# Web-Based Supporting Materials for "Qualitative evaluation of associations by the transitivity of the association signs"

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In Appendix A, we first introduce a lemma, and then give the proofs of the Theorems and Corollaries. In Appendix B, we present the three examples stated in Section 3.1 .

## Appendix A: Proofs of the Theorems and Corollaries

**Lemma 1.** If $h(y, a, r)$ is non-decreasing in $y$ and in $a$, and $S(y|a, r) = P(Y > y|A = a, R = r)$ is non-decreasing in $a$ for all $y$, then $E\{h(Y, A, R)|A = a, R = r\}$ is non-decreasing in $a$.

*Proof.* A proof is given by VanderWeele and Robins (2009, page 710, line 7). As suggested by a reviewer, we give a proof for discrete variables. Suppose $a \geq a'$

and $Y$ is discrete, taking values $-\infty = y_0 < y_1 < y_2 < \ldots < y_k$, then we have

$$
\begin{aligned}
& E\{h(Y,A,R)|A=a,R=r\} - E\{h(Y,A,R)|A=a',R=r\} \\
= & \sum_{i=1}^{k} h(y_i,a,r)P(Y=y_i|A=a,R=r) - \sum_{i=1}^{k} h(y_i,a',r)P(Y=y_i|A=a',R=r) \\
= & \sum_{i=1}^{k} h(y_i,a,r)\{S(y_{i-1}|a,r)-S(y_i|a,r)\} - \sum_{i=1}^{k} h(y_i,a',r)\{S(y_{i-1}|a',r)-S(y_i|a',r)\} \\
= & \sum_{i=1}^{k} h(y_i,a,r)\{S(y_{i-1}|a,r)-S(y_{i-1}|a',r)\} + \sum_{i=1}^{k}\{h(y_i,a,r)-h(y_i,a',r)\}S(y_{i-1}|a',r) \\
& -\sum_{i=1}^{k} h(y_i,a,r)\{S(y_i|a,r)-S(y_i|a',r)\} - \sum_{i=1}^{k}\{h(y_i,a,r)-h(y_i,a',r)\}S(y_i|a',r) \\
= & \sum_{i=2}^{k}\{h(y_i,a,r)-h(y_{i-1},a,r)\}\{S(y_{i-1}|a,r)-S(y_{i-1}|a',r)\} \\
& +\sum_{i=1}^{k}\{h(y_i,a,r)-h(y_i,a',r)\}\{S(y_{i-1}|a',r)-S(y_i|a',r)\}.
\end{aligned}
$$

The final expression is non-negative since all differences in brackets are non-negative for $a \geq a'$.

*Proof of Theorem 1.* We need only to prove that $\partial \ln f(z|x)/\partial x \geq \partial \ln f(z'|x)/\partial x$ for all $z > z'$. When $X$ is continuous, we deduce from $X \perp\!\!\!\perp Z|Y$ that

$$
\begin{aligned}
\frac{\partial \ln f(z|x)}{\partial x} &= \frac{\partial f(z|x)}{\partial x}/f(z|x) = \frac{\partial}{\partial x}\left\{\int_{-\infty}^{+\infty} f(z,y|x)dy\right\}/f(z|x) \\
&= \int_{-\infty}^{+\infty} \frac{\partial f(y|x)}{\partial x}\frac{f(z|y)}{f(z|x)}dy = \int_{-\infty}^{+\infty} \frac{\partial \ln f(y|x)}{\partial x}\frac{f(z|y)f(y|x)}{f(z|x)}dy \\
&= \int_{-\infty}^{+\infty} \frac{\partial \ln f(y|x)}{\partial x} f(y|x,z)dy \\
&= E\left\{\frac{\partial \ln f(Y|x)}{\partial x}|X=x,Z=z\right\}. \qquad (1)
\end{aligned}
$$

From $\partial^2 \ln f(y|x)/\partial y \partial x \geq 0$, we know that $\partial \ln f(y|x)/\partial x$ is non-decreasing in $y$. Again from $X \perp\!\!\!\perp Z|Y$, we have $\ln f(x,y,z) = \ln f(y) + \ln f(z|y) + \ln f(x|y)$. By condition (2) in Theorem 1, we obtain

$$
\frac{\partial^2 \ln f(y,z|x)}{\partial y \partial z} = \frac{\partial^2 \ln f(z|y,x)}{\partial y \partial z} = \frac{\partial^2 \ln f(z|y)}{\partial y \partial z} = \frac{\partial^2 \ln f(y,z)}{\partial y \partial z} \geq 0.
$$

From Property 1, we get $\partial F(y|z,x)/\partial z \leq 0$, and thus $P(Y > y|X = x, Z = z)$ is non-decreasing in $z$ for all $y$. Applying Lemma 1 to equation (1), we conclude that $\partial \ln f(z|x)/\partial x$ is non-decreasing in $z$.

When $X$ is discrete, we need only to prove that, for all $z > z'$,

$$\frac{f(z|x=1)}{f(z|x=0)} \geq \frac{f(z'|x=1)}{f(z'|x=0)},$$

or, equivalently,

$$\frac{f(z|x=1) - f(z|x=0)}{f(z|x=0)} \geq \frac{f(z'|x=1) - f(z'|x=0)}{f(z'|x=0)}.$$

We compute that

$$\begin{aligned}
\frac{f(z|x=1) - f(z|x=0)}{f(z|x=0)} &= \int_{-\infty}^{+\infty} \frac{f(z|y)\left\{f(y|x=1) - f(y|x=0)\right\}}{f(z|x=0)} dy \\
&= \int_{-\infty}^{+\infty} \frac{f(y|x=1) - f(y|x=0)}{f(y|x=0)} \frac{f(z|y)f(y|x=0)}{f(z|x=0)} dy \\
&= \int_{-\infty}^{+\infty} \frac{f(y|x=1) - f(y|x=0)}{f(y|x=0)} f(y|z, x=0) dy \\
&= E\left\{\frac{f(Y|x=1) - f(Y|x=0)}{f(Y|x=0)} | X=0, Z=z\right\}.
\end{aligned}$$

From conditions (1) and (2) in Theorem 1, we have that $\{f(y|x=1) - f(y|x=0)\}/f(y|x=0)$ is non-decreasing in $y$ and that $P(Y > y|X = x, Z = z)$ is non-decreasing in $z$ for all $y$. Thus by Lemma 1, we conclude that $f(z|x=1)/f(z|x=0) \geq f(z'|x=1)/f(z'|x=0)$.

*Proof of Theorem 2.* By $X \perp\!\!\!\perp Z|Y$, we have

$$F(z|x) = \int_{-\infty}^{+\infty} F(z|y)F(dy|x) = E\left\{F(z|Y)|X=x\right\}.$$

From conditions (1) and (2) in Theorem 2, $F(z|y)$ is non-increasing in $y$, and $P(Y > y|X = x)$ is non-decreasing in $x$ for all $y$. By Lemma 1, $F(z|x)$ is non-increasing in $x$.

*Proof of Theorem 4.* For the exponential family, we have that $\partial^2 \ln f(x,y)/\partial x \partial y = (\partial\theta_x/\partial x)/a(\phi)$ and $\partial E(Y|x)/\partial x = \partial b'(\theta_x)/\partial x = b''(\theta_x)(\partial\theta_x/\partial x) = \mathrm{var}(Y|x) \times$

$(\partial\theta_x/\partial x)/a(\phi)$. Thus we obtain that $\partial^2 \ln f(x,y)/\partial x\partial y$ and $\partial E(Y|x)/\partial x$ have the same sign, which implies the conclusion.

*Proof of Corollary 2.* The implication relationships from the signs of association measures between $X$ and $Y$ to the signs of association measures between $X$ and $Z$ can be deduced from Theorems 1 to 4. Below we show three implication relationships from the signs of measures between $X$ and $Z$ to the signs of measures between $X$ and $Y$. From result (1) of Theorem 4, we need to show that $E(Z|x)$ increasing in $x$ implies $E(Y|x)$ increasing in $x$. By $X \perp\!\!\!\perp Z|Y$, we have from the proof of Theorem 3 that

$$E(Z|x) - E(Z|x') = -\int_{-\infty}^{+\infty} \frac{\partial E(Z|y)}{\partial y}\{F(y|x) - F(y|x')\}dy.$$

We use proof by contradiction; suppose that there exists $x > x'$ such that $E(Y|x) < E(Y|x')$. Then from the property of the exponential family in Theorem 4, we have $F(y|x) > F(y|x')$ for all $y$. Because $\partial E(Z|y)/\partial y$ is strictly positive for a non-zero measure set, we get that $E(Z|x) - E(Z|x') < 0$ from Theorem 3, which contradicts the condition of a non-negative association between $X$ and $Z$.

Results (2) and (3) of Corollary 2 can be obtained immediately from the above result (1) and Theorem 4.

*Proof of Theorem 5.* We need only to prove that $\partial \ln f(z|x)/\partial x$ is non-decreasing in $z$. When $X$ is continuous, for $z > z'$, we have

$$\begin{aligned}
&\frac{\partial \ln f(z|x)}{\partial x} = \frac{\partial f(z|x)}{\partial x}/f(z|x) = \left\{\frac{\partial}{\partial x}\int_{-\infty}^{+\infty} f(z|y,x)f(y|x)dy\right\}/f(z|x)\\
&= \int_{-\infty}^{+\infty}\left\{\frac{\partial f(z|y,x)}{\partial x}\cdot\frac{f(y|x)}{f(z|x)} + \frac{\partial f(y|x)}{\partial x}\cdot\frac{f(z|y,x)}{f(z|x)}\right\}dy\\
&= \int_{-\infty}^{+\infty}\left\{\frac{\partial f(z|y,x)}{\partial x}/f(z|y,x) + \frac{\partial f(y|x)}{\partial x}/f(y|x)\right\}\cdot\frac{f(y|x)f(z|y,x)}{f(z|x)}dy\\
&= \int_{-\infty}^{+\infty}\left\{\frac{\partial \ln f(z|y,x)}{\partial x} + \frac{\partial \ln f(y|x)}{\partial x}\right\}f(y|x,z)dy.\\
&= E\left\{\frac{\partial \ln f(z|Y,x)}{\partial x}|Z=z, X=x\right\} + E\left\{\frac{\partial \ln f(Y|x)}{\partial x}|Z=z, X=x\right\}.
\end{aligned}$$

From the assumption and condition (3) in Theorem 5, $\partial \ln f(z|Y,x)/\partial x$ is non-decreasing in $y$ and $z$; from condition (1) in Theorem 5, $\partial \ln f(Y|x)/\partial x$ is non-decreasing in $y$; and from condition (2) in Theorem 5, $P(Y > y|X = x, Z = z)$ is non-decreasing in $z$ for all $y$. By Lemma 1, we have that

$$E\left\{\frac{\partial \ln f(z|Y,x)}{\partial x}|Z = z, X = x\right\} + E\left\{\frac{\partial \ln f(Y|x)}{\partial x}|Z = z, X = x\right\}$$

is non-decreasing in $z$.

When $X$ is discrete, we need only to prove that, for $z > z'$,

$$\frac{f(z|x=1)}{f(z|x=0)} \geq \frac{f(z'|x=1)}{f(z'|x=0)}.$$

We have that

$$\begin{aligned}
\frac{f(z|x=1)}{f(z|x=0)} &= \int_{-\infty}^{+\infty} \frac{f(z|y,x=1)f(y|x=1)}{f(z|x=0)}dy \\
&= \int_{-\infty}^{+\infty} \frac{f(z|y,x=1)}{f(z|y,x=0)}\frac{f(y|x=1)}{f(y|x=0)}\frac{f(z|y,x=0)f(y|x=0)}{f(z|x=0)}dy \\
&= \int_{-\infty}^{+\infty} \frac{f(z|y,x=1)}{f(z|y,x=0)}\frac{f(y|x=1)}{f(y|x=0)}f(y|z,x=0)dy \\
&= E\left\{\frac{f(z|Y,x=1)}{f(z|Y,x=0)}\frac{f(Y|x=1)}{f(Y|x=0)}|X=0, Z=z\right\}.
\end{aligned}$$

From the assumption and condition (1) in Theorem 5, $\{f(z|y,x=1)f(y|x=1)\}/\{f(z|y,x=0)f(y|x=0)\}$ is non-decreasing in $y$ and $z$. From condition (2) in Theorem 5, $P(Y > y|X = 0, Z = z)$ is non-decreasing in $z$ for all $y$. Therefore, we have $f(z|x=1)/f(z|x=0) \geq f(z'|x=1)/f(z'|x=0)$.

*Proof of Theorem 6.* For $F(z|x) = E\{F(z|Y,x)|X = x\}$, we have from the assumption and condition (2) in Theorem 6 that $F(z|y,x)$ is non-increasing in $y$ and $x$. From condition (1) in Theorem 6, $P(Y > y|X = x)$ is non-decreasing in $x$ for all $y$. Thus we have that $\partial F(z|x)/\partial x \leq 0$.

*Proof of Theorem 7.* Since $E(Z|x) = E\{E(Z|Y,x)|X = x\}$, we prove that $\partial E(z|x)/\partial x \leq 0$ using a similar argument as in the proof of Theorem 5.

*Proof of Theorem 8.* From Theorem 7, we have $\partial E(Z|x)/\partial x \geq 0, \forall x$, and then from Theorem 4, we have $\partial^2 \ln f(x,z)/\partial x \partial z \geq 0$, $\forall x, z$.

*Proof of Corollary 3.* We prove this only for Theorem 5. Obviously, the assumption $\partial^2 \ln f(x, z|y)/\partial x \partial z \geq 0$ can be evaluated by $f(x, z|y)$. Condition (1) in Theorem 5 can be evaluated by $f(x|y)$, which can be obtained after marginalizing $f(x, z|y)$ over $z$. For conditions (2) and (3), we can rewrite them as $\partial^2 \ln f(z, x|y)/\partial y \partial z \geq 0$ and $\partial^2 \ln f(x, z|y)/\partial x \partial y \geq 0$ respectively. Therefore, the assumption and conditions can all be evaluated by $f(x, z|y)$.

*Proof of Corollary 4.* From the linear model, we have $E(Z|x) = \beta_0 + \beta_1 x + \beta_2 E(Y|x)$ and $\partial E(Z|x)/\partial x = \beta_1 + \beta_2 \partial E(Y|x)/\partial x = \beta_1 + \beta_2 \beta_4$. Thus, we have $\partial E(Z|x)/\partial x \geq 0$ if $\beta_1$, $\beta_2$ and $\beta_4$ are non-negative.

Suppose $a = -\beta_2/\beta_1$, and we need only to prove the result for the case that $\beta_2 < 0$ but $\partial E(Z|Y = y)/\partial y \geq 0$. We use proof by contradiction, and suppose that $\partial E(Z|x)/\partial x < 0$ for some $x$. Then we have $\partial E(Y|x)/\partial x = \beta_4 > -\beta_1/\beta_2 = 1/a$. Since $\partial E(Z|y)/\partial y = \beta_1 \partial E(X|y)/\partial x + \beta_2 \geq 0$, we have $\partial E(X|Y = y)/\partial y \geq -\beta_2/\beta_1 = a$. From the linear model of $Y$, we have $\partial E(Y - \beta_4 X|X = x)/\partial x = 0$. We deduce that

$$\mathrm{cov}(X, Y) = \beta_4 \mathrm{var}(X) > \mathrm{var}(X)/a. \tag{2}$$

Define $b = \inf_y\{\partial E(X|Y = y)/\partial y\}$. We have $b \geq a$ and $\partial E(X - bY|Y = y)/\partial y \geq b - b = 0$. From Property 1, we get $\mathrm{cov}(X - bY, Y) \geq 0$. Thus we obtain

$$\mathrm{cov}(X, Y) = \mathrm{cov}(X - bY, Y) + b\mathrm{var}(Y) \geq b\mathrm{var}(Y) \geq a\mathrm{var}(Y). \tag{3}$$

From equations (2) and (3), we have $\mathrm{cov}(X, Y) > \{\mathrm{var}(X)\mathrm{var}(Y)\}^{1/2}$, which is impossible since the correlation coefficient cannot be larger than 1.

*Proof of Corollary 5.* We first prove results (2) and (3). Since $F(z|x) = E\{F(z|Y, x)|X = x\} = E_Y\{F(z|Y, x)\}$, and $E(Z|x) = E\{E(Z|Y, x)|X = x\} = E_Y\{E(Z|Y, x)\}$, we only need $\partial F(z|y, x)/\partial x \leq 0, \forall x, y, z$ for Theorem 6 and $\partial E(Z|y, x)/\partial x \geq 0, \forall x, y$ for Theorem 7. For result (1), according to Theorem 4, when $X$ or $Z$ is binary, the density association is equivalent to the expectation association, thus we need only $\partial^2 f(x, z|y)/\partial x \partial z \geq 0, \forall x, y, z$.

## Appendix B: Three Examples

In Example 1, we illustrate that the expectation association of $Y$ on $X$ cannot replace condition (1) of Theorem 3.

**Example 1.** *We generate data under conditional independence:* $X \sim Bernoulli(1/2)$, $\varepsilon \sim Bernoulli(p), Y = X + 2\varepsilon(1 - X), Z = I(Y = 2)$, *where* $p < 1/2$ *and* $I(\cdot)$ *is the indicator function. We have that* $E(Y|X = 1) - E(Y|X = 0) = 1 - 2p \geq 0$, $E(Z|Y = 2) - E(Z|Y = 1) = 1 - 0 \geq 0$ *and* $E(Z|Y = 1) - E(Z|Y = 0) = 0$, *but we calculate that* $E(Z|x = 1) - E(Z|x = 0) = 0 - p \leq 0$.

In Example 1, we see that $Z$ does not follow a linear model given $Y$, and thus we cannot infer the transitivity of association signs based on Corollary 1.

In Example 2, we illustrate that under $X \perp\!\!\!\perp Z|Y$, a non-negative expectation association of $Y$ on $X$ and even the most stringent non-negative density association between $Y$ and $Z$ do not imply a non-negative expectation association of $Z$ on $X$.

**Example 2.** *Assume* $X \perp\!\!\!\perp Z|Y$ *with the distributions* $P(y|x)$ *and* $P(z|y)$ *given in Table 3. Then we have* $E(Y|X = 1) - E(Y|X = 0) = 0.2$ *and*

$$\ln \frac{P(Y = y, Z = 0)P(Y = y + 1, Z = 1)}{P(Y = y, Z = 1)P(Y = y + 1, Z = 0)} \geq 0,$$

*for* $y = 0$ *and 1, but* $E(Z|X = 1) - E(Z|X = 0) = -0.32$.

Table 3: Distributions $P(y|x)$ and $P(z|y)$ for Example 2

(a)

| | $Y = 0$ | $Y = 1$ | $Y = 2$ |
|---|---|---|---|
| $X = 0$ | 0.6 | 0 | 0.4 |
| $X = 1$ | 0 | 1 | 0 |

(b)

| | $Y = 0$ | $Y = 1$ | $Y = 2$ |
|---|---|---|---|
| $Z = 0$ | 0.9 | 0.9 | 0.1 |
| $Z = 1$ | 0.1 | 0.1 | 0.9 |

In Example 3, we illustrate that under $X \perp\!\!\!\perp Z|Y$, a non-negative correlation of $X$ and $Y$ ($Y$ and $Z$) and another non-negative association measure between $Y$ and $Z$ ($X$ and $Y$ ) do not imply a non-negative correlation between $X$ and $Z$.

**Example 3.** *Assume* $X \perp\!\!\!\perp Z|Y$ *with the distributions* $P(y|x)$ *and* $P(z|y)$ *given in Table 4. Then we have* $cov(Y, Z) = 0.0017 > 0$ *and*

$$\ln \frac{P(Y = y, X = 0)P(Y = y + 1, X = 1)}{P(Y = y, X = 1)P(Y = y + 1, X = 0)} \geq 0$$

*for* $y = 0$ *and 1, but* $E(Z|X = 1) - E(Z|X = 0) = -0.005$.

Table 4: Distributions $P(y|x)$ and $P(z|y)$ for Example 3

(a)

| | $Y=0$ | $Y=1$ | $Y=2$ |
|---|---|---|---|
| $X=0$ | 0.1 | 0.1 | 0.8 |
| $X=1$ | 0.05 | 0.05 | 0.9 |

(b)

| | $Y=0$ | $Y=1$ | $Y=2$ |
|---|---|---|---|
| $Z=0$ | 0.3 | 0 | 0.2 |
| $Z=1$ | 0.7 | 1 | 0.8 |

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