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A NOTE ON THE STATIONARITY AND THE EXISTENCE OF MOMENTS OF THE GARCH MODEL

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Abstract: In the present paper we examine the strict stationarity and the existence of higher-order moments for the GARCH(p,q) model under general and tractable assumptions.

Key words and phrases: GARCH model, higher-order moments, nonlinear time series, strict stationarity,

1. Introduction

Consider the following non-linear time series model

$$\begin{cases} x_t = \epsilon_t {h_t}^{\frac{1}{2}} \\ h_t = \alpha_0 + \alpha_1 x_{t-1}^2 + \cdots + \alpha_p x_{t-p}^2 + \phi_1 h_{t-1} + \cdots + \phi_q h_{t-q}, \end{cases} \tag{1.1}$$

where $\alpha_0 > 0, \alpha_i \geq 0, i = 1, \ldots, p,\ \phi_j \geq 0, j = 1, \ldots, q,\ \{\epsilon_t\}$ is a sequence of independent identically distributed(i.i.d.) random variables with zero mean and unit variance, and ϵ_t is independent of $x_{t-s}, s > 0$.

The model (1.1) is called the GARCH(p,q) model, which is proposed by Bollerslev (1986) and is one of many generalizations of the so-called ARCH (autoregressive conditional heteroskedasticity) model proposed by Engle (1982) in the literature. The GARCH models have been widely applied in modelling monetary and financial data such as inflation rate, interest rate and stock prices. The recent review by Bollerslev et al. (1992) contains an extensive literature on this subject.

The strict stationarity and the existence of moments for a time series model are fundamental for statistical inference. Therefore, it is significant to find necessary and sufficient conditions for the strict stationarity and the existence of moments for a time series model. Bougeral and Picard (1992) gave a necessary and sufficient condition for the strict stationarity of the GARCH model. However, as they pointed out, the conditions proposed are difficult to verify and can only be checked by Monte Carlo methods. Bollerslev (1986) discussed conditions for the existence of higher-order moments for GARCH(1,1) model. So far it appears that there is not any paper about the existence of higher-order moments for the GARCH(p, q) model in the literature. The purpose of this paper is to give some sufficient conditions for the strict stationarity and the existence of moments for the GARCH(p, q) model.

2. The Strict Stationarity of the GARCH Model

First, we introduce some notation. Let $\{x_t\}$ conform to model (1.1). Define $y_t = {x_t}^2, \eta_t = {\epsilon_t}^2$, $\mathbf{Y}_t = (y_t, \ldots, y_{t-p+1}, h_t, \ldots, h_{t-q+1})^\tau, B_t = (\alpha_0\eta_t, 0, \ldots, 0, \alpha_0, 0, \ldots, 0)^\tau$, where "$\tau$" denotes the transposition of a matrix, and

$$\mathbf{A}_t = \begin{pmatrix} \alpha_1\eta_t & \cdots & \alpha_{p-1}\eta_t & \alpha_p\eta_t & \phi_1\eta_t & \cdots & \phi_{q-1}\eta_t & \phi_q\eta_t \\ 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\ \alpha_1 & \cdots & \alpha_{p-1} & \alpha_p & \phi_1 & \cdots & \phi_{q-1} & \phi_q \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

The expectation of the random matrix $\mathbf{A}_t$ is defined element-wise, hence it is obvious that $E\mathbf{A}_t$ is a constant matrix, and

$$E\mathbf{A}_t = \begin{pmatrix} \alpha_1 & \cdots & \alpha_{p-1} & \alpha_p & \phi_1 & \cdots & \phi_{q-1} & \phi_q \\ 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\ \alpha_1 & \cdots & \alpha_{p-1} & \alpha_p & \phi_1 & \cdots & \phi_{q-1} & \phi_q \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \equiv \mathbf{A}.$$

Similarly, $EB_t = B \equiv (\alpha_0, 0, \ldots, 0, \alpha_0, 0, \ldots, 0)^\tau$. Then from (1.1) we have

$$\begin{aligned} {x_t}^2 &= {\epsilon_t}^2\Big(\alpha_0 + \sum_{i=1}^{p}\alpha_i {x_{t-i}}^2 + \sum_{j=1}^{q}\phi_j h_{t-j}\Big) \\ &= \sum_{i=1}^{p}\alpha_i {\epsilon_t}^2 {x_{t-i}}^2 + \sum_{j=1}^{q}\phi_j {\epsilon_t}^2 h_{t-j} + \alpha_0 {\epsilon_t}^2, \end{aligned} \tag{2.1}$$

which implies

$$y_t = \sum_{i=1}^{p}\alpha_i\eta_t y_{t-i} + \sum_{j=1}^{q}\phi_j\eta_t h_{t-j} + \alpha_0\eta_t. \tag{2.2}$$

Thus, $\{y_t\}$ is a solution of (2.2) if and only if $\{\mathbf{Y}_t\}$ is a solution of the following stochastic difference equation

$$\mathbf{Y}_t = \mathbf{A}_t \mathbf{Y}_{t-1} + B_t. \tag{2.3}$$

Lemma 2.1. *If $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \phi_j < 1$, then the series of random vectors*

$$\sum_{k=1}^{\infty} \Big(\prod_{j=0}^{k-1} \mathbf{A}_{t-j} \Big) B_{t-k}$$

converges almost surely. Furthermore if

$$\mathbf{Y}_t = B_t + \sum_{k=1}^{\infty} \Big(\prod_{j=0}^{k-1} \mathbf{A}_{t-j} \Big) B_{t-k}, \tag{2.4}$$

then $\{\mathbf{Y}_t\}$ is a strictly stationary, vector-valued process satisfying (2.3).

Proof. By the definition of $\mathbf{A}_t$ and B_t, it is easy to see that both $\{\mathbf{A}_t\}$ and $\{B_t\}$ are sequence of independent, non-negative random vectors and $\mathbf{A}_{t-j}$ is independent of B_{t-k} for $k \neq j$. Therefore, we have

$$E\Big(\prod_{j=0}^{k-1} \mathbf{A}_{t-j} \Big) B_{t-k} = \Big(\prod_{j=0}^{k-1} E\mathbf{A}_{t-j} \Big) E B_{t-k} = \mathbf{A}^k B.$$

It is easy to verified that the characteristic polynomial of A is given by $\det(\lambda A - I) = 1 - \sum_{i=1}^{m} (\alpha_i + \phi_i) \lambda^i$, where $m = \max\{p, q\}$, and $\alpha_i = 0$, for $i > p$, $\phi_i = 0$, for $i > q$. Let $\rho(A)$ be the spectral radius of the matrix $\mathbf{A}$, hence $\rho(\mathbf{A}) < 1$ if and only if $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \phi_j < 1$. Thus, if $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \phi_j < 1$, then (see Horn and Johnson (1985))

$$\sum_{k=1}^{\infty} \mathbf{A}^k < \infty,$$

which implies that

$$\sum_{k=1}^{\infty} E\Big(\prod_{j=0}^{k-1} \mathbf{A}_{t-j} \Big) B_{t-k} < \infty,$$

and hence

$$\sum_{k=1}^{\infty} \Big(\prod_{j=0}^{k-1} \mathbf{A}_{t-j} \Big) B_{t-k} < \infty, \qquad a.s.$$

It is obvious that the vector-valued stochastic process $\{\mathbf{Y}_t\}$ defined by (2.4) is strictly stationary. Furthermore, we have

$$\mathbf{Y}_t = B_t + \mathbf{A}_t \Big[B_{t-1} + \sum_{k=2}^{\infty} (\prod_{j=0}^{k-1} \mathbf{A}_{t-j}) B_{t-k} \Big]$$

$$= B_t + \mathbf{A}_t\Big[B_{t-1} + \sum_{l=1}^{\infty}(\prod_{j=0}^{l-1}\mathbf{A}_{t-1-l})B_{t-1-l}\Big]$$
$$= B_t + \mathbf{A}_t\mathbf{Y}_{t-1}.$$

Lemma 2.2. *If* (2.3) *admits a strictly stationary solution with finite first moment, then* $\sum_{i=1}^{p}\alpha_i + \sum_{j=1}^{q}\phi_j < 1$. *Moreover, the strictly stationary solution of* (2.3) *is unique.*

Proof. By (2.3), we have

$$\begin{aligned}\mathbf{Y}_0 &= \mathbf{A}_0\mathbf{Y}_{-1} + B_0\\ &= B_0 + \mathbf{A}_0 B_{-1} + \mathbf{A}_0\mathbf{A}_{-1}\mathbf{Y}_{-2}\\ &\ \ \vdots\\ &= B_0 + \sum_{k=1}^{n-1}(\prod_{j=0}^{k-1}\mathbf{A}_{-j})B_{-k} + (\prod_{j=0}^{n-1}\mathbf{A}_{-j})\mathbf{Y}_{-n}.\end{aligned} \tag{2.5}$$

Noting that all $\mathbf{A}_n, B_n$ and $\mathbf{Y}_n$ are non-negative, $\{\mathbf{A}_t\}$ is a sequence of independent random matrices, $\mathbf{A}_{n-j}$ and B_{n-k} are independent for $k \neq j$, and $E\mathbf{Y}_0 < \infty$. By taking expectation of each side of (2.5), it follows that

$$E\mathbf{Y}_0 \geq \sum_{k=1}^{n-1} E\Big(\prod_{j=0}^{k-1}\mathbf{A}_{-j}\Big)B_{-k} = \sum_{k=1}^{n-1}\mathbf{A}^k B.$$

This shows that

$$\sum_{k=1}^{n}\mathbf{A}^k B < \infty.$$

Therefore,

$$\lim_{n\to\infty}\mathbf{A}^n B = 0. \tag{2.6}$$

Let $\{\delta_i, i = 1, \ldots, p+q\}$ be the canonical basis of R^{p+q}, i.e. $\delta_i = (\delta_{i,1}, \ldots, \delta_{i,p+q})^\tau$, where $\delta_{ij} = 0$, for $i \neq j, \delta_{ii} = 1$. If we can prove that for $1 \leq i \leq p+q$,

$$\lim_{n\to\infty}\mathbf{A}^n\delta_i = 0, \tag{2.7}$$

then (2.7) implies that $\lim_{n\to\infty}\mathbf{A}^n = 0$, which again implies that $\rho(\mathbf{A}) < 1$. As we showed before the later is equivalent to $\sum_{i=1}^{p}\alpha_i + \sum_{j=1}^{q}\phi_j < 1$, which leads to the first part of this lemma.

In fact, since $B = \alpha_0(\delta_1 + \delta_{p+1})$ and $0 < \alpha_0 < \infty$, by (2.6) and the definition of matrix $\mathbf{A}$, (2.7) holds for $i = 1$ and $i = p+1$. Again by the definition of $\mathbf{A}$ and B, $\mathbf{A}\delta_{p+q} = \phi_q(\delta_1 + \delta_{p+1})$. If $\phi_q = 0$, then $\mathbf{A}\delta_{p+q} = 0$, hence (2.7) holds. If $\phi_q > 0$, by the above equalities,

$$\lim_{n\to\infty}\mathbf{A}^n\delta_{p+q} = \lim_{n\to\infty}\mathbf{A}^{n-1}\phi_q(\delta_1 + \delta_{p+1}) = 0.$$

It is easy to see that for $2 \leq i < p$,

$$\mathbf{A}\delta_i = \alpha_i\delta_1 + \delta_{i+1} + \alpha_i\delta_{p+1}.$$

Since (2.7) holds for $i = 1$ and $i = p + 1$, by a backward recursion (2.7) holds for $i = p, p-1, \ldots, 1$, respectively. Similarly, for $p + 1 < i < p + q$,

$$\mathbf{A}\delta_i = \phi_i(\delta_1 + \delta_{p+1}) + \delta_{i+1}.$$

Noting that (2.7) holds for $i = p + q$, by a backward recursion, (2.7) holds for $i = p+q-1, p+q-2, \ldots, p+2$. Finally, (2.7) holds for any $i = 1, 2, \ldots, p+q$.

For the proof of the uniqueness, let $\{\mathbf{U}_t\}$ be another strictly stationary solution satisfying (2.3). Then $\{\mathbf{U}_t\}$ also satisfies an equation similar to (2.5). Note that the third term on the right hand side of the last line in (2.5) goes to zero in probability. Then the uniqueness follows immediately.

Theorem 2.1. *The GARCH(p, q) model* (1.1) *admits a strictly stationary solution with finite variance if and only if* $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \phi_j < 1$. *Moreover, this strictly stationary solution is also unique.*

Proof. The desired resultis obtained by combining Lemma 2.1 and Lemma 2.2.

3. The Existence of Higher-Order Moments

Let

$$L^s = \{x : \|x\|_s = E^{\frac{1}{s}}|x|^s < \infty,\ s > 0\},$$

where x is a random variable.We need the Kronecker product ($\otimes$), the direct operations ("vec" operations), the notation $\mathbf{A}^{\otimes m} = \mathbf{A} \otimes \mathbf{A} \otimes \cdots \otimes \mathbf{A}$, and the basic identity $\text{vec}(ABC) = (C^\tau \otimes A)\text{vec}(B)$. We denote the (i, j)th element of a matrix D by $(D)_{ij}$ and define $\Sigma_m = E(\mathbf{A}_t{}^{\otimes m})$.

Theorem 3.1. *Let* $\{x_t\}$ *be specified to be the strictly stationary solution of model* (1.1) *and* $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \phi_j < 1$.

(i) *If* $E|\epsilon_t|^4 < \infty$ *and* $\rho(\Sigma_2) < 1$, *then* $|x_t|^2 \in L^2$.

(ii) *If* $E|\epsilon_t|^{4(s-1)} < \infty$, *for some integer* $s > 2$, *and* $\rho(\Sigma_s) < 1$, *then* $|x_t|^2 \in L^s$.

Proof. Let $\tilde{Y}_t$ be generated according to (2.3) with the starting value $\tilde{Y}_0 = 0$. Let Y be a random variable having the same distribution as that of (2.4) with $t = 0$, which is the (marginal) distribution of the unique stationary solution of (2.3). It is clear that $\tilde{Y}_t \to Y$ in distribution. Let $\psi(Y)$ be a random variable. From weak convergence theory (Billingsley (1968)), it is known that to show $E\psi(Y) < \infty$, it suffices to show $\liminf E\psi(\tilde{Y}_t) < \infty$. Let $V(t) = E(\tilde{Y}_t)$. Then, taking expectation on both sides of (2.3), we get

$$V(t) = AV(t-1) + B.$$

It is well know that $\lim V(t)$ exists and is finite if the spectral radius of A is less than 1(cf. Subba Rao (1981)). Hence if the spectral radius of A is less than 1, $\{V(t)\}$ is bounded. Next, let $V_1(t) = E(\mathrm{vec}(\tilde{Y}_t\tilde{Y}_t^\tau))$. Then

$$\begin{aligned} V_1(t) = E(A_t \otimes A_t)V_1(t-1) + E(B_t \otimes A_t)V_1(t-1) \\ +E(A_t \otimes B_t)V_1(t-1) + \mathrm{vec}(E(B_tB_t^\tau)). \end{aligned}$$

Note that the matices $E(A_t \otimes A_t), E(B_t \otimes A_t), E(A_t \otimes B_t)$ and $\mathrm{vec}(E(B_tB_t^\tau))$ are constant and finite matrices. As $\{V(t)\}$ is bounded, it is clear that $\lim V_1(t)$ exists and is finite if the spectral radius of $\Sigma_2 = E(A_t \otimes A_t)$ is less than 1. Note that the first element of $V_1(t)$ is $E(\tilde{x}_t^4)$. Because $\tilde{x}_t^4$ converges in distribution to x_0^4 and $\lim V_1(t)$ is finite, $E(x_0^4) < \infty$. This completes the proof of (i). The proof of (ii) is similar and hence omitted.

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