional independence assumption, even if the pairs (X, Y) and (Y, Z) are both positively associated, X and Z may be negatively associated.

Table 3. Numbers of women smokers and nonsmokers in different age groups.

Y: age group	18-34		35-54		55-64		65+		all ages	
Z/X: smoker	yes	no	yes	no	yes	no	yes	no	yes	no
dead	5	6	41	19	51	40	42	105	139	230
alive	174	213	198	180	64	81	7	28	443	502
odds ratio	1.02		1.96		1.61		1.02		0.68	

Table 4. Distribution $P(x, y, z)$ for violation of transitivity.

	$Y = 1$		$Y = 0$	
	$X = 1$	$X = 0$	$X = 1$	$X = 0$
$Z = 1$	0.15	0.12	0.08	0.15
$Z = 0$	0.15	0.08	0.12	0.15

4.2. Transitivity without distributional assumptions

Here we give conditions for the transitivity of density, distribution, and expectation association signs.

Theorem 5 (Density association). *Suppose* $\partial^2 \ln f(x, z|y)/\partial x \partial z \geq 0, \forall\ x, y, z$. *If*

$$(1) \quad \frac{\partial^2 \ln f(x, y)}{\partial x \partial y} \geq 0, \qquad \forall\ x, y,$$

$$(2) \quad \frac{\partial^2 \ln f(y, z|x)}{\partial y \partial z} \geq 0, \quad \forall\ x, y, z, \ \textit{and}$$

$$(3) \quad \frac{\partial^2 \ln f(z|x, y)}{\partial x \partial y} \geq 0, \quad \forall\ x, y, z,$$

then $\partial^2 \ln f(x, z)/\partial x \partial z \geq 0$, $\forall\ x, z$.

If we have prior knowledge that the association between X and Z given Y is non-positive, we can replace Z by $Z' = -Z$ in the assumptions. In Theorem 5, (1) requires a nonnegative density association between X and Y, and (2) requires a non-negative density association between Y and Z conditional on X. However, (3) is quite different and can be interpreted as a non-negative interaction of (X, Y) on Z.

Theorem 6 (Distribution association). *Suppose* $\partial F(z|y, x)/\partial x \leq 0, \forall\ x, y, z$. *If*

$$(1) \quad \frac{\partial F(y|x)}{\partial x} \leq 0, \quad \forall\ x, y, \qquad \textit{and} \qquad (2) \quad \frac{\partial F(z|y, x)}{\partial y} \leq 0, \quad \forall\ x, y, z,$$

then $\partial F(z|x)/\partial x \leq 0$, $\forall\ x, z$.

Theorem 7 (Expectation association)**.** *Suppose* $\partial E(Z|y,x)/\partial x \geq 0, \forall\ x, y$. *If*

$$(1) \quad \frac{\partial F(y|x)}{\partial x} \leq 0, \quad \forall\ x, y, \qquad \textit{and} \qquad (2) \quad \frac{\partial E(Z|y,x)}{\partial y} \geq 0, \quad \forall\ x, y,$$

then $\partial E(Z|x)/\partial x \geq 0$, $\forall\ x$.

Comparing (2) in Theorems 5 to 7 with those in Theorems 1 to 3, we see that without the conditional independence assumption, the association between Y and Z is required to be conditional on X.

Corollary 3. *The conditions in Theorem* 5 *to* 7 *can be evaluated by* $f(x, z|y)$.

Thus we can assess the sign of the marginal association measure between X and Z by the conditional distribution $f(z, x|y)$, but we do not require the marginal distribution of Y.

By relaxing the conditional independence assumption to the assumption of non-negative association measure between X and Z given Y, the marginal associations between Y and Z are replaced by the conditional associations given X. This condition can be weakened if the models of Z and Y are linear.

Corollary 4. *Suppose* $E(Z|x, y) = \beta_0 + \beta_1 x + \beta_2 y$, $E(Y|x) = \beta_3 + \beta_4 x, \beta_1 \geq 0$, *and* $\beta_4 \geq 0$. *If* $\beta_2 \geq 0$ *or* $\partial E(Z|y)/\partial y \geq 0, \forall\ y$, *then* $\partial E(Z|x)/\partial x = \beta_1 + \beta_2\beta_4 \geq 0$ *for all* x.

Here if $\beta_2 \geq 0$, Corollary 4 is in Cochran (1938). If we have only the non-negative association between Y and Z marginally, we can still conclude that X and Z are non-negatively associated based on the linear models in Corollary 4. Thus, we do not need to observe X to judge (2) in Corollary 4. Provided X and Z are positively associated conditional on Y, if we have two populations, one with (X, Y) observed and the other with (Y, Z) observed, we can draw the conclusion about the association sign between X and Z.

When there is a randomized intervention on Y, we have $X \perp\!\!\!\perp Y$, and Theorems 5 and 7 can be simplified.

Corollary 5. *Suppose* $X \perp\!\!\!\perp Y$.

(1) *If* X *or* Z *is binary, then* $\partial^2 \ln f(x, z|y)/\partial x \partial z \geq 0, \forall\ x, y, z$ *implies* $\partial^2 \ln f(x, z) /\partial x \partial z \geq 0, \forall\ x, z$;

(2) *If* $\partial F(z|y, x)/\partial x \leq 0,\ \forall\ x, y$, *then* $\partial F(z|x)/\partial x \leq 0,\ \forall\ x$;

(3) *If* $\partial E(Z|y, x)/\partial x \leq 0,\ \forall\ x, y$, *then* $\partial E(Z|x)/\partial x \leq 0,\ \forall\ x$.

Theorems 5 to 7 can also be useful for cases with more than three variables. In addition, we can combine X and V into a vector that plays the same role as the original X.

4.3. Transitivity in the exponential family

Theorem 8. *Assume that Z given X follows an exponential family distribution, and that $\partial E(Z|y,x)/\partial x \geq 0, \forall\ x, y$. If*

$$(1) \quad \frac{\partial F(y|x)}{\partial x} \leq 0, \quad \forall\ x, y, \qquad and \qquad (2) \quad \frac{\partial E(Z|y,x)}{\partial y} \geq 0, \quad \forall\ x, y,$$

then $\partial^2 \ln f(x,z)/\partial x \partial z \geq 0$, $\forall\ x, z$.

Due to the symmetry of the density association measure, we have:

Corollary 6. *If X or Z is binary,* (3) *in Theorem* 5 *is redundant.*

In many randomized experiments, X is binary, and we need not evaluate (3) in Theorem 5.

5. Illustrations

5.1. An observational study: the national longitudinal surveys

The National Longitudinal Surveys are a set of surveys designed to gather information at multiple time points on labor market activities and other significant life events of several groups of men and women (Toomet and Henningsen (2008)). Let $X = 1$ if a subject graduated from college, and $X = 0$ otherwise. Let $Y = 1$ if a subject belonged to a union, and $Y = 0$ otherwise. Let Z be the log wage.

Suppose data are collected on the union group and the non-union group. From the data set, we have $\text{mean}(Z|X = 1, Y = 1) - \text{mean}(Z|X = 0, Y = 1) = 0.4351$ ($p < 0.001$) and $\text{mean}(Z|X = 1, Y = 0) - \text{mean}(Z|X = 0, Y = 0) = 0.3328$ ($p < 0.001$). Thus the college indicator X has a positive expectation association on the log wage for both groups. Since there is no information on the distribution of Y from the conditional sampling given Y, we cannot determine whether the college indicator X also has a positive association on the log wage Z marginally. To do so, we check the conditions in Theorem 7 as follows. For (1), we have the difference of observed frequencies: $\hat{P}(X = 1|Y = 1) - \hat{P}(X = 1|Y = 0) = 0.0626$ ($p < 0.001$), thus we have $\partial F(y|x)/x \leq 0$ since X and Y are binary; for (2), we have $\text{mean}(Z|X = 1, Y = 1) - \text{mean}(Z|X = 1, Y = 0) = 0.1196$ ($p < 0.001$) and $\text{mean}(Z|X = 0, Y = 1) - \text{mean}(Z|X = 0, Y = 0) = 0.2218$ ($p < 0.001$). Thus we can conclude that there is a positive expectation association of the college indicator X on the log wage Z in the population.

Using the data set containing the joint distribution of (X, Y, Z), we obtain $\text{mean}(Z|X = 1) - \text{mean}(Z|X = 0) = 0.4211$, which confirms our conclusion. If we would like to assume that Z follows an exponential family distribution conditional on X, then we can deduce from Corollary 2 that the density and distribution associations between X and Z are positive.

5.2. A randomized experiment: the job search intervention study

The Job Search Intervention Study is a randomized field experiment that investigates the efficacy of a job training intervention on unemployed workers (Vinokur and Schul (1997); Tingley et al. (2012)). The program is designed not only to increase reemployment among the unemployed, but also to enhance the mental health of the job seekers. In the JOBS II field experiment, 1,801 unemployed workers are randomly assigned to a treatment group ($X = 1$) and a control group ($X = 0$). Those in the treatment group participate in job-skills workshops; in the workshops, respondents learned job-searching skills and coping strategies for dealing with setbacks in the job-search process. Those in the control group receive a booklet describing job-searching tips. In the follow-up interviews, two key outcome variables are measured: a continuous variable Y denoting the job-search self-efficacy, and a continuous variable Z denoting depressive symptoms based on the Hopkins Symptom Checklist.

We randomly split the data set into two subsets with the same number of observations, and suppose that only X and Y were observed in the first subset and only Y and Z were observed in the second subset. The goal is to qualitatively evaluate the effect of the treatment X on the depressive symptoms Z. We assume the linear regression model: $E(Z'|x, y) = \beta_0 + \beta_1 x + \beta_2 y$ and $\beta_1 > 0$, where $Z' = -Z$. This is confirmed by the full data set ($\hat{\beta}_1 = 0.0408$ with p-value=0.36).

For (1) of Corollary 4, the linear model $E(Y|x) = \beta_3 + \beta_4 x$ is saturated because X is binary, and we obtaine $\hat{\beta}_4 = 0.1416$ with p-value=0.023 from the first subset containing X and Y.

To check (2) of Corollary 4, we use both parametric and nonparametric approaches. First, we use the linear model $E(Z'|y) = \beta_5 + \beta_6 y$ and obtain the estimate of $\partial E(Z'|y)/\partial y = \beta_6$: $\hat{\beta}_6 = 0.2882$ (p-value <0.001) from the second data subset of observed variables Y and Z'. Next we use local polynomial regression to get the nonparametric estimations of $E(Z'|y)$ and $\partial E(Z'|y)/\partial y$, shown in Figure 2. The scatterplot of Y and Z' is also shown on the left side of Figure 2. We can see that $\partial E(Z'|y)/\partial y \geq 0$ for any y. Both approaches confirm (2) of Corollary 4, and we conclude that X has a non-negative association with Z'.

From the full data set, we find that mean$(Z|X = 1)-$ mean$(Z|X = 0) =$ 0.06335 (p-value =0.17), thus confirming our conclusion. Although the p-value for β_1 is not significant, this example is useful for illustrating Corollary 4, because Corollary 4 is about only the signs of parameters and does not require the significance of the parameters.

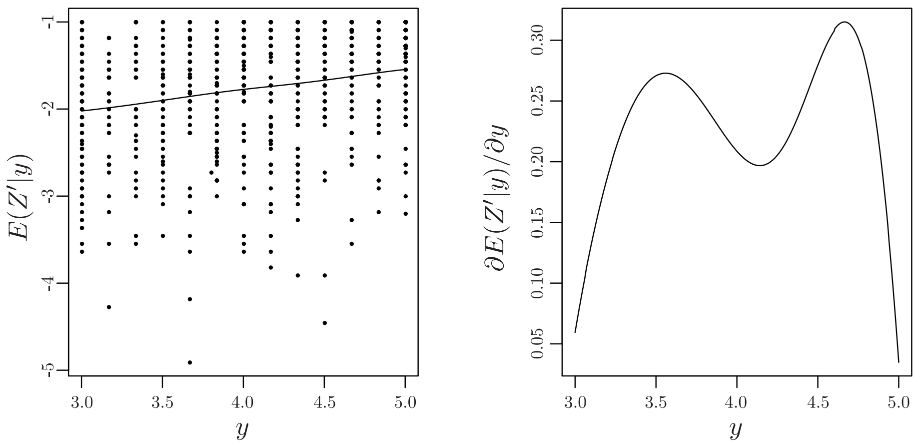

Figure 2. The curves of $E(Z'|y)$ and $\partial E(Z'|y)/\partial y$ estimated by the local polynomial regression with the Gaussian kernel and bandwidth 0.4.

6. Discussion and Extensions

We have discussed the transitivity of the signs of four association measures with and without the conditional independence assumption. These four have different stringencies, and more stringent association measures have stronger transitivity. The signs or directions of stringent association measures are more easily kept for transitivity under the conditional independence assumption. We proposed conditions for the transitivity of association measures, and showed that these conditions are necessary and sufficient if the intermediate variable Y is from an exponential family given X. The conditions can be checked by data based on marginal distributions or conditional distributions from different studies.

The problem of transportability (Frangakis and Rubin (2002); Rubin (2004); Pearl and Bareinboim (2011)) is related to the transitivity discussed in this paper. Transportability refers to two populations sharing some features in common, but transitivity refers to variables in a single population which are not observed jointly. If we treat these variables as from different populations, then transitivity can be viewed as transportability. Pearl and Bareinboim's approaches for transportability are for quantitative inference, and they require more restrictive conditions for the similarity between different populations. These conditions may not be satisfied in practice.

As pointed out by a referee, several applications and generalizations beyond our paper are possible. The density association is used mostly for binary variables, and, among continuous distribution, it has been defined explicitly only for the multivariate normal distribution. Therefore, it may be of interest to find other continuous distributions which have nice properties for the density association. In this paper, the results do not require a particular DAG, and we discuss two DAGs related to our results. There are four different types of decomposable chain graphs discussed by Drton (2009), and it is worthwhile to discuss applications to these chain graphs. It may also be interesting to simplify these conditions for transitivity using the distributional results of Roverato (2013) and the conditions for traceability of paths in regression graphs in Wermuth (2012).

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