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A NOTE ON HYPOTHESIS TESTING IN STOCHASTIC REGRESSION MODELS

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Abstract. Consider the multiple regression model $y_n = \sum_i \beta_i x_{ni} + \epsilon_n$, $n = 1, 2, \ldots$, where the ϵ_n are unobservable random errors; $\beta_1, \ldots, \beta_p$ are unknown parameters and y_n is the observed response corresponding to the design vector $\underset{\sim}{x}_n = (x_{n1}, \ldots, x_{np})'$. Lai & Wei (1982) established results concerning the strong consistency and asymptotic normality of the least squares estimate of $\underset{\sim}{\beta} = (\beta_1, \ldots, \beta_p)'$ where $\{\epsilon_n\}$ is a martingale difference sequence and some regularity conditions are satisfied. We obtain the same asymptotic normality result under weaker conditions, and also establish the test of linear hypothesis and the strong consistency of the constrained least squares estimate of $\underset{\sim}{\beta}$ under $H'\underset{\sim}{\beta} = \underset{\sim}{h}$.

Keywordsandphrases. Asymptotic normality, constrained least squares, linear hypothesis, martingales, stochastic regressors, strong consistency.

1. Introduction

Consider the multiple regression model

$$y_n = \beta_1 x_{n1} + \beta_2 x_{n2} + \cdots + \beta_p x_{np} + \epsilon_n, \quad n = 1, 2, \ldots, \tag{1.1}$$

where the ϵ_n are unobservable random errors; $\beta_1, \ldots, \beta_p$ are unknown parameters and y_n is the observed response corresponding to the design vector $\underset{\sim}{x}_n = (x_{n1}, \ldots, x_{np})'$. Let $X_n = (x_{ij})$, $1 \le i \le n$, $1 \le j \le p$, $Y_n = (y_1, \ldots, y_n)'$ and $\underset{\sim}{\epsilon}_n = (\epsilon_1, \ldots, \epsilon_n)'$. Then $X_n' = (\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n)$. The regression model (1.1) can be written as $Y_n = X_n \underset{\sim}{\beta} + \underset{\sim}{\epsilon}_n$; and $\underset{\sim}{b}_n = (b_{n1}, \ldots, b_{np})' = (X_n' X_n)^{-1} X_n' Y_n$ is the least squares estimate of $\underset{\sim}{\beta} = (\beta_1, \ldots, \beta_p)'$ based on the observations $\underset{\sim}{x}_1, y_1, \underset{\sim}{x}_2, y_2, \ldots, \underset{\sim}{x}_n, y_n$ assuming that $X_n' X_n$ is nonsingular. We shall assume that $\{\epsilon_n\}$ is a martingale difference sequence with respect to an increasing sequence of σ-fields $\{\mathcal{F}_n\}$, i.e. ϵ_n is $\mathcal{F}_n$-measurable and $E(\epsilon_n \mid \mathcal{F}_{n-1}) = 0$ for every n. We shall also assume that $\underset{\sim}{x}_n$ is $\mathcal{F}_{n-1}$-measurable. Therefore, the design vector $\underset{\sim}{x}_n$ at stage n may depend on the previous observations $\underset{\sim}{x}_1, y_1, \ldots, \underset{\sim}{x}_{n-1}, y_{n-1}$.

Now rewrite the least squares estimate $\underset{\sim}{b}_n$ of $\underset{\sim}{\beta}$ as follows:

$$\underset{\sim}{b}_n = \underset{\sim}{\beta} + \left(\sum_1^n \underset{\sim}{x}_i \underset{\sim}{x}_i' \right)^{-1} \sum_1^n \underset{\sim}{x}_i \epsilon_i.$$

The statistical properties of the least squares estimate $\underset{\sim}{b}_n$ are related to the random matrix $X_n'X_n$ and the martingale transform $\sum_1^n \underset{\sim}{x}_i\epsilon_i$. Lai & Wei (1982) established results concerning the strong consistency and asymptotic normality of $\underset{\sim}{b}_n$. Previously, Anderson and Taylor (1979) and Christopei and Helmes (1980) also established the strong consistency of $\underset{\sim}{b}_n$ under stronger conditions. Wei (1985), among other results, also discussed asymptotic properties of $\underset{\sim}{b}_n$ under a reparametrization of model (1.1).

In this note we study the hypothesis testing problem for the stochastic regression model (1.1). More precisely, we consider the problem of testing the linear hypothesis $H'\underset{\sim}{\beta} = \underset{\sim}{h}$ against $H'\underset{\sim}{\beta} \neq \underset{\sim}{h}$, where H' is a $k \times p$ fixed matrix ($k < p$) with rank k and $\underset{\sim}{h}$ is a $k \times 1$ known vector. The strong consistency problem of the constained least squares estimate under $H'\underset{\sim}{\beta} = \underset{\sim}{h}$ will also be studied.

In Section 2, we review some important results in the stochastic regression model and give an example to motivate the need for weak convergence under weaker conditions than those of Lai & Wei (1982). In Section 3, the proof of the new result will be given. Then, in Section 4, we apply the new result to test the hypothesis $H_0 : H'\underset{\sim}{\beta} = \underset{\sim}{h}$ against $H_1 : H'\underset{\sim}{\beta} \neq \underset{\sim}{h}$. Finally, a summary is presented in Section 5.

2. Review

Assume that the errors $\{\epsilon_n\}$ in the regression model (1.1) form a martingale difference sequence w.r.t. the σ-fields $\{\mathcal{F}_n\}$ such that

$$\sup_n E\{|\epsilon_n|^\alpha \mid \mathcal{F}_{n-1}\} < \infty \quad \text{a.s. for some } \alpha > 2 \tag{2.1}$$

and

$$\lim_{n\to\infty} E(\epsilon_n^2 \mid \mathcal{F}_{n-1}) = \sigma^2 \quad \text{a.s. for some constant } \sigma. \tag{2.2}$$

An important special case is where the $\{\epsilon_n\}$ are independent random variables with zero means, variance σ^2, and $\sup_n E|\epsilon_n|^\alpha < \infty$ for some $\alpha > 2$. The following theorems give conditions on the stochastic regressors $\underset{\sim}{x}_n$ that ensure the strong consistency and the asymptotic normality of the least squares estimate $\underset{\sim}{b}_n$ established by Lai and Wei (1982).

Theorem A. *Assume the regression model* (1.1). *Suppose that* (2.1) *holds and* $\lambda_{\min}(n) \to \infty$ *a.s.,* $\log \lambda_{\max}(n) = o(\lambda_{\min}(n))$ *a.s., where* $\lambda_{\min}(n)$ *and* $\lambda_{\max}(n)$ *denote the minimum and maximum eigenvalues of* $X_n'X_n$ *respectively. Then* $\underset{\sim}{b}_n \to \underset{\sim}{\beta}$ *a.s.*

Theorem B. *Suppose that, in the regression model* (1.1), $\{\epsilon_n\}$ *is a martingale difference sequence w.r.t. an increasing sequence of* σ*-fields* $\{\mathcal{F}_n\}$ *such that* (2.1) *and* (2.2) *hold. Moreover assume, for each* n*, that the design vector*

$\underset{\sim}{x}_n = (x_{n1}, \ldots, x_{np})'$ *at stage* n *is* $\mathcal{F}_{n-1}$*-measurable and that there exists a non-random positive definite symmetric matrix* B_n *such that* $B_n^{-1}\left(\sum_1^n \underset{\sim}{x}_i \underset{\sim}{x}_i'\right)^{1/2} \xrightarrow{p} I_p$ *and* $\max_{1\le i\le n} \|B_n^{-1}\underset{\sim}{x}_i\| \xrightarrow{p} 0$. *Then* $\left(\sum_1^n \underset{\sim}{x}_i \underset{\sim}{x}_i'\right)^{1/2}(\underset{\sim}{b}_n - \underset{\sim}{\beta}) \xrightarrow{\mathcal{D}} N(\underset{\sim}{0}, \sigma^2 I_p)$, *where* $\xrightarrow{p}$ *and* $\xrightarrow{\mathcal{D}}$ *denote, respectively, the convergence in probability and in distribution.*

However, the assumption of positive definiteness on B_n seems unnecessarily strict and the computation of $B_n^{-1}(X_n'X_n)^{1/2}$ at stage n in Theorem B is extremely cumbersome. This makes Theorem B difficult to apply. In Section 3, the results of Theorem B are obtained under much weaker conditions on B_n. Before stating the new result on the weak convergence of $\underset{\sim}{b}_n$, we demonstrate an example to motivate the need for weakening the conditions about B_n as follows.

Example. Consider the time series model

$$y_t = ry_{t-1} + \alpha(r-1)t - \alpha + \epsilon_t, \quad t = 1, 2, \ldots, \tag{2.3}$$

where $\{\epsilon_t\}$ are i.i.d. random variables with mean zero and variance one. Let $z_t = y_t + \alpha(t+1)$. Then we have

$$z_t = rz_{t-1} + \epsilon_t \tag{2.4}$$

and

$$\begin{bmatrix} z_{t-1} \\ t \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ t \\ 1 \end{bmatrix} = A\underset{\sim}{x}_t, \quad A = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $\underset{\sim}{x}_t$ is the vector notation in model (1.1) and in Theorem B. Now from (2.3) we know that $\underset{\sim}{\beta} = (r, \alpha(r-1), -\alpha)'$ and

$$X_n'X_n = \begin{bmatrix} \sum_1^n y_{i-1}^2 & \sum_1^n i\, y_{i-1} & \sum_1^n y_{i-1} \\ \sum_1^n i\, y_{i-1} & \sum_1^n i^2 & \sum_1^n i \\ \sum_1^n y_{i-1} & \sum_1^n i & n \end{bmatrix}.$$

Assume that $0 < r < 1$, then by (2.4) and (A2.7) in Wei (1987), we have $\sum_1^n z_i^2 \sim n/(1-r^2)$ where "$\sim$" denotes order equivalence. Note that

$$\begin{aligned}\sum_1^n z_i &= z_0 r(1-r^n)/(1-r) + \sum_{t=1}^n \sum_{i=1}^n r^{t-i}\epsilon_i \\ &= z_0 r(1-r^n)/(1-r) + \sum_1^n \epsilon_i(1-r^{n-i+1})/(1-r).\end{aligned}$$

Let $u_i = (1-r^{n-i+1})$, and by using $\sum_1^n u_i^2 \sim n$ and (2.8) in Lai & Wei (1982) we obtain $\sum_1^n \epsilon_i u_i/n \xrightarrow{p} 0$ as $n \to \infty$; thus $\sum_1^n z_i = o_p(n)$. Further, since by the same argument, we have $\sum_1^n iz_i = r\sum_1^{n-1} iz_i + r\sum_1^{n-1} z_i + \sum_1^{n-1} i\epsilon_i = r\sum_1^{n-1} iz_i + o(n^2)$, this implies $\sum_1^n iz_i \xrightarrow{p} 0$ as $n \to \infty$.

Taking

$$C_n = \begin{bmatrix} [(1-r^2)/n]^{1/2} & 0 & 0 \\ 0 & (3/n^3)^{1/2} & 0 \\ 0 & 0 & n^{-1/2} \end{bmatrix},$$

and by $B_n^{-1} = C_n A$ we have

$$B_n^{-1}\sum_1^n \begin{bmatrix} z_{t-1} \\ t \\ 1 \end{bmatrix} [z_{t-1}, t, 1](B_n') \xrightarrow{p} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3^{1/2}2^{-1} \\ 0 & 3^{1/2}2^{-1} & 1 \end{bmatrix},$$

and $\max_{1\le t\le n} \underset{\sim}{x}_t'(B_n')^{-1}B_n^{-1}\underset{\sim}{x}_t = \max_{1\le t\le n}\{(1-r^2)z_{t-1}^2/n + 3t^2/n^3 + 1/n\} \xrightarrow{p} 0$ as $n\to\infty$, which satisfies the same condition of Theorem B but $(\sum_1^n \underset{\sim}{x}_i\underset{\sim}{x}_i')^{1/2}$ does not need to be computed here. Therefore the matrix B_n does not satisfy the positive definite condition of Theorem B and it does avoid the difficulty of evaluating $(\sum_1^n \underset{\sim}{x}_i\underset{\sim}{x}_i')^{1/2}$; this is why we need to weaken the conditions of Theorem B in the following section.

3. New Result

Before stating the new result, we first state a corollary of the result given in Hall & Heyde (1980, pp.58).

Corollary 1. *Let $\{S_{ni};\ \mathcal{F}_{ni},\ 1\le i\le k_n,\ n\ge 1\}$ be a zero-mean, square-integrable martingale array with difference X_{ni}, and let η^2 be an a.s. finite random variable. If for all $\epsilon>0$, $\sum_{i=1}^{k_n} E\left[X_{ni}^2 I_{\{|X_{ni}|>\epsilon\}} \mid \mathcal{F}_{n,i-1}\right] \xrightarrow{p} 0$, and $V_{n,k_n}^2 = \sum_{i=1}^{k_n} E\left[X_{ni}^2 \mid \mathcal{F}_{n,i-1}\right] \xrightarrow{p} \eta^2$, and if $\mathcal{F}_{n,i}\subseteq\mathcal{F}_{n+1,i}$, $1\le i\le k_n$, $n\ge 1$ then $S_{n,k_n} = \sum_{i=1}^{k_n} X_{ni} \Rightarrow Z$ (stably), where Z has characteristic function $E\left[\exp\{-\frac{1}{2}\eta^2t^2\}\right]$.*

Next, we discuss the following convergence result. Since it was only briefly mentined in Wei (1983), we give a proof.

Theorem 1. *Under the regression model (1.1), assume that $\{\epsilon_n\}$ is a martingale difference sequence w.r.t. an increasing sequence of σ-fields $\{\mathcal{F}_n\}$ and that $\{\epsilon_n\}$ satisfies (2.1) and (2.2). Suppose that for each n there exists a nonsingular matrix B_n such that $B_n^{-1}X_n'X_n(B_n')^{-1} \xrightarrow{p} \Gamma$, where Γ is positive definite and $\max_{1\le i\le n}\underset{\sim}{x}_i'(B_nB_n')^{-1}\underset{\sim}{x}_i \xrightarrow{p} 0$. Then*

(i) $(B_n^{-1}X_n'X_n(B_n')^{-1},\ B_n^{-1}X_n'X_n(b_n-\underset{\sim}{\beta})) \xrightarrow{\mathcal{D}} (\Gamma, \Gamma^{1/2}\underset{\sim}{N})$,

(ii) $(\underset{\sim}{b}_n-\underset{\sim}{\beta})'X_n'X_n(\underset{\sim}{b}_n-\underset{\sim}{\beta}) \xrightarrow{\mathcal{D}} \sigma^2\chi_p^2$,

where $\underset{\sim}{N}\sim N(\underset{\sim}{0}, \sigma^2 I_p)$ and χ_p^2 denotes the chi-squared distribution with p degrees of freedom.

Proof. Let $\mathcal{F}_{n,k} = \mathcal{F}_k$, $X_{nk} = \underset{\sim}{c}' B_n^{-1} \underset{\sim}{x}_k \epsilon_k$, where $\underset{\sim}{c}$ is a fixed constant vector. Since $\max_{1\leq i\leq n} \underset{\sim}{x}_i'(B_nB_n')^{-1}\underset{\sim}{x}_i \xrightarrow{p} 0$, we can assume without loss of generality that $B_n^{-1}\underset{\sim}{x}_k$ is bounded. This in turn implies that

$$E[X_{nk} \mid \mathcal{F}_{k-1}] = E[\underset{\sim}{c}' B_n^{-1} \underset{\sim}{x}_k \epsilon_k \mid \mathcal{F}_{k-1}] = 0 \quad \text{and} \sup_{1\leq k\leq n} E[|X_{nk}|^\alpha \mid \mathcal{F}_{k-1}] < \infty \text{ a.s.}$$

for some $\alpha > 2$. Note that

$$(1) \qquad \sum_{i=1}^n E[X_{ni}^2 \mid \mathcal{F}_{n,i-1}] = \sum_{i=1}^n E[(\underset{\sim}{c}' B_n^{-1} \underset{\sim}{x}_i \epsilon_i)^2 \mid \mathcal{F}_{i-1}]$$
$$= \underset{\sim}{c}' B_n^{-1} \Big\{ \sum_1^n \underset{\sim}{x}_i \underset{\sim}{x}_i' E[\epsilon_i^2 \mid \mathcal{F}_{i-1}] \Big\} (B_n^{-1})' \underset{\sim}{c} \xrightarrow{p} \underset{\sim}{c}' \sigma^2 \Gamma \underset{\sim}{c}.$$

(2) $\forall\, \delta > 0$, since $\sum_{i=1}^n E[X_{ni}^2 \mid \mathcal{F}_{n,i-1}] \xrightarrow{p} \|\underset{\sim}{c}'\Gamma^{1/2}\|^2\sigma^2$, we get

$$\sum_{i=1}^n E[X_{ni}^2 I_{\{|X_{ni}|>\delta\}} \mid \mathcal{F}_{n,i-1}] = \sum_{i=1}^n E[X_{ni}^2 I_{\{|X_{ni}|>\delta\}} \mid \mathcal{F}_{i-1}]$$
$$\leq \sum_{i=1}^n E[|X_{ni}|^\alpha \mid \mathcal{F}_{i-1}]/\delta^{\alpha-2} = \sum_{i=1}^n |\underset{\sim}{c}' B_n^{-1} \underset{\sim}{x}_i|^\alpha E[|\epsilon_i|^\alpha \mid \mathcal{F}_{i-1}]/\delta^{\alpha-2} \xrightarrow{p} 0.$$

This is because $\sup_{1\leq k\leq n} E[|\epsilon_k|^\alpha \mid \mathcal{F}_{k-1}]/\delta^{\alpha-2} = O_p(1)$, $\sum_{i=1}^n |\underset{\sim}{c}' B_n^{-1} \underset{\sim}{x}_i|^2 = O_p(1)$ and

$$\sup_{1\leq k\leq n} |\underset{\sim}{c}' B_n^{-1} \underset{\sim}{x}_k|^{\alpha-2} \leq \|\underset{\sim}{c}\|^{\alpha-2} \sup_{1\leq k\leq n} \|B_n^{-1} \underset{\sim}{x}_k\|^{\alpha-2}$$
$$= \|\underset{\sim}{c}\|^{\alpha-2} \Big[\sup_{1\leq k\leq n} \underset{\sim}{x}_k' (B_n')^{-1} B_n^{-1} \underset{\sim}{x}_k \Big]^{(\alpha-2)/2} \xrightarrow{p} 0.$$

Thus, (1) and (2) verify the conditions of Corollary 1.

By Corollary 1, we have, for any given constant vector $\underset{\sim}{c}$, that

$$\underset{\sim}{c}' B_n^{-1} \sum_i^n \underset{\sim}{x}_i \epsilon_i = \underset{\sim}{c}' B_n^{-1} \sum_1^n \underset{\sim}{x}_i \underset{\sim}{x}_i' (\underset{\sim}{b}_n - \underset{\sim}{\beta}) = \sum_{i=1}^n X_{ni} \Rightarrow Z \text{ (stably)},$$

where $Z \sim N(0, \|\underset{\sim}{c}'\Gamma^{1/2}\|^2\sigma^2)$. That is, $Z = \|\underset{\sim}{c}'\Gamma^{1/2}\| W = \underset{\sim}{c}'\Gamma^{1/2}\underset{\sim}{N}$, where $W \sim N(0,\sigma^2)$ and $\underset{\sim}{N} \sim N(\underset{\sim}{0}, \sigma^2 I_p)$. By the Cramer-Wold Theorem we conclude that $(\Gamma, B_n^{-1}X_n'X_n(\underset{\sim}{b}_n - \underset{\sim}{\beta})) \xrightarrow{\mathcal{D}} (\Gamma, \Gamma^{1/2}\underset{\sim}{N})$. Since $B_n^{-1}X_n'X_n(B_n')^{-1} \xrightarrow{p} \Gamma$, $(B_n^{-1}X_n'X_n(B_n')^{-1},\, B_n^{-1}X_n'X_n(\underset{\sim}{b}_n - \underset{\sim}{\beta})) \xrightarrow{\mathcal{D}} (\Gamma,\, \Gamma^{1/2}\underset{\sim}{N})$ and

$$(\underset{\sim}{b}_n - \underset{\sim}{\beta})' X_n' X_n (\underset{\sim}{b}_n - \underset{\sim}{\beta})$$
$$= \{B_n^{-1} X_n' X_n (\underset{\sim}{b}_n - \underset{\sim}{\beta})\}' \{B_n' (X_n' (X_n' X_n)^{-1} B_n\} \{B_n^{-1} (X_n' X_n) (\underset{\sim}{b}_n - \underset{\sim}{\beta})\}$$
$$\xrightarrow{\mathcal{D}} \underset{\sim}{N}' \Gamma^{1/2} \Gamma^{-1} \Gamma^{1/2} \underset{\sim}{N} = \underset{\sim}{N}' \underset{\sim}{N} = \sigma^2 \chi_p^2.$$

4. Hypothesis Testing

In this section we test the hypothesis $H_0 : H'\underset{\sim}{\beta} = \underset{\sim}{h}$ against $H_1 : H'\underset{\sim}{\beta} \neq \underset{\sim}{h}$, where H' is a $k \times p$ fixed matrix ($k < p$) with rank k and $\underset{\sim}{h}$ is a $k \times 1$ known vector.

Let $\underset{\sim}{b}_n^*$ be the constrained least squares estimate (CLS) of $\underset{\sim}{\beta}$ under the restriction $H'\underset{\sim}{\beta} = \underset{\sim}{h}$. Using the Lagrange multiplier $\underset{\sim}{\lambda}$, we have

$$\underset{\sim}{b}_n^* = (X_n'X_n)^{-1}X_n'Y_n - (X_n'X_n)^{-1}H\underset{\sim}{\lambda} = \underset{\sim}{b}_n - (X_n'X_n)^{-1}H\underset{\sim}{\lambda} \quad \text{and} \quad H'\underset{\sim}{b}_n^* = \underset{\sim}{h}.$$

Hence

$$\underset{\sim}{b}_n^* = \underset{\sim}{b}_n - (X_n'X_n)^{-1}H[H'(X_n'X_n)^{-1}H]^{-1}(H'\underset{\sim}{b}_n - \underset{\sim}{h}).$$

The following result gives the strong consistency of the CLS $\underset{\sim}{b}_n^*$.

Theorem 2. *Suppose that in the regression model* (1.1), $\{\epsilon_n\}$ *is a martingale difference w.r.t. an increasing sequence of* σ*-fields* $\{\mathcal{F}_n\}$ *such that* (2.1), (2.2) *hold and* $\lambda_{\min}(n) \to \infty$ *a.s.,* $\log \lambda_{\max}(n) = o(\lambda_{\min}(n))$ *a.s.. Then* $\underset{\sim}{b}_n^* \to \underset{\sim}{\beta}$ *a.s..*

Proof. Let $Z_n = X_n'X_n$, $Q_n = H'Z_n^{-1}H$. Then

$$\underset{\sim}{b}_n^* - \underset{\sim}{b}_n = -Z_n^{-1}HQ_n^{-1}H'(\underset{\sim}{b}_n - \underset{\sim}{\beta}) = -Z_n^{-1/2}Z_n^{-1/2}HQ_n^{-1}H'\ Z_n^{-1}X_n'\underset{\sim}{\epsilon}_n.$$

Since $Z_n^{-1} - Z_n^{-1}H\ Q_n^{-1}H'Z_n^{-1} = (I - Z_n^{-1}H\ Q_n^{-1}H')Z_n^{-1}(I - HQ_n^{-1}H'\ Z_n^{-1})$ is positive semidefinite,

$$\begin{aligned}\|\underset{\sim}{b}_n^* - \underset{\sim}{b}_n\|^2 &\leq \|Z_n^{-1/2}\|^2 \cdot \|Z_n^{-1/2}HQ_n^{-1}H'Z_n^{-1}X_n'\underset{\sim}{\epsilon}_n\|^2 \\ &\leq \lambda_{\min}^{-1}(n) \cdot \underset{\sim}{\epsilon}_n'X_n(Z_n^{-1}HQ_n^{-1}H'Z_n^{-1})X_n'\underset{\sim}{\epsilon}_n \\ &\leq \lambda_{\min}^{-1}(n) \cdot \underset{\sim}{\epsilon}_n'X_nZ_n^{-1}X_n'\underset{\sim}{\epsilon}_n = \lambda_{\min}^{-1}(n) \cdot O(\log \lambda_{\max}(n)).\end{aligned}$$

Therefore $\underset{\sim}{b}_n^* \to \underset{\sim}{\beta}$ a.s. under $H'\underset{\sim}{\beta} = \underset{\sim}{h}$ by Theorem A.

Now we define $R_0^2 = \min_{\underset{\sim}{\beta}}(Y_n - X_n\underset{\sim}{\beta})'(Y_n - X_n\underset{\sim}{\beta})$ and $R_1^2 = \min_{H'\underset{\sim}{\beta}=\underset{\sim}{h}}(Y_n - X_n\underset{\sim}{\beta})'\ (Y_n - X_n\underset{\sim}{\beta})$. Then $R_0^2 = (Y_n - X_n\underset{\sim}{b}_n)'(Y_n - X_n\underset{\sim}{b}_n) = \tilde{\underset{\sim}{\epsilon}}_n'\tilde{\underset{\sim}{\epsilon}}_n$ and $R_1^2 = (Y_n - X_n\underset{\sim}{b}_n^*)'(Y_n - X_n\underset{\sim}{b}_n^*) = \tilde{\underset{\sim}{\epsilon}}_n^{*'}\tilde{\underset{\sim}{\epsilon}}_n^*$. Here $\tilde{\underset{\sim}{\epsilon}}_n = (I - P_n)\underset{\sim}{\epsilon}_n$, $P_n = X_nX_n^+$ is the projection operator on $R(X_n)$, $X_n^+ = (X_n'X_n)^{-1}X_n'$, $\tilde{\underset{\sim}{\epsilon}}_n^* = (I - P_w)\underset{\sim}{\epsilon}_n$ and $P_w = \{X_n - X_n^{+'}H[H'(X_n'X_n)^{-1}H]^{-1}H'\}X_n^+$, is the projection operator on $\mathcal{L} = \{X_n\underset{\sim}{\beta} \mid H'\underset{\sim}{\beta} = \underset{\sim}{h}\}$. By straightforward calculation,

$$R_1^2 - R_0^2 = \underset{\sim}{\epsilon}_n'(P_n - P_w)\underset{\sim}{\epsilon}_n = \underset{\sim}{\epsilon}_n'P_n^*\underset{\sim}{\epsilon}_n' = \hat{\underset{\sim}{\epsilon}}_n^{*'}\hat{\underset{\sim}{\epsilon}}_n^*. \tag{4.1}$$

Here $P_n^* = P_n - P_w$ is the projection operator on the orthogonal complement of $\mathcal{L}$ in $R(X_n)$ and $\hat{\underset{\sim}{\epsilon}}_n^* = P_n^*P_n\underset{\sim}{\epsilon}_n = P_n^*X_nX_n^+\underset{\sim}{\epsilon}_n = X_n^*(\underset{\sim}{b}_n - \underset{\sim}{\beta})$ since $P_n^*P_n = P_nP_n^* = P_n^*$, where $X_n^* = P_n^*X_n$.

According to (4.1) and the following Lemma, we have an immediate consequence that $(R_1^2 - R_0^2)/\sigma^2 \rightarrow \chi_k^2$. Therefore, suppose we have a consistent estimate $\hat{\sigma}_n^2$ of σ^2; the straightforward application of this result is that the hypothesis $H_0 : H'\underset{\sim}{\beta} = \underset{\sim}{h}$ will be rejected at level α if $(R_1^2 - R_0^2)/\hat{\sigma}_n^2 > \chi_{k,1-\alpha}^2$ for large n, where $\tilde{\chi}_{k,1-\alpha}^2$ is the $(1-\alpha) \times 100\%$ percentage point of the χ_k^2. The following result can be easily derived.

Lemma. *Under the assumptions of Theorem 1, and the condition*

$$B_n^{-1} X_n^{*\prime} X_n^* (B_n')^{-1} \xrightarrow{p} \Gamma^{1/2} \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix} \Gamma^{1/2} \tag{4.2}$$

then

$$(\underset{\sim}{b}_n - \underset{\sim}{\beta})' X_n^{*\prime} X_n^* (\underset{\sim}{b}_n - \underset{\sim}{\beta}) \xrightarrow{\mathcal{D}} \sigma^2 \chi_k^2,$$

where U is a k-dimensional idempotent matrix with rank k and Γ is positive definite.

Moreover, we can obtain an estimate of σ^2 as follows.

Theorem 3. *Under the assumptions of Theorem 1, we have*

$$\hat{\sigma}_n^2 = \sum_{i=1}^{n} (y_i - \underset{\sim}{x}_i' \underset{\sim}{b}_n)^2 / n \xrightarrow{p} \sigma^2 \quad as \quad n \rightarrow \infty.$$

Proof. Note that

$$\begin{aligned} \hat{\sigma}_n^2 &= \frac{1}{n}(Y_n - X_n \underset{\sim}{b}_n)'(Y_n - X_n \underset{\sim}{b}_n) = \frac{1}{n}[\underset{\sim}{\epsilon}_n' \underset{\sim}{\epsilon}_n - \underset{\sim}{\epsilon}_n' X_n (X_n' X_n)^{-1} X_n' \underset{\sim}{\epsilon}_n] \\ &= \frac{1}{n}[\underset{\sim}{\epsilon}_n' \underset{\sim}{\epsilon}_n - (\underset{\sim}{b}_n - \underset{\sim}{\beta})' X_n' X_n (\underset{\sim}{b}_n - \underset{\sim}{\beta})]. \end{aligned}$$

Since $(\underset{\sim}{b}_n - \underset{\sim}{\beta})' X_n' X_n (\underset{\sim}{b}_n - \underset{\sim}{\beta}) \xrightarrow{\mathcal{D}} \sigma^2 \chi_\rho^2$ as $n \rightarrow \infty$ and $\sum_{i=1}^n [\epsilon_i^2 - E(\epsilon_i^2 \mid \mathcal{F}_{i-1})] = o(n)$ a.s. (see Chow (1965)), we have $\hat{\sigma}_n^2 \xrightarrow{p} \sigma^2$ as $n \rightarrow \infty$.

5. Summary

We also note that the constrained least squares estimate for inequality constraints can be approached as in the above discussion and the corresponding strong (or weak) consistency can be investigated. For brevity, we omit these discussions. Thus, in this note, using the result of Theorem 1, we investigate the estimate of $\underset{\sim}{\beta}$ under the constraint $H'\underset{\sim}{\beta} = \underset{\sim}{h}$ and establish the strong consistency, and weak convergence of this estimate. Finally, we obtain an approximated χ^2 test for testing $H'\underset{\sim}{\beta} = \underset{\sim}{h}$ against the alternative $H'\underset{\sim}{\beta} \neq \underset{\sim}{h}$.

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